Number representation and symbolic dynamics

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PART I

Signed 2-expansions

Expansions in base 2

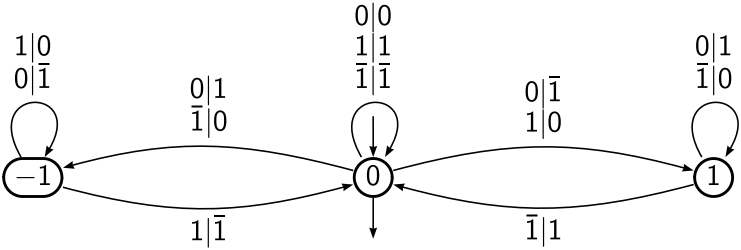
Every integer $N \geqslant 0$ has an expansion in base 2

$$N = \sum_{j=0}^{K} d_j 2^j = d_K \cdots d_1 d_0.$$

with $d_j \in \{0, 1\}$, which is unique up to leading zeros.

Using negative digits, there is redundancy and the number of non-zero digits can often be reduced:

$$7 = 4 + 2 + 1 = 111. = 100\bar{1}. = 8 - 1 \qquad (\bar{1} = -1)$$



Redundancy automaton base 2

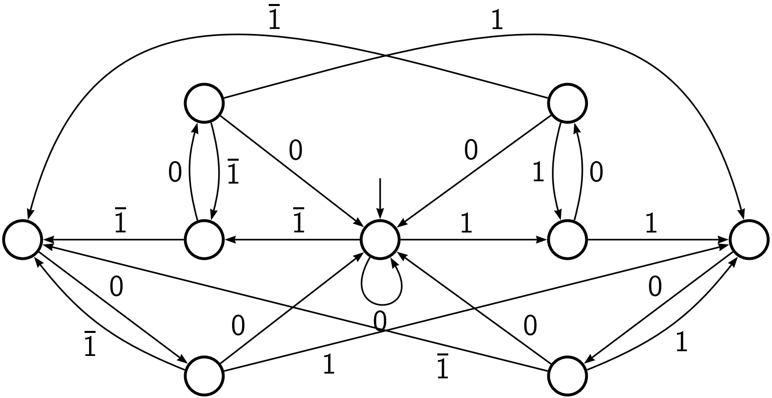
Expansions of minimal weight in base 2

Problem: Find an expansion of N of minimal weight $\sum_{j=0}^{K} |d_j|$.
(cf. Hamming weight: number of non-zero digits d_j, equal to this weight if $d_j \in \{-1, 0, 1\}$).

Heuberger (2004): $d_K \cdots d_1 d_0 \in \{\bar{1}, 0, 1\}^*$ is a signed 2-expansion of minimal weight if and only if it contains none of the factors

$$11(01)^*1, \ 1(0\bar{1})^*\bar{1}, \ \bar{1}\bar{1}(0\bar{1})^*\bar{1}, \ \bar{1}(01)^*1.$$

and the opposites.



Signed 2-expansions of minimal weight

Canonical expansions of minimal weight in base 2

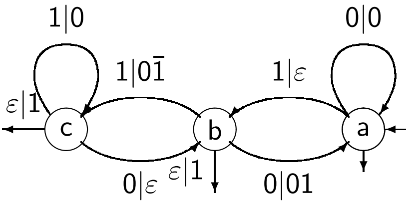
Booth (1951), Reitwiesner (1960), ...

Non-Adjacent Form (NAF)

Every integer N has a unique expansion $N = \sum_{j=0}^{K} d_j 2^j$ with $d_j \in \{-1, 0, 1\}$ such that $d_{j-1} = d_{j+1} = 0$ if $d_j \neq 0$. The weight of this expansion is minimal among all expansions of N in base 2.

Right-to-left recoding: every factor of form 01^n, with $n \geqslant 2$, is transformed into $10^{n-1}\bar{1}$.

The Booth recoding is a right subsequential function



Applications of the NAF

- multiplication
- internal representation for dividers: base 4 with digits in $\{\bar{2}, \ldots, 2\}$
- computations on elliptic curves and cryptography

Transformation for the NAF

Ergodic properties of the dynamical system associated with the NAF [Dajani, Kraaikamp, Liardet 2006]
Transformation giving the NAF

$$\begin{aligned} T : [-2/3, 2/3) &\rightarrow [-2/3, 2/3) \\ x &\mapsto 2x - \lfloor (3x+1)/2 \rfloor \end{aligned}$$

The digit x_j is $x_j = \lfloor (3T^{j-1}(x) + 1)/2 \rfloor$.

The expected number of non-zero digits in a NAF of length n is $n/3$, Arno and Wheeler (1993).

PART II

Beta-shift

Beta-numeration (Rényi, Parry)

Base $\beta > 1$, $x \in [0,1]$
β-expansion of x: greedy algorithm
$r_0 := x$
$x_n := \lfloor \beta r_{n-1} \rfloor$; $r_n := \beta r_{n-1} - x_n$.
Then $x = \sum_{n \geqslant 1} x_n \beta^{-n}$, with $x_n \in A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$

$d_\beta(x) = x_1 x_2 \cdots$ is the greedy β-expansion of x.
It is the greatest representation in the lexicographic order.
If β is an integer, it is the standard β-ary numeration.

If the sequence (x_n) ends in 0^ω, it is said finite.

Redondancy A number may have several β-representations.
Example $\varphi = (1+\sqrt{5})/2$, $A_\varphi = \{0,1\}$, $d_\varphi(1) = 11$.
$x = 3 - \sqrt{5}$, $d_\varphi(x) = 10010^\omega$.
The factor 11 is forbidden in $d_\varphi(x)$.
Other φ-representations of x:
01110^ω, $100(01)^\omega$, $011(01)^\omega$, $\ldots$

β-transformation

$$\begin{aligned} T_\beta : [0,1] &\rightarrow [0,1) \\ x &\mapsto \beta x \pmod 1 \end{aligned}$$

$d_\beta(x) = x_1 x_2 \cdots$ with $x_n = \lfloor \beta T_\beta^{n-1}(x) \rfloor$.

T_β has a unique invariant measure μ_β which is absolutely continuous with respect to the Lebesgue measure on $[0,1]$.
μ_β is ergodic and is the unique measure of maximal entropy (Rényi 1957).

β-shift

σ shift on $A_\beta^{\mathbb{N}}$: $\sigma((x_i)_{i\geqslant 1}) = (x_{i+1})_{i\geqslant 1}$.

$D_\beta = \{d_\beta(x) \mid x \in [0,1[\,\}$ is a shift-invariant subset of $A_\beta^{\mathbb{N}}$.
β-shift S_β = topological closure of D_β.

Example

- $\beta = 2$ and $A_\beta = \{0,1\}$ then S_β is the full 2-shift $= \{0,1\}^{\mathbb{N}}$
- $\varphi = (1+\sqrt{5})/2$ and $A_\varphi = \{0,1\}$ then S_φ is the golden mean shift $= \{s \in \{0,1\}^{\mathbb{N}}$ with no factor $11\}$.

$d_\beta(1) = (t_n)_{n\geqslant 1}$ greedy β-expansion of 1

$$d^*_\beta(1) := \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is infinite} \\ (t_1 \cdots t_{m-1}(t_m - 1))^\infty & \text{if } d_\beta(1) = t_1 \cdots t_{m-1} t_m \text{ is finite.} \end{cases}$$

Theorem (Parry 1960)

$s = (s_n)_{n\geqslant 1}$ with $s_n \in \mathbb{N}$.

- *s is the greedy β-expansion of some $x \in [0,1)$ if and only if*

$$\forall k \geqslant 0, \quad \sigma^k(s) <_{lex} d^*_\beta(1)$$

- *s is the greedy β-expansion of 1 for some $\beta > 1$ if and only if*

$$\forall k \geqslant 1, \quad \sigma^k(s) <_{lex} s$$

Remark The nature of the β-shift is entirely determined by $d_\beta(1)$ the greedy β-expansion of 1.

Entropy of the β-shift

Topological entropy of S_β

$$h(S_\beta) = \lim_{n\to\infty} \frac{1}{n} \log B(n) = \log \beta$$

where $B(n)$ = number of words of S_β of length n.

Symbolic dynamical systems

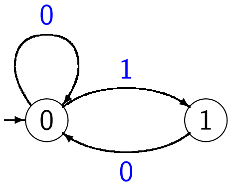
$S \subseteq A^{\mathbb{N}}$ symbolic dynamical system = closed shift-invariant subset
$F(S) \subseteq A^*$ = set of finite factors (admissible blocks) of S.
$X(S) \subseteq A^*$ set of minimal forbidden words.
The symbolic dynamical system S is completely defined by the set of its factors $F(S)$.

- S is of finite type if $X(S)$ is finite. Equivalent to $F(S)$ recognizable by a local finite automaton.
- S is sofic if $X(S)$ is recognizable by a finite automaton. Equivalent to $F(S)$ recognizable by a finite automaton.
- S is coded if there exists a prefix code $Y \subset A^*$ such that $F(S) = F(Y^*)$. Equivalent to $S = \overline{Y^\omega}$.
- S is specified if $\exists k : \forall u, v \in F(S), \exists w \in F(S)$, $|w| = k$, such that $uwv \in F(S)$
- S is synchronizing if $\exists w \in F(S)$ such that if $uw \in F(S)$ and $wv \in F(S)$, then $uwv \in F(S)$.

First properties of the β-shift

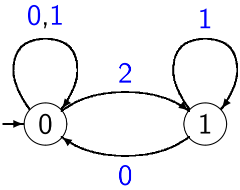
S_β is coded [Blanchard and Hansel 1986]
If $d_\beta(1) = (t_i)_{i \geqslant 1}$ is infinite, set
$Y = \{t_1 \cdots t_{n-1} a \mid 0 \leqslant a < t_n,\ n \geqslant 1\}$
If $d_\beta(1) = t_1 \cdots t_m$, set
$Y = \{t_1 \cdots t_{n-1} a \mid 0 \leqslant a < t_n,\ 1 \leqslant n \leqslant m\}$.
S_β is coded by Y.

S_β is of finite type iff $d_\beta(1)$ is finite [Ito and Takahashi 1974]
Example $\varphi = (1 + \sqrt{5})/2$, $d_\varphi(1) = 11$.
$\{11\}$ = minimal forbidden words.



Local automaton for $F(S_\varphi)$

S_β is sofic iff $d_\beta(1)$ is eventually periodic [Bertrand 1977]
Example $\gamma = (3+\sqrt{5})/2$, then $d_\gamma(1) = 21^\omega$. Minimal forbidden words $= 21^*2$.



Automaton for $F(S_\gamma)$

This automaton is not local.

The β-shift and the Chomsky hierarchy

Languages recognizable by finite state automaton $\subsetneq$
Context-free languages $\subsetneq$
Context-sensitive languages $\subsetneq$
Recursive languages $\subsetneq$
Recursively enumerable languages.

Context-sensitive languages are recognizable by a linear bounded automaton. A linear bounded automaton is a Turing machine which uses only a finite portion of the tape, whose length is a linear function of the length of the initial input.

Recursive languages are recognizable by a Turing machine which halts on every input.

Recursively enumerable languages are recognizable by a Turing machine.

Results of K. Johnson 1999

- $F(S_\beta)$ is context-free iff it is recognizable by a finite automaton
- $F(S_\beta)$ is context-sensitive iff $d_\beta(1)$ is generated by a linear bounded automaton
- $F(S_\beta)$ is recursive iff $d_\beta(1)$ is generated by a TM which halts on every input iff $d_\beta(1)$ is generated by a TM
- There exist non-recursive S_β:
  $d_2(\beta)$ is generated by a TM $\iff$ $d_\beta(1)$ is generated by a TM

Generated by a TM: on input 0^n the machine computes the first n digits of $d_\beta(1)$.

Context-sensitive β-shift

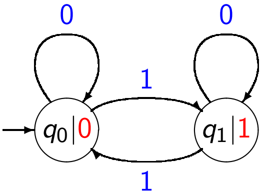
Proposition (K. Johson)

Every β-shift generated by a constant length morphism is context-sensitive.

The Thue-Morse morphism

$$0 \to 01 \quad 1 \to 10$$

has a fixed point $(k_n)_{n \geqslant 0} = 01101001 \cdots$ which is 2-automatic in the sense of Mendès France.



2-automaton generating the Thue-Morse sequence

k_n is the output corresponding to the path with input $\langle n \rangle_2$.
Let κ such that $d_\kappa(1) = 1101001 \cdots$ the shifted Thue-Morse sequence. Then S_κ is a context-sensitive shift.
κ is the Komornik-Loreti constant; it is the smallest univoque number (1 has only one writing in base κ).

Algebraic integers

Pisot number algebraic integer such that every conjugate is < 1 in modulus.
Salem number algebraic integer such that every conjugate is $\leqslant 1$ in modulus, and the equality is attained.
Perron number algebraic integer β such that every conjugate is $< \beta$ in modulus.
Example Integers, the golden ratio, $(3+\sqrt{5})/2$ are Pisot numbers.

If $d_\beta(1)$ is eventually periodic, β is called a Parry number.
If $d_\beta(1)$ is finite, β is called a simple Parry number.
If β is Pisot then $d_\beta(1)$ is eventually periodic and thus S_β is sofic [A. Bertrand 1986].
If S_β is sofic then β is Perron [Lind 1984].
If β is Salem of degree 4 then $d_\beta(1)$ is eventually periodic [Boyd 1989].
Open problem for Salem of degree $\geqslant 6$.

Blanchard classification 1989

Class 1: $d_\beta(1)$ is finite $\iff$ S_β is finite
Class 2: $d_\beta(1)$ is infinite eventually periodic $\iff$ S_β is sofic
Class 3: $d_\beta(1)$ does not contain arbitrarily large strings of 0's and $d_\beta(1)$ is not eventually periodic $\iff$ S_β is specified [A. Bertrand 1986]
Class 4: $d_\beta(1)$ does not contain some admissible words but contains arbitrarily large strings of 0's $\iff$ S_β is synchronizing [A. Bertrand 1986]
Class 5: $d_\beta(1)$ contains all the admissible words

Transcendental numbers in Class 3

The Komornik-Loreti constant κ, such that

$$d_\kappa(1) = 1101001\cdots$$

is the shifted Thue-Morse sequence,

- is transcendental [Allouche and Cosnard 2000]
- is $(2, \kappa)$-automatic
- belongs to Class 3.

β is self-Sturmian if $d_\beta(1)$ is Sturmian, i.e. the number of factors of length n in $d_\beta(1)$ is equal to $n+1$.

Theorem (Chi and Kwon 2004)

Every self-Sturmian number is transcendental and in Class 3.

Example The Fibonacci word $f = 01001010010010\cdots$ is the fixed point of the morphism

$$0 \to 01 \quad 1 \to 0$$

The word $1f = 101001010010010\cdots$ is the fixed point of the morphism

$$1 \to 10 \quad 0 \to 100$$

It is Sturmian, but not automatic.
β such that $d_\beta(1) = 101001010010010\cdots$ is transcendental.

Transcendental numbers in Class 4

β is self-lacunary if exist $\delta > 0$ and a sequence $u = (u_n)_{n \geqslant 1}$ of positive integers with

$$u_1 = 1 \text{ and } \frac{u_{n+1}}{u_n} \geqslant 1 + \delta \text{ for } n \geqslant 1$$

such that

$$1 = \sum_{n \geqslant 1} \frac{1}{\beta^{u_n}}$$

Theorem (Adamczewski and Bugeaud 2007)

Every self-lacunary number is transcendental and in Class 4.

Define $(a_k)_{k \geqslant 1}$ as $a_k = 1$ if $k \in u$ and 0 otherwise, and let $d_\beta(1) = a_1 a_2 \cdots$ Then β is transcendental and in Class 4.

Gaps in $d_\beta(1)$, β algebraic

β is not a simple Parry number. $d_\beta(1) = t_1 t_2 \cdots$ Assume there exists a sequence of positive integers $(r_n)_{n \geqslant 1}$ and an increasing sequence of positive integers $(s_n)_{n \geqslant 1}$ such that

$$t_{s_n+1} = t_{s_n+2} = \cdots = t_{s_n+r_n} = 0$$

Problem Estimation of the gaps in $d_\beta(1)$, β algebraic number, i.e. asymptotic behaviour of r_n/s_n.

If b is leading coefficient of the minimal polynomial of β, and $\beta_1 = \beta, \beta_2, \ldots, \beta_d$ are the roots, the Mahler measure of β is

$$M(\beta) = |b| \prod_{i=1}^{i=d} \max\{|\beta_i|, 1\}$$

Theorem (Verger-Gaugry 2006)

Let $\beta > 1$ be an algebraic number. Then

$$\limsup_{n\to\infty} \frac{r_n}{s_n} \leqslant \frac{M(\beta)}{\log \beta} - 1$$

PART III

Signed beta-expansions

(with Wolfgang Steiner 2007)

Redundancy automaton base β

Let $\beta > 1$ be a real number, $c \geqslant \lfloor \beta \rfloor$ a fixed integer, and

$$Z_\beta(c) = \left\{ z_1 \cdots z_n \mid n \geqslant 1,\ |z_j| \leqslant c,\ \sum_{j=1}^{n} z_j \beta^{-j} = 0 \right\}.$$

If β is a Pisot number, then for every $c \geqslant \lfloor \beta \rfloor$ the set $Z_\beta(c)$ is recognized by a finite automaton. [Frougny 1992]

Proposition

If β is a Pisot number, then the set

$$\left\{ (x_1 \cdots x_n, y_1 \cdots y_n) \in A_\beta^* \times A_\beta^* \mid n \geqslant 1,\ \sum_{j=1}^{n} x_j \beta^{-j} = \sum_{j=1}^{n} y_j \beta^{-j} \right\}$$

is recognizable by a finite automaton.

β-expansions of minimal weight

Weight of $x = x_1 \cdots x_n$ is $\sum_{j=1}^{n} |x_j|$.

$x = x_1 \cdots x_n \in A_\beta^*$ is β-heavy if it is not minimal in weight, i.e., if there exists $y = y_\ell \cdots y_r \in A_\beta^*$ with

$$\sum_{j=\ell}^{r} y_j \beta^{-j} = \sum_{j=1}^{n} x_j \beta^{-j} \quad \text{and} \quad \sum_{j=\ell}^{r} |y_j| < \sum_{j=1}^{n} |x_j|.$$

If $x_1 \cdots x_{n-1}$ and $x_2 \cdots x_n$ are not β-heavy, x is strictly β-heavy.

Condition (D)

(D): $\beta > 1$ and $P(\beta) = 0$ for some polynomial
$P(X) = X^d - b_1 X^{d-1} - \cdots - b_d \in \mathbb{Z}[X]$ with $b_1 > \sum_{j=2}^{d} |b_j|$

Proposition

If β satisfies (D), then β is a Pisot number.

Proposition (Akiyama, Rao, Steiner 2004)

Let β satisfy (D), and $x_1 \cdots x_n \in \mathbb{Z}^*$ such that $|.x_1 \cdots x_n| < 1$. Then there exists a word $y_0 \cdots y_m \in \{-\lfloor\beta\rfloor, \ldots, \lfloor\beta\rfloor\}^*$ such that $y_0 . y_1 \cdots y_m = .x_1 \cdots x_n$ and $\sum_{j=0}^{m} |y_j| \leqslant \sum_{j=1}^{n} |x_j|$.

Theorem

If β satisfies (D), then the set of signed β-expansions of minimal weight is recognized by a finite automaton, which is computable.

$\beta = \frac{1+\sqrt{5}}{2}$

Greedy β-expansions are not minimal in weight

$$0101001. = 10\bar{1}1001. = 1000\bar{1}01. = 10000\bar{1}0.$$

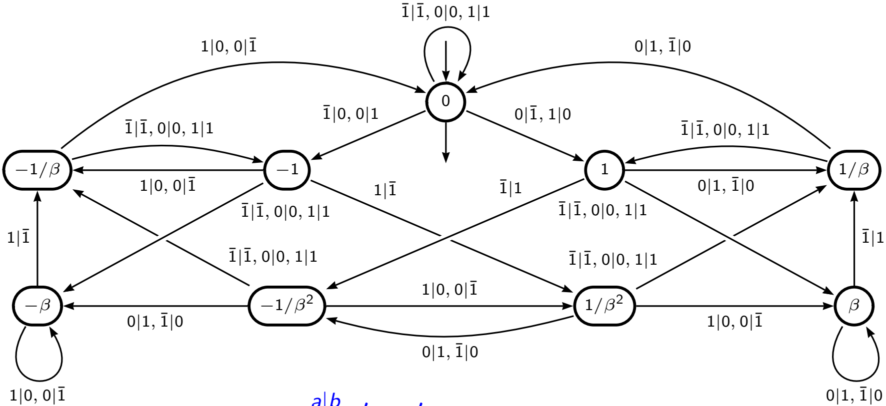
Theorem

If $\beta = \frac{1+\sqrt{5}}{2}$, then the set of strictly β-heavy words is

$$1(0100)^*1 \;\cup\; 1(0100)^*0101 \;\cup\; 1(00\bar{1}0)^*\bar{1} \;\cup\; 1(00\bar{1}0)^*0\bar{1} \;\cup$$
$$\bar{1}(0\bar{1}00)^*\bar{1} \;\cup\; \bar{1}(0\bar{1}00)^*0\bar{1}0\bar{1} \;\cup\; \bar{1}(0010)^*1 \;\cup\; \bar{1}(0010)^*01.$$

If $\cdots \epsilon_{-1}\epsilon_0\epsilon_1 \cdots$ does not contain any of these factors, then $\cdots \epsilon_{-1}\epsilon_0 . \epsilon_1 \cdots$ is a signed β-expansion of minimal weight.

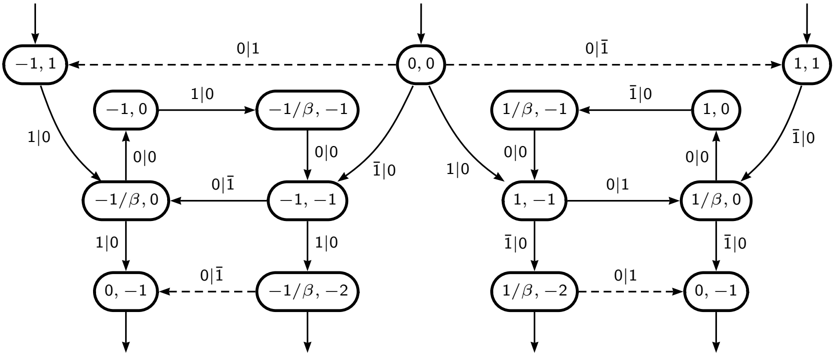
Redundancy automaton $\beta = \frac{1+\sqrt{5}}{2}$



$$s \xrightarrow{a|b} s' : \; s' = \beta s + a - b$$

If $s_0 = 0$, $s_{j-1} \xrightarrow{x_j|y_j} s_j$, $1 \leqslant j \leqslant n$, then $s_j = x_1 \cdots x_j . - y_1 \cdots y_j .$, and $.x_1 \cdots x_n = .y_1 \cdots y_n$ if and only if $s_n = 0$.

The strictly β-heavy words are the inputs of the following transducer. The outputs are corresponding lighter words (if the path is completed by dashed arrows such that it runs from $(0,0)$ to $(0,-1)$).



$$(s,\delta) \xrightarrow{a|b} (s',\delta') : \; s' = \beta s + a - b, \; \delta' = \delta + |b| - |a|$$

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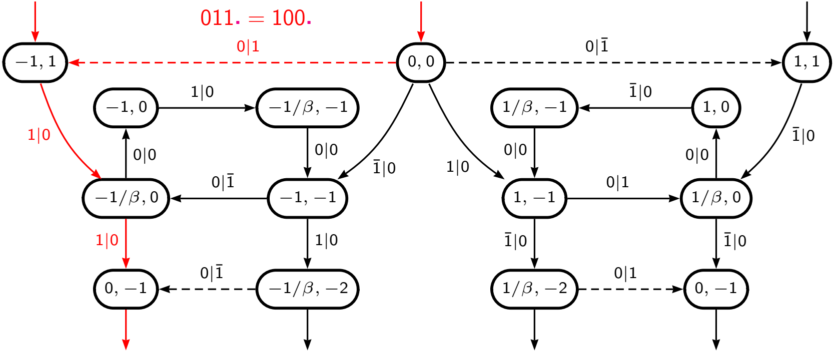

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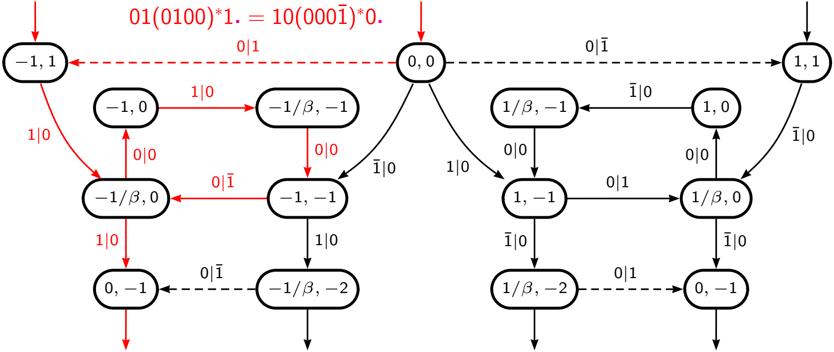
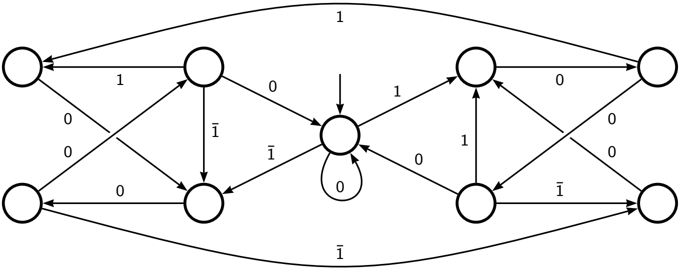


$$(s,\delta) \xrightarrow{a|b} (s',\delta') : \; s' = \beta s + a - b, \; \delta' = \delta + |b| - |a|$$

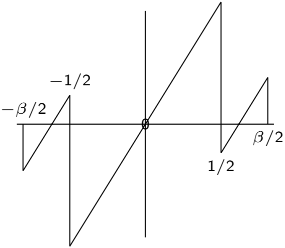
Theorem

For $\beta = \frac{1+\sqrt{5}}{2}$, the signed β-expansions of minimal weight are given by the following automaton, where all states are terminal.



Transformation providing some signed β-expansion of minimal weight, $\beta = \frac{1+\sqrt{5}}{2}$

$$T : [-\beta/2, \beta/2) \to [-\beta/2, \beta/2),\ T(x) = \beta x - \lfloor x + 1/2 \rfloor$$



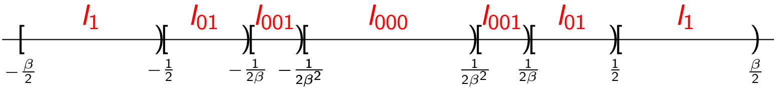
Proposition

If $x \in [\beta/2, \beta/2)$ and $x_j = \lfloor T^{j-1}(x) + 1/2 \rfloor$, then $x = .x_1 x_2 \cdots$ is a signed β-expansion of minimal weight avoiding the factors 11, 101, $1\bar{1}$, $10\bar{1}$, $100\bar{1}$ *and their opposites.*

Remark. Heuberger (2004) excluded (for the Fibonacci numeration system) the factor 1001 instead of $100\bar{1}$. This can be achieved by $T(x) = \beta x - \left\lfloor \frac{\beta^2+1}{2\beta} x + \frac{1}{2} \right\rfloor$ on $\left[\frac{-\beta^2}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}\right)$, $\frac{\beta^2}{\beta^2+1} = .(1000)^\omega$.

Markov chain of digits

Let $T(x) = \beta x - \lfloor x + 1/2 \rfloor$, and $I_{000}, I_{001}, I_{01}, I_1$ as follows



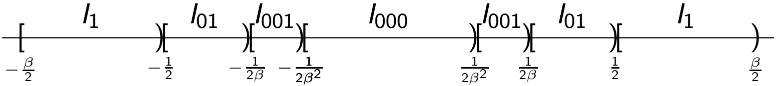
The sequence of random variables $(X_k)_{k \geqslant 0}$ defined by

$$
\begin{aligned}
&\Pr[X_0 = j_0, \ldots, X_k = j_k] \\
&= \lambda(\{x \in [-\beta/2, \beta/2) : x \in I_{j_0}, T(x) \in I_{j_1}, \ldots, T^k(x) \in I_{j_k}\})/\beta \\
&\qquad = \lambda(I_{j_0} \cap T^{-1}(I_{j_1}) \cap \cdots \cap T^{-k}(I_{j_k}))/\beta
\end{aligned}
$$

(where λ denotes the Lebesgue measure) is a Markov chain since

$$T(I_{000}) = I_{000} \cup I_{001} = T(I_1), \; T(I_{001}) = I_{01}, \; T(I_{01}) = I_1$$

and $T(x)$ is linear on each I_j.



The matrix of transition probabilities is

$$(\Pr[X_k = j \mid X_{k-1} = i])_{i,j \in \{000,001,01,1\}} = \begin{pmatrix} 1/\beta & 1/\beta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2/\beta^2 & 1/\beta^3 & 0 & 0 \end{pmatrix}$$

the stationary distribution vector is $(2/5, 1/5, 1/5, 1/5)$. Therefore

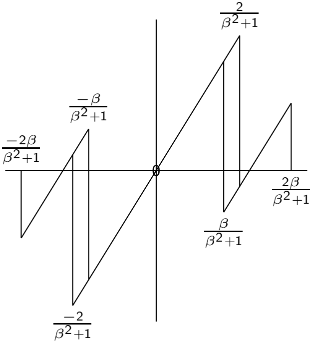
$$\Pr[X_k = 1] = \lambda(\{x \in [-\beta/2, \beta/2) : T^k(x) \in I_1\}) \to 1/5,$$

i.e., the expected number of non-zero digits in a signed β-expansion of minimal weight of length n is $n/5 + \mathcal{O}(1)$.
(cf. greedy β-expansions $n/(\beta^2 + 1)$, base 2 minimal expansions $n/3$)

Branching transformation providing all signed β-expansions of minimal weight, $\beta = \frac{1+\sqrt{5}}{2}$

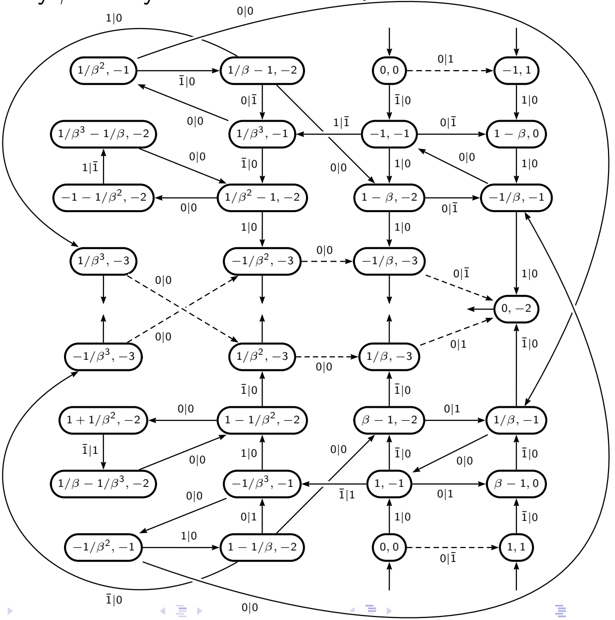
$$T : [-\tfrac{2\beta}{\beta^2+1}, \tfrac{2\beta}{\beta^2+1}) \to [-\tfrac{2\beta}{\beta^2+1}, \tfrac{2\beta}{\beta^2+1}),\ T^j(x) = \beta x - x_j \text{ with}$$

$$x_j = \begin{cases} 1 & \text{if } \frac{2}{\beta^2+1} < T^{j-1}(x) < \frac{2\beta}{\beta^2+1} \\ 0 \text{ or } 1 & \text{if } \frac{\beta}{\beta^2+1} < T^{j-1}(x) < \frac{2}{\beta^2+1} \\ 0 & \text{if } \frac{-\beta}{\beta^2+1} < T^{j-1}(x) < \frac{\beta}{\beta^2+1} \\ -1 \text{ or } 0 & \text{if } \frac{-2}{\beta^2+1} < T^{j-1}(x) < \frac{-\beta}{\beta^2+1} \\ -1 & \text{if } \frac{-2\beta}{\beta^2+1} < T^{j-1}(x) < \frac{-2}{\beta^2+1} \end{cases}$$

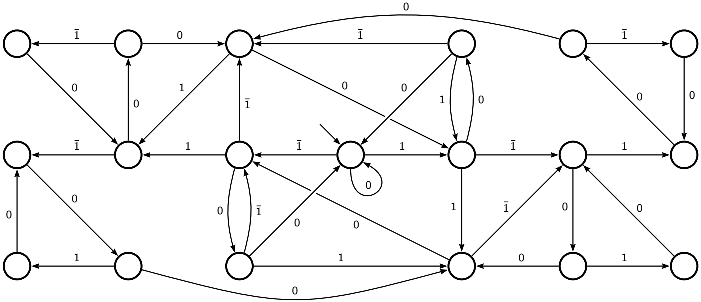


Tribonacci number $\beta^3 = \beta^2 + \beta + 1$

The strictly β-heavy words are the inputs of this automaton.



The signed β-expansions of minimal weight are given by the following automaton where all states are terminal.



Particular signed β-expansions of minimal weight, $\beta^3 = \beta^2 + \beta + 1$

$$\begin{aligned} T : [\frac{-\beta^2}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}) &\to [\frac{-\beta^2}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}) \\ x &\mapsto \beta x - \lfloor \frac{\beta^2+1}{2\beta} x + \frac{1}{2} \rfloor \end{aligned}$$

Proposition

If $x \in \left[\frac{-\beta^2}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}\right)$ *and* $x_j = \lfloor \frac{\beta^2+1}{2\beta} T^{j-1}(x) + 1/2 \rfloor$, *then* $x = .x_1 x_2 \cdots$ *is a signed* β*-expansion of minimal weight avoiding the factors* 11, $1\bar{1}$, $10\bar{1}$ *and their opposites.*

One can show that the expected number of non-zero digits in a signed β-expansion of minimal weight of length n is asymptotically $n\beta^3/(\beta^5+1)$ (with $\beta^3/(\beta^5+1) = .(0011010100)^\omega \approx 0.28219$).