Univoque numbers

Christiane Frougny

Joint work with
Jean-Paul Allouche and Kevin Hare

Univoque numbers

$\beta > 1$ is univoque if there exists a unique sequence of integers $(s_n)_{n\geqslant 1}$, with $0 \leqslant s_n < \beta$, such that

$$1 = \sum_{n\geqslant 1} s_n \beta^{-n}$$

2 is univoque, as $1 = \bullet 111\cdots$

$\frac{1+\sqrt{5}}{2}$ is not univoque since

$1 = \bullet 11 = \bullet (10)^n 11 = \bullet (10)^\infty$

Greedy expansions

$\beta > 1$. $x \in [0, 1]$.

Greedy algorithm of Rényi

$r_0 := x$

$x_n := \lfloor \beta r_{n-1} \rfloor$

$r_n := \{\beta r_{n-1}\}$.

Then $x = \sum_{n \geqslant 1} x_n \beta^{-n}$.

$x_n \in A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$

$d_\beta(x) = x_1 x_2 \cdots$ is the greedy β-expansion of x.
It is the greatest representation in the lexicographic order.

$d_\beta(1) = (e_n)_{n\geqslant 1}$ greedy β-expansion of 1

$$d^*_\beta(1) := \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is infinite} \\ (e_1 \cdots e_{m-1}(e_m - 1))^\infty & \text{if } d_\beta(1) = e_1 \cdots e_{m-1} e_m \text{ is finite.} \end{cases}$$

Theorem 1. (Parry) *$s = (s_n)_{n\geqslant 1}$ in $A_\beta^{\mathbb{N}_+}$.*

- *s is the greedy β-expansion of some $x \in [0, 1)$ if and only if*

$$\forall k \geqslant 0, \quad \sigma^k(s) <_{lex} d^*_\beta(1)$$

- *s is the greedy β-expansion of 1 for some $\beta > 1$ if and only if*

$$\forall k \geqslant 1, \quad \sigma^k(s) <_{lex} s$$

Lazy expansions

$B := \sum_{n\geqslant 1} \frac{(\lceil\beta\rceil-1)}{\beta^n} = \frac{(\lceil\beta\rceil-1)}{\beta-1}$.

Lazy algorithm:

$r_0 := x$

$x_n := \max(0, \lceil \beta r_{n-1} - B \rceil)$

$r_n := \beta r_{n-1} - x_n$.

Then $x = \sum_{n\geqslant 1} x_n \beta^{-n}$

$\ell_\beta(x) = x_1 x_2 \cdots$ is the lazy β-expansion of x.
It is the smallest representation of x in the lexicographical order.

$s = (s_n)_{n \geqslant 1}$ in $A_\beta^{\mathbb{N}_+}$.

$\overline{s_n} := (\lceil \beta \rceil - 1) - s_n$ the complement of s_n, and $\bar{s} := (\overline{s_n})_{n \geqslant 1}$.

Theorem 2. (Erdős, Joó and Komornik; Dajani and Kraaikamp) *$s = (s_n)_{n \geqslant 1}$ in $A_\beta^{\mathbb{N}_+}$.*

- *s is the lazy β-expansion of some $x \in [0, 1)$ if and only if*

$$\forall k \geqslant 0, \quad \sigma^k(\bar{s}) <_{lex} d_\beta^*(1)$$

- *s is the lazy β-expansion of 1 for some $\beta > 1$ if and only if*

$$\forall k \geqslant 1, \quad \sigma^k(\bar{s}) <_{lex} s$$

$$s = \ell_\beta(x) \iff \bar{s} = d_\beta(B - x)$$

Example

$\psi_1 = \frac{1+\sqrt{5}}{2}$ the golden ratio.

Greedy β-expansion $d_{\psi_1}(1) = 11$.

$d^*_{\psi_1}(1) = (10)^\infty$.

A greedy expansion of $x \in [0,1)$ does not have the factor 11.

Lazy expansion $\ell_{\psi_1}(1) = 01^\infty$.

A lazy expansion of $x \in [0,1)$ does not have the factor 00.

Univoque numbers

β is univoque if there exists a unique sequence of integers $(s_n)_{n \geqslant 1}$, with $0 \leqslant s_n < \beta$, such that

$$1 = \sum_{n \geqslant 1} s_n \beta^{-n}$$

So

$$d_\beta(1) = \ell_\beta(1)$$

Remark: $\sigma^k(\bar{s}) \leqslant_{lex} s \iff \bar{s} \leqslant_{lex} \sigma^k(s)$.

$$\Gamma = \{s \in \{0,1\}^{\mathbb{N}_+} \mid \forall k \geqslant 1,\ \bar{s} \leqslant_{lex} \sigma^k(s) \leqslant_{lex} s\}$$

Γ set of binary self-bracketed sequences.

$$\Gamma_{strict} = \{s \in \{0,1\}^{\mathbb{N}_+} \mid \forall k \geqslant 1,\ \bar{s} <_{lex} \sigma^k(s) <_{lex} s\}$$

Γ_{strict} set of binary strictly self-bracketed sequences.

If $\sigma^k(s) = s$ or $\sigma^k(s) = \bar{s}$ for some $k \geqslant 1$ then the sequence s is periodic.

Theorem 3. (Erdős, Joó and Komornik) *A sequence in $\{0,1\}^{\mathbb{N}_+}$ is the unique β-expansion of 1 for a univoque number β in $(1,2)$ if and only if it is strictly self-bracketed.*

$$\mathcal{U} = \{\beta \in (1,2) : d_\beta(1) \in \Gamma_{strict}\}$$

set of univoque numbers in $(1,2)$

There exists a smallest univoque number, the Komornik-Loreti constant $\kappa \approx 1.787231$ and $d_\kappa(1) = (t_n)_{n\geqslant 1}$, where $(t_n)_{n\geqslant 1} = 11010011\ldots$ is the shifted Thue-Morse sequence.

Thue-Morse sequence: $0 \to 01;\ \ 1 \to 10$

The Komornik-Loreti constant κ is transcendental (Allouche and Cosnard).

$\widetilde{\mathcal{U}} = \{\beta \in (1,2) : d_\beta(1) \text{ is finite and } d^*_\beta(1) \text{ is periodic self} - \text{bracketed}\}$

$\psi_1 = \frac{1+\sqrt{5}}{2}$ the golden ratio.

Greedy β-expansion $d_{\psi_1}(1) = 11$

$d^*_{\psi_1}(1) = (10)^\infty$

Lazy expansion $\ell_{\psi_1}(1) = 01^\infty$.

The golden ratio is the smallest element of $\widetilde{\mathcal{U}}$.

Pisot and Salem numbers

A Pisot number is an algebraic integer > 1 such that all its algebraic conjugates (other than itself) have modulus < 1. The set of Pisot numbers is denoted by S.

S is closed (Salem), and has a smallest element, which is the root > 1 of the polynomial $x^3 - x - 1$ (approx. 1.3247).

A Salem number is an algebraic integer > 1 such that all its algebraic conjugates have modulus $\leqslant 1$, with at least one conjugate on the unit circle.

Theorem 4. (Bertrand; Schmidt) *Let β be a Pisot number. A number x of $[0, 1]$ has a (finite or infinite) eventually periodic greedy β-expansion if and only if it belongs to $\mathbb{Q}(\beta)$.*

Corollary 1. *Let β be a Pisot number. A number x of $[0, 1]$ has an eventually periodic lazy β-expansion if and only if it belongs to $\mathbb{Q}(\beta)$.*

If β is a Salem number of degree 4, then $d_\beta(1)$ is eventually periodic (Boyd).

Conjecture: holds true for degree 6.

A Parry number is a number β such that $d_\beta(1)$ is eventually periodic. If $d_\beta(1)$ is finite, it is a simple Parry number.

Limit points of Pisot numbers

Theorem 5. (Amara) *The limit points of S in $(1,2)$ are the following:*

$$\varphi_1 = \psi_1 < \varphi_2 < \psi_2 < \varphi_3 < \chi < \psi_3 < \varphi_4 < \cdots < \psi_r < \varphi_{r+1} < \cdots < 2$$

Minimal Polynomial	Pisot Number	Greedy expansion	Lazy expansion	Comment
$x^{r+1} - 2x^r + x - 1$	φ_r	$1^r 0^{r-1} 1$	$1^{r-1} 0 1^\infty$	periodic s-b
$x^{r+1} - x^r - \cdots - 1$	ψ_r	1^{r+1}	$(1^r 0)^\infty$	periodic s-b
$x^4 - x^3 - 2x^2 + 1$	χ	$11(10)^\infty$	$11(10)^\infty$	univoque

Questions

Is the set of univoque Pisot numbers in $(1, 2)$ closed?

Is there a smallest univoque Pisot number?

Preliminary combinatorial results

$\Gamma = \{s \in \{0,1\}^{\mathbb{N}_+} \mid \forall k \geqslant 1,\ \overline{s} \leqslant_{lex} \sigma^k(s) \leqslant_{lex} s\}$ is a closed set.

Lemma 1. (Allouche)

- *If b in Γ begins with $u\overline{u}$ then $b = (u\overline{u})^\infty$.*
- *If $b = (z0)^\infty$ is in Γ, then*

$$\Phi(b) := (z1\overline{z}0)^\infty$$

belongs to Γ, and there is no element of Γ lexicographically between b and $\Phi(b)$.

Corollary 2. *Let $b = (z0)^\infty$. The sequence $(\Phi^{(n)}(b))_{n \geqslant 0}$ is a sequence of elements of Γ that converges to a limit $\Phi^{(\infty)}(b)$ in Γ. The only elements of Γ lexicographically between b and $\Phi^{(\infty)}(b)$ are the $\Phi^{(k)}(b)$, $k \geqslant 0$.*

Lemma 2. *A sequence of* Γ *of the form* $(w0)^\infty$ *cannot be a limit from above of a non-eventually constant sequence of elements of* Γ*.*

Take $b = d^*_{\psi_r}(1) = (1^r0)^\infty$. Then $\Phi(b) = (1^r10^r0)^\infty = d^*_{\varphi_{r+1}}(1)$.

We say that $\varphi_{r+1} = \Phi(\psi_r)$.

Let π_r defined by $d^*_{\pi_r}(1) = \Phi^{(\infty)}((1^r0)^\infty)$, that is, $\pi_r = \Phi^\infty(\psi_r)$.

Proposition 1. *The number* π_r *is univoque. Between* ψ_r *and* $\pi_r = \Phi^{(\infty)}(\psi_r)$ *the only real numbers belonging to* $\mathcal{U}$ *or* $\widetilde{\mathcal{U}}$ *are the numbers* φ_{r+1}, $\Phi(\varphi_{r+1})$, $\Phi^{(2)}(\varphi_{r+1})$, *etc. They all belong to* $\widetilde{\mathcal{U}}$*.*

Limit points of univoque numbers

Proposition 2. *The limit of a sequence of real numbers belonging to $\mathcal{U}$ belongs to $\mathcal{U}$ or $\widetilde{\mathcal{U}}$.*

The φ_r cannot be limit points of numbers in $\mathcal{U}$, because $d^*_{\varphi_r}(1) = (1^r 0^r)^\infty$, and if $s = 1^r 0^r w \in \Gamma$ then $s = (1^r 0^r)^\infty$.

The ψ_r $(r \geqslant 2)$ are limit points of numbers in $\mathcal{U}$: for instance numbers with expansion $(1^r 0)^n (10)^\infty$.

Proposition 3.

(i) *Let* $t = (t_n)_{n \geqslant 1} = 11010011\ldots$ *be the shifted Thue-Morse sequence, and let* τ_{2^k} *such that* $d_{\tau_{2^k}}(1) = t_1 \cdots t_{2^k}$*. Then* $(\tau_{2^k})_{k \geqslant 1}$ *converges from below to the Komornik-Loreti constant* κ*. The numbers* τ_{2^k} *are simple Parry numbers belonging to* $\widetilde{\mathcal{U}}$*.*

(ii) *There exists a sequence of univoque Parry numbers* $(\delta_{2^k})_{k \geqslant 1}$ *defined by*

$$d_{\delta_{2^k}}(1) = t_1 \cdots t_{2^k-1}(1\overline{t_1} \cdots \overline{t_{2^k-1}})^{\infty}$$

that converges to κ *from above.*

Pisot and Salem of small degree in $(1,2)$

- The golden ratio $\varphi_1 = \psi_1$ is the smallest element of $\widetilde{\mathcal{U}}$.
- There is no univoque Pisot number of degree 2 or 3.
- The number χ is the unique Pisot number of degree 4 which is univoque.
- There exists a unique Salem number of degree 4 which is univoque.
- Salem numbers of degrees 4 and 6 that are greater than the Komornik-Loreti constant κ are univoque.

First result

Theorem 6. *There exists a smallest Pisot number in the set $\mathcal{U}$.*

Proof. $\theta := \inf(S \cap \mathcal{U})$. $\theta \in S$, since S is closed.

θ is in $\mathcal{U}$ or in $\widetilde{\mathcal{U}}$.

Suppose θ is in $\widetilde{\mathcal{U}}$. Then $d^*_\theta(1) = (w0)^\infty$.

Then θ would be a limit point of elements of $(S \cap \mathcal{U})$.

But $(w0)^\infty$ cannot be limit from above of elements of Γ. □

Regular and irregular Pisot numbers

The Pisot numbers approaching φ_r, ψ_r or χ are called *regular Pisot numbers*, and are described by Talmoudi.

Further, Talmoudi showed that, for all $\varepsilon > 0$, there are only a finite number of Pisot numbers in $(1, 2 - \varepsilon)$, that are not regular. These are called the *irregular Pisot numbers*.

For any interval $[a, b]$, with $b < 2$, an algorithm of Boyd finds all Pisot numbers in the interval. If $[a, b]$ contains a limit point θ, then there exists an $\varepsilon > 0$ such that all Pisot numbers in $[\theta - \varepsilon, \theta + \varepsilon]$ are *regular* Pisot numbers of a known form. Boyd's algorithm detects these regular Pisot numbers.

- $\varphi_1 = \psi_1 = \frac{1+\sqrt{5}}{2}$ smallest element of $\widetilde{\mathcal{U}}$
- $\varphi_2 \approx 1.754877$
- $\kappa \approx 1.787231$ smallest element of $\mathcal{U}$
- $\psi_2 \approx 1.839286$ "Tribonacci" number
- $\varphi_3 \approx 1.866760$
- $\chi \approx 1.905166$ univoque
- $\psi_3 \approx 1.927562$ "Quadrinacci" number

Since

$$\Phi^2(\psi_2) = \Phi(\varphi_3) \approx 1.870556$$

there are no univoque numbers between ψ_2 and 1.8705. (Note that $1.8705 > \varphi_3$.)

Approaching ψ_2 from below

$P_{\psi_r}(x) = x^{r+1} - x^r - \cdots - 1$ minimal polynomial of ψ_r.

$A_{\psi_r}(x) = x^{r+1} - 1$ and $B_{\psi_r}(x) = \frac{x^r - 1}{x - 1}$ two polynomials associated with $P_{\psi_r}(x)$.

For sufficiently large n, $P_{\psi_r}(x)x^n \pm A_{\psi_r}(x)$ and $P_{\psi_r}(x)x^n \pm B_{\psi_r}(x)$ admit a unique root between 1 and 2, which is a Pisot number.

$P_{\psi_r}(x)x^n - A_{\psi_r}(x)$ and $P_{\psi_r}(x)x^n - B_{\psi_r}(x)$ approach ψ_r from above.

$P_{\psi_r}(x)x^n + A_{\psi_r}(x)$ and $P_{\psi_r}(x)x^n + B_{\psi_r}(x)$ approach ψ_r from below.

By computation of the expansions we obtain

Proposition 4. *There exists a neighborhood* $[\psi_2 - \varepsilon, \psi_2 + \varepsilon]$ *that contains no univoque Pisot numbers.*

Approaching χ

$P_\chi(x) = x^4 - x^3 - 2x^2 + 1$ minimal polynomial of χ.

$A_\chi(x) = x^3 + x^2 - x - 1$ and $B_\chi(x) = x^4 - x^2 + 1$.

$P_\chi(x)x^n - A_\chi(x)$ and $P_\chi(x)x^n - B_\chi(x)$ approach χ from above.

$P_\chi(x)x^n + A_\chi(x)$ and $P_\chi(x)x^n + B_\chi(x)$ approach χ from below.

Theorem 7. *There are only a finite number of univoque Pisot numbers less than* χ.

Theorem 8. *The univoque Pisot number* χ *is the smallest limit point of univoque Pisot numbers. It is a limit point from above of regular univoque Pisot numbers.*

Univoque Pisot numbers less than χ

All univoque Pisot numbers less than χ are either in $[\kappa, \psi_2]$, or in $[\pi_2, \chi]$.

Boyd's algorithm:

227 (irregular) Pisot numbers in $[1.78, 1.85] \supset [\kappa, \psi_2]$

303 in $[1.87, 1.91] \supset [\pi_2, \chi]$

Theorem 9. *There are exactly two univoque Pisot numbers less than χ. They are*

- $1.880000\cdots$ *the root in* $(1,2)$ *of* $x^{14}-2x^{13}+x^{11}-x^{10}-x^7+x^6-x^4+x^3-x+1$ *with univoque expansion* $111001011(1001010)^\infty$.

- $1.886681\cdots$ *the root in* $(1,2)$ *of* $x^{12}-2x^{11}+x^{10}-2x^9+x^8-x^3+x^2-x+1$ *with univoque expansion* $111001101(1100)^\infty$

For each r, there are regular Pisot numbers between ψ_r and 2 with expansion $1^{r+1}(0^{n-r-1}1^r0)^\infty$ that are univoque for $r+1 \leqslant n < 2(r+1)$.

The ψ_r are limit points of the set of regular Pisot numbers. Moreover $\psi_r \to 2$ as $r \to \infty$.

Theorem 10. *2 is a limit point of* $S \cap \mathcal{U}$*.*