Two applications of the spectrum of numbers

Christiane Frougny*and Edita Pelantová†

Abstract

Let the base β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite alphabet of digits. The *A-spectrum* of β is the set $S_A(\beta) = \{\sum_{k=0}^{n} a_k\beta^k \mid n \in \mathbb{N},\ a_k \in A\}$. We show that the spectrum $S_A(\beta)$ has an accumulation point if and only if 0 has a particular (β, A)-representation, said to be *rigid*.

The first application is restricted to the case that $\beta > 1$ and the alphabet is $A = \{-M, \dots, M\}$, $M \geq 1$ integer. We show that the set $Z_{\beta,M}$ of infinite (β, A)-representations of 0 is recognizable by a finite Büchi automaton if and only if the spectrum $S_A(\beta)$ has no accumulation point. Using a result of Akiyama-Komornik and Feng, this implies that $Z_{\beta,M}$ is recognizable by a finite Büchi automaton for any positive integer $M \geq \lceil\beta\rceil - 1$ if and only if β is a Pisot number. This improves the previous bound $M \geq \lceil\beta\rceil$.

For the second application the base and the digits are complex. We consider the on-line algorithm for division of Trivedi and Ercegovac generalized to a complex numeration system. In on-line arithmetic the operands and results are processed in a digit serial manner, starting with the most significant digit. The divisor must be far from 0, which means that no prefix of the (β, A)-representation of the divisor can be small. The numeration system (β, A) is said to *allow preprocessing* if there exists a finite list of transformations on the divisor which achieve this task. We show that (β, A) allows preprocessing if and only if the spectrum $S_A(\beta)$ has no accumulation point.

Key words: spectrum, Pisot number, Büchi automaton
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1 Introduction

The so-called *beta-numeration* has been introduced by Rényi in [21] and studied by Parry in [20] in the case that β is a real number, $\beta > 1$, and since then there are been many works in this domain, in connection with number theory, dynamical systems, and automata theory, see the survey [12] or more recent [22] for instance.

For $\beta > 1$ and $M \geq 1$ an integer, the following spectrum

$$X_M(\beta) = \{\sum_{k=0}^{n} a_k\beta^k \mid n \in \mathbb{N},\ a_k \in \{0, 1, \dots, M\}\}$$

has been introduced by Erdős, Joó and Komornik [8].

Since $X_M(\beta)$ is discrete its elements can be arranged into an increasing sequence

$$0 = x_0 < x_1 < \cdots$$

Denote $\ell_M(\beta) = \liminf_{k\to\infty}(x_{k+1} - x_k)$. Numerous works have been devoted to the study of this value, see in particular the introduction and the results of [1].

*IRIF, UMR 8243 CNRS and Université Paris-Diderot
†Doppler Institute for Mathematical Physics and Applied Mathematics, and Department of Mathematics, FNSPE, Czech Technical University in Prague

More generally, let β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite alphabet of digits. The *A-spectrum* of β is the set

$$S_A(\beta) = \Big\{ \sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N},\ a_k \in A \Big\}.$$

Recently Feng answered an open question raised in [8], see also [1], on the density of the spectrum of β when β is real and the digits are consecutive integers:

Theorem 1.1 ([9])**.** *Let $\beta > 1$ and let $A = \{-M, \ldots, M\}$, M an integer ≥ 1. Then the spectrum $S_A(\beta)$ is dense in $\mathbb{R}$ if and only if $\beta < M+1$ and β is not a Pisot number.*

Feng has obtained the following corollary: $\ell_M(\beta) = 0$ if and only if $\beta < M+1$ and β is not a Pisot number.

In this paper we use the concept of spectrum of a number to solve two problems arising in beta-numeration.

Let β and the digits of A be complex. The topological properties of the spectrum are linked with a particular representation of 0. Let $z_1 z_2 \cdots$ be a (β, A)-representation of 0, that is to say, $\sum_{i \geq 1} z_i \beta^{-i} = 0$. It is said to be *rigid* if $0{\boldsymbol .}z_1 z_2 \cdots z_j \neq 0{\boldsymbol .}0 z'_2 \cdots z'_j$ for all $j \geq 2$ and for all $z'_2 \cdots z'_j$ in A^*. The term "rigid" comes from the preprocessing motivation, see Section 5.

We first prove that the spectrum $S_A(\beta)$ has an accumulation point if and only if 0 has a rigid (β, A)-representation, Theorem 3.5.

Then we obtain some results when the base is a complex Pisot number, which extend the real case covered by Garsia [13]. Let β be a complex number, and let $A \subset \mathbb{Q}(\beta)$ containing 0. If β is real and if β or $-\beta$ is a Pisot number, or if $\beta \in \mathbb{C} \setminus \mathbb{R}$ is a complex Pisot number then $S_A(\beta)$ has no accumulation point, Theorem 3.6.

The first question we address in this work is the one of the recognizability by a finite Büchi automaton of the set of infinite β-representations of 0 when β is a real number and the digits are integer.

The set of infinite β-representations of 0 on the alphabet $\{-M, \ldots, M\}$, $M \geq 1$ integer, is denoted

$$Z_{\beta,M} = \{ z_1 z_2 \cdots \mid \sum_{i \geq 1} z_i \beta^{-i} = 0,\ z_i \in \{-M, \ldots, M\} \}.$$

The following result has been formulated in [12]:

Theorem 1.2. *Let $\beta > 1$. The following conditions are equivalent:*

1. *the set $Z_{\beta,M}$ is recognizable by a finite Büchi automaton for every integer M,*
2. *the set $Z_{\beta,M}$ is recognizable by a finite Büchi automaton for one integer $M \geq \lceil \beta \rceil$,*
3. *β is a Pisot number.*

(3) implies (1) is proved in [10], (1) implies (3) is proved in [2] and (2) implies (1) is proved in [11].

Note that in [7] Bugeaud has shown, using (1) implies (3) of Theorem 1.2, that if β is not a Pisot number then there exists an integer M such that $\ell_M(\beta) = 0$.

In this paper we first prove that the set $Z_{\beta,M}$ is recognizable by a finite Büchi automaton if and only if the spectrum $S_A(\beta)$ has no accumulation point, Theorem 4.2.

By [1] or [9] it is known that, for $A = \{-M, \ldots, M\}$, the spectrum $S_A(\beta)$ has an accumulation point if and only if $\beta < M + 1$ and β is not Pisot.

This result together with Theorem 4.2 proves the conjecture stated in [12]:
If the set $Z_{\beta,\lceil\beta\rceil-1}$ is recognizable by a finite Büchi automaton then β must be a Pisot number.

Moreover we obtain a simpler proof of the implication (2) ⇒ (3) of Theorem 1.2. Note that the value $M = \lceil\beta\rceil - 1$ is the best possible as $Z_{\beta,M}$ is reduced to $\{0^\omega\}$ if $M < \lceil\beta\rceil - 1$.

Normalization in base β is the function which maps any β-representation on the canonical alphabet $A_\beta = \{0, \ldots, \lceil\beta\rceil - 1\}$ of a number $x \in [0, 1]$ onto the greedy β-expansion of x. Since the set of greedy β-expansions of the elements of $[0, 1]$ is computable by a finite Büchi automaton when β is a Pisot number, see [4], the following result holds true:
Normalization in base $\beta > 1$ is computable by a finite Büchi automaton on the alphabet $A_\beta \times A_\beta$ if and only if β is a Pisot number.

The second utilisation of the notion of spectrum occurs in the on-line algorithm for division in a complex base.

On-line arithmetic, introduced in [25] for an integer base, is a mode of computation where operands and results are processed in a digit serial manner, starting with the most significant digit. To generate the first digit of the result, the first δ digits of the operands are required. The integer δ is called the delay of the algorithm. One of the interests of the functions that are on-line computable is that they are continuous for the usual topology on the set of infinite words on a finite alphabet.

In [5, 6] we have extended the original on-line algorithm of Trivedi-Ercegovac to a complex base. The algorithm for on-line division in a complex numeration system (β, A) has two parameters: the delay $\delta \in \mathbb{N}$ and $D > 0$, the minimal value (in modulus) of the divisor.

When making division, we need that the divisor stays away from 0. By definition of the on-line algorithm, this means that the value of all the prefixes of the divisor $d_1 d_2 \cdots$ must be greater in absolute value than $D > 0$, so the divisor must be preprocessed before making the division.

We say that a complex numeration system (β, A) *allows preprocessing* if there exists a finite list of transformations on the (β, A)-representation of the divisor which achieve this task, see Definition 5.1.

We show that a complex numeration system (β, A) allows preprocessing if and only if the spectrum $S_A(\beta)$ has no accumulation point, Theorem 5.4.

2 Preliminaries

2.1 Words and automata

Let A be a finite alphabet. A *finite word* w on A is a finite concatenation of letters from A, $w = w_1 \cdots w_n$ with w_i in A. The set of all finite words over A is denoted by A^*. An *infinite word* w on A is an infinite concatenation of letters from A, $w = w_1 w_2 \cdots$ with w_i in A. The set of all infinite words over A is denoted by $A^{\mathbb{N}}$. The infinite concatenation $uuu \cdots$ is noted u^ω. If $w = uv$, u is a *prefix* of w.

An *automaton* $\mathcal{A} = (A, Q, I, T)$ over the alphabet A is a directed graph labeled by letters of A, with a denumerable set Q of vertices called *states*. $I \subseteq Q$ is the set of *initial* states, and $T \subseteq Q$ is the set of *terminal* states. The automaton is said to be *finite* if the set of states Q is finite.

An infinite path of $\mathcal{A}$ is said to be *successful* if it starts in I and goes infinitely often through T. The set of infinite words *recognized* by $\mathcal{A}$ is the set of labels of its successful infinite paths. An automaton used to recognize infinite words in this sense is called a *Büchi automaton*.

2.2 Numeration

Let β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite set, the alphabet of digits. We say that (β, A) is a *numeration system*. A (β, A)-*representation* of a number z is an infinite word $z_1 z_2 \cdots$ such that $z = \sum_{i=1}^{+\infty} z_i \beta^{-i}$ with z_i in A. It should be noted that here we do not make any hypothesis on the fact that every complex number has, or does not have, a (β, A)-representation. This is a difficult problem, studied by many authors, see the pioneering works of Knuth [17], Kátai and Kovács [16], Gilbert [14], Thurston [24] for instance.

We now recall some definitions and results on the so-called *beta-numeration*, see [12] or [22] for a survey. Let $\beta > 1$ be a real number. Any real number $x \in [0, 1]$ can be represented by a greedy algorithm as $x = \sum_{i=1}^{+\infty} x_i \beta^{-i}$ with x_i in the *canonical alphabet* $A_\beta = \{0, \ldots, \lceil \beta \rceil - 1\}$ for all $i \geq 1$. The greedy sequence $(x_i)_{i \geq 1}$ corresponding to a given real number x is the greatest in the lexicographical order, and is said to be the β-*expansion* of x, see [21]. It is denoted by $d_\beta(x) = (x_i)_{i \geq 1}$. When the expansion ends in infinitely many 0's, it is said to be *finite*, and the 0's are omitted.

The greedy β-expansion of 1 is denoted $d_\beta(1) = (t_i)_{i \geq 1}$. When it is finite, of the form $d_\beta(1) = t_1 \cdots t_m$, the *quasi-greedy* β-expansion of 1 is defined as $d^*_\beta(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$. If it is infinite, set $d^*_\beta(1) = d_\beta(1)$. The sequence $d^*_\beta(1)$ is the lexicographically greatest infinite representation of 1 in the base β and the alphabet $\mathbb{N}$. It is known from [20] that a sequence of integers $x_1 x_2 \cdots$ is the greedy β-expansion of some x from $[0, 1]$ if and only if, for all $j \geq 1$, $x_j x_{j+1} \cdots$ is less than or equal to $d^*_\beta(1)$ in the lexicographic order.

Notation: The numerical value $y_{m-1}\beta^{m-1} + \cdots + y_0 + y_{-1}\beta^{-1} + y_{-2}\beta^{-2} + \cdots$ is denoted by $y_{m-1} \cdots y_0 \centerdot y_{-1} y_{-2} \cdots$.

2.3 Numbers

A number $\beta > 1$ such that $d_\beta(1)$ is eventually periodic is a *Parry number*. It is a *simple* Parry number if $d_\beta(1)$ is finite.

A *Pisot number* is an algebraic integer greater than 1 such that all its Galois conjugates have modulus less than 1. Every Pisot number is a Parry number, see [3] and [23].

A *complex Pisot number* is an algebraic integer β such that $|\beta| > 1$ and such that all its Galois conjugates different from its complex conjugate $\overline{\beta}$ have modulus less than 1.

3 Spectrum and rigid representation of 0

Let β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite alphabet. We introduce the A-spectrum of β as

$$S_A(\beta) = \Big\{ \sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N},\ a_k \in A \Big\}.$$

The topological properties of $S_A(\beta)$ are linked with a particular representation of 0.

Definition 3.1. Let $z_1 z_2 \cdots$ be a β-representation of 0 on A, that is to say, $\sum_{i\geq 1} z_i \beta^{-i} = 0$. It is said to be *rigid* if $0{\scriptstyle\bullet} z_1 z_2 \cdots z_j \neq 0{\scriptstyle\bullet}0 z_2' \cdots z_j'$ for all $j \geq 2$ and for all $z_2' \cdots z_j'$ in A^*.

Example 3.2. The signed digit (-1) is denoted $\overline{1}$. In base 2 with alphabet $\{\overline{1}, 0, 1\}$, 0 has two representations, namely $0 = 0{\scriptstyle\bullet}1\overline{1}\,\overline{1}\,\overline{1}\,\overline{1}\cdots = 0{\scriptstyle\bullet}\overline{1}\,1\,1\,1\,1\cdots$. They are not rigid, since $0{\scriptstyle\bullet}1\overline{1} = 0{\scriptstyle\bullet}01$ and $0{\scriptstyle\bullet}\overline{1}1 = 0{\scriptstyle\bullet}0\overline{1}$.

Definition 3.3. Let $z_1 z_2 \cdots$ be a (β, A)-representation of 0. For n in $\mathbb{N}$, its *n-th tail* is $r_n = 0{\scriptstyle\bullet} z_{n+1} z_{n+2} z_{n+3} \cdots$.

Lemma 3.4. *Let $z_1 z_2 \cdots$ be a (β, A)-representation of* 0.

1. *If the sequence $(r_n)_{n\in\mathbb{N}}$ is injective, then the spectrum $S_A(\beta)$ has an accumulation point.*
2. *If the representation of* 0 *is rigid, then the sequence $(r_n)_{n\in\mathbb{N}}$ is injective.*

Proof. Since $0 = 0{\scriptstyle\bullet} z_1 z_2 z_3 \cdots$, the n^{th} tail $r_n = \sum_{k=1}^{+\infty} z_{n+k}\beta^{-k} = -\sum_{k=0}^{n-1} z_{n-k}\beta^k$. It means that $-r_n$ belongs to the spectrum $S_A(\beta)$ and moreover

$$|r_n| \leq \frac{\alpha}{|\beta| - 1}, \qquad \text{where} \;\; \alpha = \max\{|a| \,:\, a \in A\}.$$

1) If the sequence $(r_n)_{n\in\mathbb{N}}$ is injective, then the ball centered at 0 with radius $\frac{\alpha}{|\beta|-1}$ contains infinitely many elements $(-r_n)$ of the spectrum, and thus the spectrum has an accumulation point.

2) Suppose that the representation of 0 is rigid. We show by contradiction the injectivity of $(r_n)_{n\in\mathbb{N}}$. Let us assume that $r_i = r_j$ for some indices $i < j$. Then $\sum_{k=0}^{j-1} z_{j-k}\beta^k = \sum_{k=0}^{i-1} z_{i-k}\beta^k$ and thus $0{\scriptstyle\bullet} z_1 z_2 \cdots z_j = 0{\scriptstyle\bullet} \underbrace{0\cdots 0}_{(j-i)\text{ times}} z_1 \cdots z_i$ — a contradiction with the rigidity of the representation of zero. ◻

Theorem 3.5. *Let β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite alphabet. The spectrum $S_A(\beta)$ has an accumulation point if and only if* 0 *has a rigid (β, A)-representation.*

Proof. $(\Rightarrow)$ Let s be an accumulation point of $S_A(\beta)$. There exists an injective sequence $(x^{(n)})_{n\in\mathbb{N}}$ of points from $S_A(\beta)$ such that $\lim(x^{(n)})_{n\in\mathbb{N}} = s$. For any $x \in S_A(\beta)$ denote

$$\rho(x) = \min\{n \in \mathbb{N} : x = \sum_{k=0}^{n} a_k \beta^k, \text{ with } a_k \in A\}.$$

Set $\rho_n = \rho(x^{(n)})$, then $x^{(n)} = \sum_{k=0}^{\rho_n} x_k^{(n)}\beta^k$. The sequence $(\rho_n)_{n\in\mathbb{N}}$ is unbounded, as there exists only a finite number of strings of a given length over a finite alphabet. Without loss of generality assume that $(\rho_n)_{n\in\mathbb{N}}$ is strictly increasing. Clearly,

$$\frac{x^{(n)}}{\beta^{1+\rho_n}} = 0{\scriptstyle\bullet} x_{\rho_n}^{(n)} \cdots x_2^{(n)} x_1^{(n)} x_0^{(n)} 0000 \cdots \to 0 \tag{3.1}$$

since the nominators tend to s. The fact that $A^{\mathbb{N}}$ endowed with the product topology is a compact space implies the existence of a string $x_1 x_2 x_3 \cdots$ which is the limit of a subsequence of $(x_{\rho_n}^{(n)} \cdots x_2^{(n)} x_1^{(n)} x_0^{(n)} 0^\omega)_{n\in\mathbb{N}}$. It means that for any $N \in \mathbb{N}$ one can find $n \in \mathbb{N}$ such that $\rho_n > N$ and $x_{\rho_n}^{(n)} \cdots x_2^{(n)} x_1^{(n)} x_0^{(n)}$ is a prefix of $x_1 x_2 x_3 \cdots$. The definition of ρ_n and the fact (3.1) forces $0\raisebox{0.1ex}{\textbf{.}}x_1x_2x_3\cdots$ to be a rigid representation of 0.

($\Leftarrow$) Let $0 = 0\raisebox{0.1ex}{\textbf{.}}z_1z_2z_3\cdots$ be a rigid representation of zero. Then by Point 2 of Lemma 3.4, the sequence of its tails is injective and by Point 1 of the same lemma, the spectrum has an accumulation point. ☐

We now turn to the Pisot case. The real case is due to Garsia [13], and we follow his idea.

Theorem 3.6. *Let β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{Q}(\beta)$ containing 0.*

1. *If β is real and if β or $-\beta$ is Pisot*
2. *or if $\beta \in \mathbb{C} \setminus \mathbb{R}$ is complex Pisot*

then $S_A(\beta)$ has no accumulation point.

Proof. Let $\beta = \beta_1$ be a complex Pisot number of degree r with conjugates $\beta_2 = \overline{\beta_1}, \beta_3, \ldots, \beta_r$, i.e. $|\beta_k| < 1$ for $k = 3, 4, \ldots, r$. We denote $\sigma_k : \mathbb{Q}(\beta_1) \to \mathbb{Q}(\beta_k)$ the isomorphism induced by $\beta_1 \mapsto \beta_k$. As A is finite there exists $q \in \mathbb{N}$ such that qA belongs to the ring of integers of the field $\mathbb{Q}(\beta)$. In particular, the norm $N(qa) = q^r \prod_{k=1}^{r} |\sigma_k(a)|$ is an integer for any letter a in A.

Consider $x, y \in S_A(\beta)$, $x \neq y$. Then the difference between x and y can be expressed as $x - y = v = \sum_{j=0}^{n} b_j\beta^j$, for some n in $\mathbb{N}$ and b_j in $A - A$.

Let us denote $A_k = \max\{|\sigma_k(a)| : a \in A\}$. For $k = 3, 4, \ldots, r$, the modulus of the k-th conjugate of v satisfies

$$|\sigma_k(v)| \le \sum_{j=0}^{n} |b_j|.|\beta_k|^j \le 2A_k \sum_{j=0}^{\infty} |\beta_k|^j = 2A_k \tfrac{|\beta_k|}{1-|\beta_k|}.$$

Since β and qb_k are algebraic integers, qv is an algebraic integer as well and its norm is a rational non-zero integer. Compute the norm of qv

$$1 \le |N(qv)| = q^r \prod_{k=1}^{r} |\sigma_k(v)| \le q^r v\,\overline{v} \prod_{k=3}^{r} |\sigma_k(v)| \le (2q)^r v\overline{v} \prod_{k=3}^{r} \frac{A_k|\beta_k|}{1-|\beta_k|}\,.$$

It means that the squared distance $v\overline{v}$ of two different points from the spectrum $S_A(\beta)$ is bounded from below by the constant $(2q)^{-r} \prod_{k=3}^{r} \frac{1-|\beta_k|}{A_k|\beta_k|}$. Consequently, the spectrum has no accumulation point.

The case β real is analogous. ☐

If the base β is real and the alphabet is a symmetric set of consecutive integers, Theorem 3.5 together with the following theorem answers completely the question of the existence of a rigid representation of zero.

Theorem 3.7 (Akiyama and Komornik [1], Feng [9])**.** *Let $\beta > 1$ and let $A = \{-M, \ldots, M\}$. Then $S_A(\beta)$ has an accumulation point if and only if $\beta < M + 1$ and β is not Pisot.*

If the base β is real but the alphabet is not symmetric we have only the following partial observation.

Proposition 3.8. *Let $\beta > 1$ and $\{-1, 0, 1\} \subset A = \{m, \ldots, 0, \ldots, M\} \subset \mathbb{Z}$.*

1. *Zero has a non-trivial (β, A)-representation if and only if $\beta \le \max\{M + 1, -m + 1\}$.*

2. *If $\beta \le \max\{M+1, -m+1\}$, and β is not a Parry number, then zero has a rigid (β, A)-representation.*

Proof. Let Let $d_\beta(1) = t_1 t_2 t_3 \cdots$ be the greedy expansion of 1. Then $\beta - 1 \le t_1 < \beta$, $t_i \le t_1$ and

$$0 = 0.\overline{1} t_1 t_2 t_3 \cdots = 0.1 \overline{t_1}\, \overline{t_2}\, \overline{t_3} \cdots$$

We have two non-trivial representations of 0 over the alphabets $\{-\lceil\beta\rceil+1, \ldots, \overline{1}, 0, 1\}$ and $\{\overline{1}, 0, 1, \ldots, \lceil\beta\rceil - 1\}$ respectively.

Therefore, if $\{-1, 0, 1, \ldots, t_1\} \subset A$ or $\{-t_1, \ldots, -1, 0, 1, \} \subset A$, zero has a non-trivial (β, A)-representation. Let us note that $t_1 \in A$ means $M \ge t_1 \ge \beta - 1$. Similarly $-t_1 \in A$ implies $m \le -t_1 \le -\beta + 1$.

On the other hand, let $M < \beta - 1$ and $m > -\beta + 1$. Then for $z = \sum_{k\ge1} z_i \beta^{-i}$ with $z_i \in A$ and $z_1 \ge 1$, we have $z \ge \frac{1}{\beta} + \sum_{i\ge2} \frac{m}{\beta} = \frac{\beta-1+m}{\beta(\beta-1)} > 0$. Analogously, if $z_1 \le -1$, then $z < 0$. Consequently, 0 has only the trivial representation.

Now assume that β is not a Parry number. Then the sequence of the n^{th} tails of the β-expansion of 1, $r_n = 0.t_{n+1}t_{n+2}\cdots$, is injective. By Lemma 3.4 and Theorem 3.5, zero has a rigid (β, A)-representation. □

Remark 3.9. A numeration system with negative base $-\beta < -1$ and an alphabet $A_{-\beta} = \{0, \ldots, \lfloor\beta\rfloor\}$ was introduced by Ito and Sadahiro in [15]. Liao and Steiner in [19] defined an Yrrap number as an analogy of a Parry number for numeration systems with negative base. This definition implies that if β is not Yrrap, then there exists a rigid $(-\beta, A)$-representation of 0 over the alphabet $A = \{1, \ldots, \lfloor\beta\rfloor + 1\}$.

4 A problem in automata theory

4.1 Representations of 0

Let β be a real number > 1 We consider infinite β-representations of 0 on an alphabet of the form $\{-M, \ldots, M\}$, $M \ge 1$ integer. Let

$$Z_{\beta,M} = \{z_1 z_2 \cdots \mid \sum_{i\ge1} z_i \beta^{-i} = 0,\ z_i \in \{-M, \ldots, M\}\}$$

be the set of infinite words having value 0 in base β on the alphabet $\{-M, \ldots, M\}$.

Proposition 3.8 says that 0 has a non-trivial representation only if $M \ge \lceil\beta\rceil - 1$. Therefore, we consider only M satisfying this inequality.

Note that, if $Z_{\beta,M}$ is recognizable by a finite Büchi automaton, then, for every $c < M$, $Z_{\beta,c} = Z_{\beta,M} \cap \{-c, \ldots, c\}^{\mathbb{N}}$ is recognizable by a finite Büchi automaton as well.

We briefly recall the construction of the (not necessarily finite) Büchi automaton recognizing $Z_{\beta,M}$, see [10] and [12]:

- the set of states is $Q_M \subset \{\sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N},\ a_k \in \{-M, \ldots, M\}\} \cap [-\frac{M}{\beta-1}, \frac{M}{\beta-1}]$
- for $s, t \in Q_M$, $a \in \{-M, \ldots, M\}$ there is an edge

$$s \xrightarrow{a} t \iff t = \beta s + a$$

- the initial state is 0
- all states are terminal.

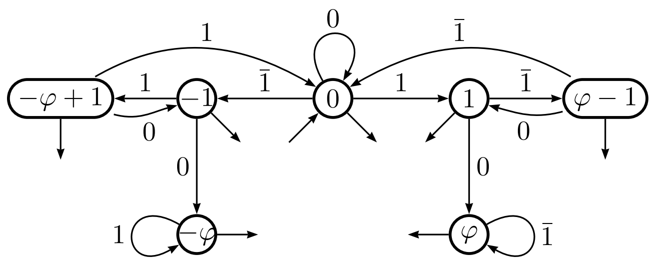

Figure 1: Finite Büchi automaton recognizing $Z_{\varphi,1}$ for $\varphi = \frac{1+\sqrt{5}}{2}$.

Example 4.1. Take $\beta = \varphi = \frac{1+\sqrt{5}}{2}$ the Golden Ratio. It is a Pisot number, with $d_\varphi(1) = 11$. A finite Büchi automaton recognizing $Z_{\varphi,1}$ is designed in Figure 1. The initial state is 0, and all the states are terminal.

Theorem 4.2. *Let $\beta > 1$ and $A = \{-M, \dots, M\}$ with M a fixed integer ≥ 1. The set $Z_{\beta,M}$ is recognizable by a finite Büchi automaton if and only if the spectrum $S_A(\beta)$ has no accumulation point.*

Proof. To any string $z = z_1z_2 \cdots \in Z_{\beta,M}$ we assigne the sequence of polynomials $P_n^{(z)}(X) = z_1X^{n-1} + z_2X^{n-2} + \cdots + z_{n-1}X + z_n$. Denote $R_n^{(z)}$ the remainder of the Euclidean division of the polynomial $P_n^{(z)}(X)$ by the polynomial $(X - \beta)$. It means that there exists a polynomial $Q_n^{(z)}(X)$ such that $P_n^{(z)}(X) = (X-\beta)Q_n^{(z)}(X) + R_n^{(z)}$. Clearly $P_n^{(z)}(\beta) = R_n^{(z)}$. Denote $R = \{R_n^{(z)} : z \in Z_{\beta,M} \text{ and } n \in \mathbb{N}\}$.

As $z = z_1z_2 \cdots$ is a (β, A)-representation of 0, the value $P_n^{(z)}(\beta) = -0{\bullet}z_{n+1}z_{n+2} \cdots$ belongs to the spectrum $S_A(\beta)$ and $-P_n^{(z)}(\beta)$ is the n^{th} tail r_n of the (β, A)-representation of 0. Consequently,

$$R \subset S_A(\beta) \quad \text{and} \quad R \quad \text{is bounded.} \tag{4.1}$$

To prove the theorem, we apply Proposition 3.1 from [10]. It says that $Z_{\beta,M}$ is recognizable by a finite Büchi automaton if and only if the set R is finite.

($\Leftarrow$) If $Z_{\beta,M}$ is not recognizable by finite automaton, then R is infinite and by (4.1) the spectrum has an accumulation point.

($\Rightarrow$) If $S_A(\beta)$ has an accumulation point, then by Theorem 3.5, zero has a rigid representation $z_1z_2 \cdots \in Z_{\beta,M}$. By Point 2 of Lemma 3.4, the sequence of its tails (r_n) is injective. Since $-r_n = P_n^{(z)}(\beta) = R_n^{(z)} \in R$, the set R is not finite and therefore $Z_{\beta,M}$ is not recognizable by finite automaton. □

Combining Theorems 3.7 and 4.2, we answer a conjecture raised in [12] and obtain the following result.

Theorem 4.3. *Let $\beta > 1$. The following conditions are equivalent:*

1. *the set $Z_{\beta,M}$ is recognizable by a finite Büchi automaton for every positive integer M,*
2. *the set $Z_{\beta,M}$ is recognizable by a finite Büchi automaton for one $M \geq \lceil\beta\rceil - 1$,*
3. *β is a Pisot number.*

Remark 4.4. The fact that, if β is not a Pisot number, then the set $Z_{\beta,M}$ is not recognizable by a finite Büchi automaton for any $M \geq \lceil\beta\rceil$ was already settled in Theorem 1.2, but the proof given above is more direct than the original one.

4.2 Normalization

Normalization in base β is the function which maps a β-representation on the canonical alphabet $A_\beta = \{0, \ldots, \lceil\beta\rceil - 1\}$ of a number $x \in [0,1]$ onto the greedy β-expansion of x. From the Büchi automaton $\mathcal{Z}$ recognizing the set of representations of 0 on the alphabet $\{-\lceil\beta\rceil + 1, \ldots, \lceil\beta\rceil - 1\}$, one constructs a Büchi automaton (a converter) $\mathcal{C}$ on the alphabet $A_\beta \times A_\beta$ that recognizes the set of couples on A_β that have the same value in base β, as follows:

$$s \xrightarrow{(a,b)} t \text{ in } \mathcal{C} \iff s \xrightarrow{a-b} t \text{ in } \mathcal{Z},$$

see [12] for details. Obviously $\mathcal{C}$ is finite if and only if $\mathcal{Z}$ is finite.

Then we take the intersection of the set of second components with the set of greedy β-expansions of the elements of $[0,1]$, which is recognizable by a finite Büchi automaton when β is a Pisot number, see [4]. Thus the following result holds true.

Corollary 4.5. *Normalization in base $\beta > 1$ is computable by a finite Büchi automaton on the alphabet $A_\beta \times A_\beta$ if and only if β is a Pisot number.*

5 On-line division in complex base

5.1 Trivedi-Ercegovac algorithm

On-line arithmetic, introduced in [25], is a mode of computation where operands and results are processed in a digit serial manner, starting with the most significant digit. To generate the first digit of the result, the first δ digits of the operands are required. The integer δ is called the delay of the algorithm.

In [5, 6] we have extended the original on-line algorithm of Trivedi-Ercegovac to the complex case.

The algorithm for on-line division in a complex numeration system (β, A) has two parameters: the delay $\delta \in \mathbb{N}$ and $D > 0$, the minimal value (in modulus) of the divisor.

The (β, A)-representation of the nominator is $n = \sum_{i=1}^{\infty} n_i \beta^{-i}$, of the divisor is $d = \sum_{i=1}^{\infty} d_i \beta^{-i}$, and of their quotient $q = \sum_{i=1}^{\infty} q_i \beta^{-i}$. Partial sums are denoted by $N_k = \sum_{i=1}^{k} n_i \beta^{-i}$, $D_k = \sum_{i=1}^{k} d_i \beta^{-i}$, and $Q_k = \sum_{i=1}^{k} q_i \beta^{-i}$.

The inputs of the algorithm are two infinite strings $0{\bullet}n_1 n_2 \cdots n_\delta n_{\delta+1} n_{\delta+2} \cdots$ with $n_i \in A$ and $n_1 = n_2 = \cdots = n_\delta = 0$ and $0{\bullet}d_1 d_2 \cdots$ with $d_i \in A$ satisfying $|D_j| \geq D$ for all $j \in \mathbb{N}$, $j \geq 1$.

The output is a string $q_1 q_2 q_3 \cdots$ corresponding to a (β, A)-representation of the quotient $q = n/d = 0{\bullet}q_1 q_2 q_3 \cdots$. The settings of the algorithm ensure that the representation of q starts behind the fractional point.

Set $W_0 = q_0 = Q_0 = 0$. Then, for $k \geq 1$ compute

$$W_k = \beta(W_{k-1} - q_{k-1} D_{k-1+\delta}) + (n_{k+\delta} - Q_{k-1} d_{k+\delta})\beta^{-\delta}.$$

The k-th digit q_k of the representation of the quotient is evaluated by a function Select, function of the values of the auxiliary variable W_k and the interim representation $D_{k+\delta}$, so that

$$q_k = \mathrm{Select}(W_k, D_{k+\delta}) \in A\,.$$

It can be shown that for any $k \geq 1$:

$$W_k = \beta^k (N_{k+\delta} - Q_{k-1} D_{k+\delta})\,.$$

Moreover, if the sequence (W_k) is bounded, then $q = \lim_{k\to\infty} Q_k = \frac{n}{d}$.

Conditions on the system (β, A) so that the definition of the function Select ensures the correctness of the on-line division algorithm are given in [5, 6].

5.2 Preprocessing of divisors

When making division, we need that the divisor stays away from 0. By definition of the on-line algorithm, this means that the value of all the prefixes of the divisor $d_1d_2\cdots$ must be greater in absolute value than a parameter $D > 0$.

Definition 5.1. We say that a complex numeration system (β, A) *allows preprocessing* if there exists $D > 0$ and a finite list $\mathcal{L}$ of identities of the type $0{\bullet}w_k\cdots w_0 = 0{\bullet}0u_{k-1}\cdots u_0$ with digits in A such that any string $d_1d_2\cdots$ on A without prefix $w_k\cdots w_0$ from $\mathcal{L}$ satisfies $|0{\bullet}d_1d_2\cdots d_j| > D$ for all $j \in \mathbb{N}$.

We must have at least $d_1 \neq 0$ after preprocessing, so the preprocessing consists first of all in shifting the fractional point to the most significant non-zero digit of the (β, A)-representation of the divisor. Of course, after preprocessing the value of the original divisor w has been changed into a new one d which is just a shift of the original one, that is to say $d = w\beta^k$ for some k in $\mathbb{Z}$. This will have to be taken into account to give the result of the division.

If zero has only the trivial (β, A)-representation the situation is simple. This fact can be equivalently rewritten as

$$\inf \mathcal{R} > 0, \quad \text{where} \quad \mathcal{R} = \Big\{ \big|\sum_{i\geq 1} z_i\beta^{-i}\big| \ : \ z_1 \neq 0, z_i \in A \Big\}.$$

In this case the numeration system (β, A) allows preprocessing, since we can take $D = \inf \mathcal{R}$ and the list of rewriting rules is empty.

Example 5.2. If $\beta = 4$ and $A = \{\overline{2}, \overline{1}, 0, 1, 2\}$, then zero has only the trivial representation and for D one can take $\frac{1}{12} = \min \mathcal{R}$.

Example 5.3. If $\beta = 2$ and $A = \{\overline{1}, 0, 1\}$, zero has two non-trivial representations $0 = 0{\bullet}1\overline{1}\,\overline{1}\,\overline{1}\,\overline{1}\cdots = 0{\bullet}\overline{1}\,1\,1\,1\,1\cdots$. Therefore, preprocessing is a little bit more sophisticated. Consider the list

$$0{\bullet}\overline{1}1 = 0{\bullet}0\overline{1} \quad \text{and} \quad 0{\bullet}1\overline{1} = 0{\bullet}01$$

If a string $d_1d_2\cdots$ has no prefix $\overline{1}1$ neither $1\overline{1}$, then

$$|0{\bullet}d_1d_2\cdots d_j| \geq 0{\bullet}10\overline{1}\,\overline{1}\,\overline{1}\cdots = \tfrac{1}{4}$$

and thus one can take $D = \frac{1}{4}$.

Theorem 5.4. *A complex numeration system (β, A) allows preprocessing if and only if the spectrum $S_A(\beta)$ has no accumulation point.*

The result is proved by the following three lemmas, in which we use the notation

$$H = \max\{|\sum_{i\geq 1} d_i\beta^{-i}| \ : \ d_i \in A \ \text{ for all } i \in \mathbb{N}\}.$$

Lemma 5.5. *If 0 has a rigid $(\beta, \mathcal{A})$-representation then the numeration system $(\beta, \mathcal{A})$ does not allow preprocessing.*

Proof. Let $0 = 0{\bullet}z_1z_2z_3\cdots$ be a rigid representation of 0. Assume that preprocessing is possible with $D > 0$. Find j such that $\frac{H}{|\beta|^j} < D$. Consider the number $0{\bullet}z_1z_2z_3\cdots z_j000\cdots$. Since the representation of zero is rigid, no prefix of the string $z_1z_2z_3\cdots z_j$ is contained in the list of the rewriting rules. But $|0{\bullet}z_1z_2z_3\cdots z_j| = |0{\bullet}\underbrace{00\cdots0}_{j-\text{times}}z_{j+1}z_{j+2}\cdots| < \frac{H}{|\beta|^j} < D$ — a contradiction. ∎

Lemma 5.6. *Let us assume that $S_A(\beta)$ has no accumulation point and fix $K > 0$. Then there exists $m \in \mathbb{N}$ such that any string $x_{m-1}x_{m-2}\cdots x_1x_0$ of length m over A satisfies either*

$$|x_{m-1}\beta^{m-1} + x_{m-2}\beta^{m-2} + \cdots + x_1\beta + x_0| \geq K$$

or there exists a string $y_{k-1}x_{k-2}\cdots y_1y_0$ of length $k < m$ over A such that

$$x_{m-1}\beta^{m-1} + x_{m-2}\beta^{m-2} + \cdots + x_1\beta + x_0 = y_{k-1}\beta^{k-1} + y_{k-2}\beta^{k-2} + \cdots + y_1\beta + y_0\,.$$

Proof. Since $S_A(\beta)$ has no accumulation point, the set $P = \{z \in S_A(\beta) : |z| < K\}$ is finite. Denote $m = 1 + \max\{\rho(z) : z \in P\}$. Let $x = x_{m-1}\beta^{m-1} + x_{m-2}\beta^{m-2} + \cdots + x_1\beta + x_0$. Obviously, $x \in S_A(\beta)$. Then either $|x| \geq K$ or $x \in P$ and thus $x = y_k\beta^{k-1} + y_{k-2}\beta^{k-2} + \cdots + y_1\beta + y_0$, where $k \leq \max\{\rho(z) : z \in P\} \leq m-1$. □

Lemma 5.7. *If $S_A(\beta)$ has no accumulation point, then there exists $D > 0$ and $m \in \mathbb{N}$ such that for all infinite strings $d_1d_2\cdots$ over A one has*

1. *either* $|0{\bullet}d_1d_2\cdots d_j| \geq D$ *for all* $j \in \mathbb{N}$,
2. *or* $0{\bullet}d_1d_2\cdots d_m = 0{\bullet}0d_2'd_3'\cdots d_m'$ *for some string* $d_2'd_3'\cdots d_m' \in A^*$.

Proof. Let us take $\mu > 0$ and apply Lemma 5.6 with $K = H + \mu$ to get $m \in \mathbb{N}$. Denote $\mathcal{D} = \{|0{\bullet}d_1d_2\cdots d_j| : j < m \text{ and } 0{\bullet}d_1d_2\cdots d_j \neq 0{\bullet}0d_2'\cdots d_j'\}$. The set $\mathcal{D}$ is finite and does not contain zero. Therefore, $D' = \min \mathcal{D} > 0$.

To prove the lemma, consider an infinite string $d_1d_2\cdots$ and assume that $0{\bullet}d_1d_2\cdots d_m \neq 0{\bullet}0d_2'd_3'\cdots d_m'$ for all strings $d_2'd_3'\cdots d_m' \in A^*$. We distinguish two cases

- $j < m$, $j \in \mathbb{N}$. Then $0{\bullet}d_1d_2\cdots d_j \neq 0{\bullet}0d_2'\cdots d_j'$, otherwise $0{\bullet}d_1d_2\cdots d_m = 0{\bullet}0d_2'd_3'\cdots d_j'd_{j+1}\cdots d_m$ — a contradiction. Therefore, $|0{\bullet}d_1d_2\cdots d_j| \geq D'$.
- $j \geq m, j \in \mathbb{N}$. Then

$$|0{\bullet}d_1d_2\cdots d_j| \geq |0{\bullet}d_1d_2\cdots d_m| - \tfrac{1}{|\beta|^m}|0{\bullet}d_{m+1}d_{m+2}\cdots d_j| \geq \tfrac{1}{|\beta|^m}K - \tfrac{1}{|\beta|^m}H = \frac{\mu}{|\beta|^m}$$

Thus we can set $D = \min\left\{D', \frac{\mu}{|\beta|^m}\right\}$. □

The previous lemma gives a hint for creating the list of rewriting rules. We take the index m found by the lemma and inspect all strings $d_1d_2\cdots d_m$ over A. If $0{\bullet}d_1d_2\cdots d_m = 0{\bullet}0d_2'd_3'\cdots d_m'$ for some string $d_2'd_3'\cdots d_m'$ we put it into the list.

Example 5.8. Let $\beta = \varphi = \frac{1+\sqrt{5}}{2}$ and $A = \{\overline{1}, 0, 1\}$. The minimal polynomial of φ is $X^2 - X - 1$. In this numeration system, 0 has countably many finite representations and uncountably many infinite representations. As the alphabet is symmetric, the rewriting rules appear in pairs. For example, as $10\overline{1}$ can be rewritten to 010, also $\overline{1}01$ can be rewritten to $0\overline{1}0$. To shorten our list, we put into it only one rule of each pair, namely the rule, where the first digit is 1. First we consider the list

$\mathcal{L}_0$: $10\overline{1} \longrightarrow 010$, $1\overline{1}0 \longrightarrow 001$, $1\overline{1}\overline{1} \longrightarrow 000$.

Claim: If no rule from $\mathcal{L}_0$ can be applied to the string $d_1d_2\cdots$, then $|d| \geq D = \frac{1}{\varphi^5}$, where $d = 0{\bullet}d_1d_2\cdots$.

Proof. WLOG $d_1 = 1$.

If $d_2 = 0$, then ≥ 0 and thus $|d| \geq \frac{1}{\varphi} - \sum_{k\geq 4}^{\infty}\varphi^{-k} = \frac{1}{\varphi} - \frac{1}{\varphi^2} = \frac{1}{\varphi^3} \geq D$.

If $d_2 = 1$, then $|d| \geq \frac{1}{\varphi} + \frac{1}{\varphi^2} - \sum_{k\geq 3}^{\infty}\varphi^{-k} = 1 - \frac{1}{\varphi} - = \frac{1}{\varphi^2} \geq D$.

If $d_2 = \overline{1}$, then $d_3 = 1$. Therefore, $|d| \geq \frac{1}{\varphi} - \frac{1}{\varphi^2} + \frac{1}{\varphi^3} - \sum_{k\geq 4}^{\infty}\varphi^{-k} = \frac{1}{\varphi^5} \geq D$. □

We can extend the list of rewriting rules to increase the lower bound D. Let us consider the whole families of rules $\mathcal{L}$:

$(1\bar{1})^k 0 \longrightarrow 00(10)^{k-1}1$ for $k \geq 1$.
$(1\bar{1})^k \bar{1} \longrightarrow 00(10)^{k-1}0$ for $k \geq 1$.
$(1\bar{1})^k 10\bar{1} \longrightarrow 01(00)^k 0$ for $k \geq 0$.
$(1\bar{1})^k 100 \longrightarrow 01(00)^k 1$ for $k \geq 0$.
$(1\bar{1})^k 11 \longrightarrow 01(00)^{k-1}10$ for $k \geq 1$.
$10\bar{1}^k 0 \longrightarrow 0^{k+1}11$ for $k \geq 0$.
$10\bar{1}^k 1 \longrightarrow 0^k 101$ for $k \geq 1$.

Claim: If no rule from $\mathcal{L}$ can be applied to the string $d_1 d_2 \cdots$, then $|d| \geq D = \frac{1}{\varphi^2}$, where $d = 0{\bullet}d_1 d_2 \cdots$.

Proof. WLOG $d_1 = 1$. Our string has a prefix 11 or a prefix $(1\bar{1})^k 101$ for $k \geq 0$. Therefore either

$$|d| \geq 0{\bullet}11(\bar{1})^\omega = \tfrac{1}{\varphi^2} \qquad \text{or} \qquad |d| \geq 0{\bullet}(1\bar{1})^k 101(\bar{1})^\omega = \tfrac{1}{\varphi^2} + \tfrac{1}{\varphi^{2k+3}}.$$

□

Some examples where the base is a complex number can be found in [6].

6 Comments and open questions

6.1 F-number

In [18] Lau defined for $1 < \beta < 2$ the following notion, that we extend to any $\beta > 1$.

Definition 6.1. Let $\beta > 1$ and $B_\beta = \{-\lceil\beta\rceil + 1, \ldots, \lceil\beta\rceil - 1\}$ be the symmetrized alphabet of the canonical alphabet A_β. Then β is said to be a *F-number* if the set

$$L_{(\lceil\beta\rceil-1)}(\beta) = S_{B_\beta}(\beta) \cap \Big[-\frac{\lceil\beta\rceil - 1}{\beta - 1}, \frac{\lceil\beta\rceil - 1}{\beta - 1}\Big]$$

is finite.

Feng proved in [9] that $1 < \beta < 2$ is a F-number if and only if it is a Pisot number. This property extends readily to any $\beta > 1$. Another way of proving it consists in realizing that the set of states $Q_{(\lceil\beta\rceil-1)}$ of the automaton for $Z_{\beta,\lceil\beta\rceil-1}$ is included into $L_{(\lceil\beta\rceil-1)}(\beta)$.

6.2 Open questions

- A motivation for introducing the notion of "rigid representation of zero" comes from on-line division in a numeration system (β, A). A more elementary question is "Has zero a non-trivial (β, A)-representation"? The answer is easy for real bases and alphabets of the form $A = \{m, \ldots, 0, \ldots, M\}$, see Proposition 3.8. The same question for complex bases is an open problem.
- In the case that the base is real and the alphabet is $A = \{-M, \ldots, M\}$, Theorem 4.2 says that recognizability by a finite automaton is equivalent to the fact that the spectrum $S_A(\beta)$ has no accumulation point.

  An analogous result can be proved for complex bases as well. But for complex bases the question about the existence of accumulation points in the spectrum $S_A(\beta)$ is not yet investigated. Nevertheless, it is often easy to check that a (β, A)-representation of 0 is **not** rigid.

- If $\beta > 1$ is a non-Pisot base then $A = \{-\lceil\beta\rceil + 1, \ldots, \lceil\beta\rceil - 1\}$ is the smallest symmetric alphabet of consecutive integers for which the spectrum $S_A(\beta)$ has an accumulation point. What is the minimal size of an alphabet $A = \{-M, \ldots, M\} \subset \mathbb{Z}$ for which the spectrum of a non-Pisot complex number β has an accumulation point?

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