Representation of *p*-adic numbers in rational base numeration systems

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Numeration
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Introduction to *p*-adic numbers

Base $p = 10$

$$\frac{1}{3} =_{10} .33333\cdots$$

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Introduction to p-adic numbers

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- $\frac{1}{3} = 7 + 6\sum_{i\geqslant 1}(10)^i$ using the formal sum $\sum_{i\geqslant 0} X^i = \frac{1}{1-X}$
- Arithmetic as usual

$$\begin{array}{cccccccc} \cdots & 6 & 6 & 6 & 6 & 7 & \\ \cdots & & & & & 3 & \times \\ \hline \cdots & 0 & 0 & 0 & 0 & 1 & = \end{array}$$

p-adic numbers

The set of p-adic integers is $\mathbb{Z}_p = \sum_{i \geqslant 0} a_i p^i$ with a_i in $\mathcal{A}_p = \{0, 1, \ldots, p-1\}$. Notation: $\cdots a_2 a_1 a_0 \centerdot \in \mathbb{Z}_p$.

The set of p-adic numbers is $\mathbb{Q}_p = \sum_{i \geqslant -k_0} a_i p^i$ with a_i in $\mathcal{A}_p$. Notation: $\cdots a_2 a_1 a_0 \centerdot a_{-1} \cdots a_{-k_0} \in \mathbb{Q}_p$.

$\mathbb{Q}_p$ is a commutative ring, and $\mathbb{Z}_p$ is a subring of $\mathbb{Q}_p$.

p-adic valuation $v_p : \mathbb{Z} \setminus \{0\} \to \mathbb{Z}$ is given by

$$n = p^{v_p(n)} n' \quad \text{with} \quad p \nmid n'.$$

For $x = \frac{a}{b} \in \mathbb{Q}$

$$x = p^{v_p(x)} \frac{a'}{b'} \quad \text{with} \quad (a', b') = 1 \quad \text{and} \quad (b', p) = 1.$$

The p-metric is defined on $\mathbb{Q}$ by $d_p(x, y) = p^{-v_p(x-y)}$. It is an ultrametric distance.

Example

$p = 10$

Take $z_n =_{10} \overbrace{6\cdots6}^{n}7$.
Then

$$z_n - \frac{1}{3} = \frac{2}{3}(10)^{n+1}$$

and $v_{10}(z_n - \frac{1}{3}) = n + 1$ thus

$$d_{10}(z_n, \frac{1}{3}) = (10)^{-n-1} \to 0$$

Hence the 10-adic representation of $\frac{1}{3}$ is $\cdots 66667$.

$p = 10$

Take $s = \cdots 109376_{\bullet}$ and $t = \cdots 890625_{\bullet}$

$$
\begin{array}{ccccccccl}
\cdots & 1 & 0 & 9 & 3 & 7 & 6 & = s \\
\cdots & 8 & 9 & 0 & 6 & 2 & 5 & = t \\
\hline
\cdots & 0 & 0 & 0 & 0 & 0 & 1 & = s + t
\end{array}
$$

Hence $s + t =_{10} 1$.

$$
\begin{array}{ccccccccc}
\cdots & 1 & 0 & 9 & 3 & 7 & 6 & = s \\
\cdots & 8 & 9 & 0 & 6 & 2 & 5 & = t \\
\hline
\cdots & 5 & 4 & 6 & 8 & 8 & 0 & \\
\cdots & 1 & 8 & 7 & 5 & 2 & . & \\
\cdots & 6 & 2 & 5 & 6 & . & . & \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\hline
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & = s \times t
\end{array}
$$

Hence $st =_{10} 0$ thus $\mathbb{Z}_{10}$ is not a domain.

$$
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\cdots & 1 & 0 & 9 & 3 & 7 & 6 & = s \\
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\cdots & 5 & 4 & 6 & 8 & 8 & 0 & \\
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\cdots & 6 & 2 & 5 & 6 & . & . & \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
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\cdots & 1 & 8 & 7 & 5 & 2 & . & \\
\cdots & 6 & 2 & 5 & 6 & . & . & \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\hline
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & = s \times t
\end{array}$$

Hence $st =_{10} 0$ thus $\mathbb{Z}_{10}$ is not a domain.

One has also $s^2 = s$, $t^2 = t$.

One can prove that

- $s =_2 0$ and $s =_5 1$
- $t =_2 1$ and $t =_5 0$.

Computer arithmetic

2's complement

In base 2, on n positions. $n = 8$:

$$\begin{aligned} 01111111 &= 127 \\ 00000001 &= 1 \\ 00000000 &= 0 \\ 11111111 &= -1 \\ 10000000 &= -128 \end{aligned}$$

Troncation of 2-adic numbers.

p prime

$\mathbb{Q}_p$ is a field if and only if p is prime.
$\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to d_p.

p-adic absolute value on $\mathbb{Q}$:

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-v_p(x)} & \text{otherwise.} \end{cases}$$

$\mathbb{Q}_p$ is the quotient field of the ring of p-adic integers
$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leqslant 1\}$.

p-expansion of p-adic numbers

p prime

Algorithm

Let $x = \frac{s}{t}$, s an integer and t a positive integer.

(i) *If $s = 0$, return the empty word $\mathbf{a} = \varepsilon$.*

(ii) *If t is co-prime to p, put $s_0 = s$ and for all $i \in \mathbb{N}$ define s_{i+1} and $a_i \in \mathcal{A}_p$ by*

$$\frac{s_i}{t} = \frac{ps_{i+1}}{t} + a_i .$$

Return $\mathbf{a} = \cdots a_2 a_1 a_0$.

(iii) *If t is not co-prime to p, multiply $\frac{s}{t}$ by p until xp^ℓ is of the form $\frac{s'}{t'}$, where t' is co-prime to p. Then apply (ii) returning $\mathbf{a}' = \cdots a'_2 a'_1 a'_0$. Return*

$$\mathbf{a} = \cdots a_1 a_0 \,\mathbf{.}\, a_{-1} a_{-2} \cdots a_{-\ell} = \cdots a'_\ell \,\mathbf{.}\, a'_{\ell-1} \cdots a'_1 a'_0 .$$

$\mathbf{a}$ is said to be the *p-expansion* of x and denoted by $< x >_p$.

$\ell = v_p(x)$

Theorem

Let $x \in \mathbb{Q}_p$. Then the p-representation of x is

1. *uniquely given,*
2. *finite if, and only if, $x \in \mathbb{N}$,*
3. *eventually periodic if, and only if, $x \in \mathbb{Q}$.*

Rational base numeration system

(Akiyama, Frougny, Sakarovitch 2008)
$p > q \geqslant 1$ co-prime integers.

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Representation of the natural integers:

Algorithm (MD algorithm)

Let s be a positive integer. Put $s_0 = s$ and for all $i \in \mathbb{N}$:

$$qs_i = ps_{i+1} + a_i \quad \textit{with} \quad a_i \in \mathcal{A}_p$$

Return $\frac{1}{q}\frac{p}{q}$-expansion of s: $< s >_{\frac{1}{q}\frac{p}{q}} = a_n \cdots a_1 a_0$.

$$s = \sum_{i=0}^{n} \frac{a_i}{q} \left(\frac{p}{q}\right)^i$$

$\frac{1}{q}\frac{p}{q}$-expansions – properties

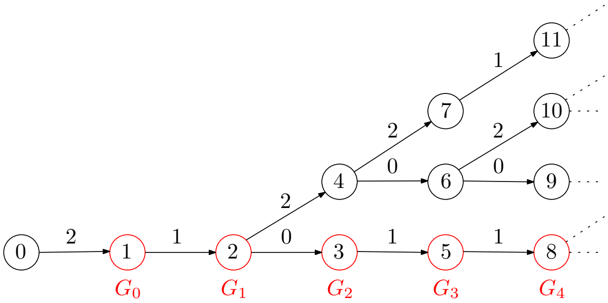
$$L_{\frac{1}{q}\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid w \text{ is } \frac{1}{q}\frac{p}{q}\text{-expansion of some } s \in \mathbb{N}\}$$

- $L_{\frac{1}{q}\frac{p}{q}}$ is prefix-closed,
- any $u \in \mathcal{A}_p^+$ is a suffix of some $w \in L_{\frac{1}{q}\frac{p}{q}}$,
- $L_{\frac{1}{q}\frac{p}{q}}$ is not context-free (if $q \neq 1$),
- $\pi : \mathcal{A}_p^+ \mapsto \mathbb{Q}$ the evaluation map. If $v, w \in L_{\frac{1}{q}\frac{p}{q}}$, then

$$v \preceq w \quad \Leftrightarrow \quad \pi(v) \leqslant \pi(w).$$

$T_{\frac{1}{q}\frac{p}{q}}$ – tree of nonnegative integers

$p = 3, q = 2$



Children of the vertex n are given by $\frac{1}{q}(pn + a) \in \mathbb{N}, a \in \mathcal{A}_p$.

$$G_0 = 1, \quad G_{i+1} = \left\lceil \frac{p}{q} G_i \right\rceil$$

In AFS 2008, the right infinite words that label the infinite paths of $T_{\frac{1}{q}\frac{p}{q}}$ are defined as the admissible $\frac{1}{q}\frac{p}{q}$-expansions of positive real numbers, of the form $.a_{-1}a_{-2}\cdots$, and it is proved that every positive real in $[0,1]$ has exactly one $\frac{1}{q}\frac{p}{q}$-expansion, but for an infinite countable subset of reals which have more than one such expansion.
No periodic expansion.

Connection with the problem of the distribution of the powers of a rational number modulo 1 (Mahler 1968).

MD algorithm – the negative case

Let s be a negative integer. The $\frac{1}{q}\frac{p}{q}$-expansion of s is $< s >_{\frac{1}{q}\frac{p}{q}} = \cdots a_2 a_1 a_0$. from the MD algorithm:

$$s_0 = s, \quad q s_i = p s_{i+1} + a_i.$$

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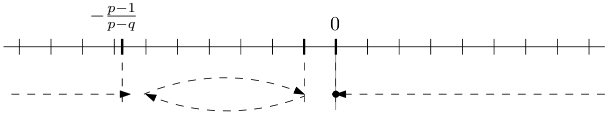
$$s_0 = s, \quad q s_i = p s_{i+1} + a_i.$$

Properties of $(s_i)_{i \geqslant 0}$:

(i) $(s_i)_{i \geqslant 1}$ is negative,

(ii) if $s_i < -\frac{p-1}{p-q}$, then $s_i < s_{i+1}$,

(iii) if $-\frac{p-1}{p-q} \leqslant s_i < 0$, then $-\frac{p-1}{p-q} \leqslant s_{i+1} < 0$.



In which fields does it work?

$$s = s_n \left(\frac{p}{q}\right)^n + \sum_{i=0}^{n-1} \frac{a_i}{q} \left(\frac{p}{q}\right)^n$$

We want:

$$\left| s - \sum_{i=0}^{n-1} \frac{a_i}{q} \left(\frac{p}{q}\right)^n \right|_r = |s_n|_r \left| \left(\frac{p}{q}\right)^n \right|_r \to 0 \quad \text{as } n \to \infty$$

Hence, if $p = r_1^{j_1} r_2^{j_2} \cdots r_k^{j_k}$, the $\frac{1}{q}\frac{p}{q}$-expansion of s "works" only in $\mathbb{Q}_{r_\ell}, \ell = 1, \ldots, k$. The speed of convergence is then $\approx r^{-j_\ell n}$.

$\frac{1}{q}\frac{p}{q}$-expansions of negative integers

Proposition

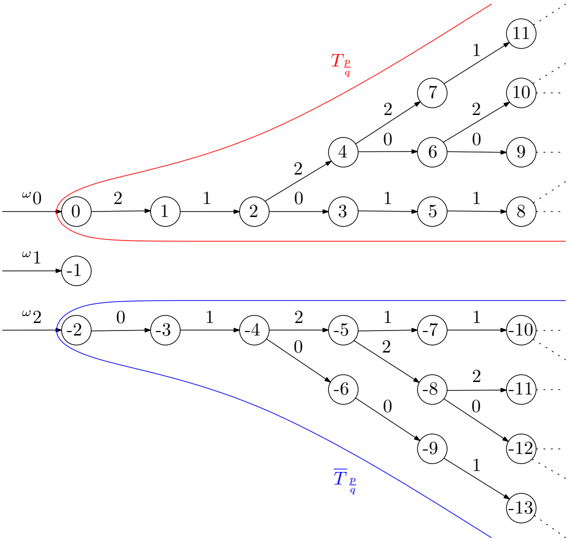
Let k be a positive integer, and denote $B = \left\lfloor \frac{p-1}{p-q} \right\rfloor$. Then:

(i) *if $k \leqslant B$, then $< -k >_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}b$ with $b = k(p-q)$,*

(ii) *otherwise, $< -k >_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}bw$ with $w \in \mathcal{A}_p^+$ and $b = B(p-q)$.*

$\overline{T}_{\frac{1}{q}\frac{p}{q}}$ – tree of negative integers

$p = 3, q = 2$



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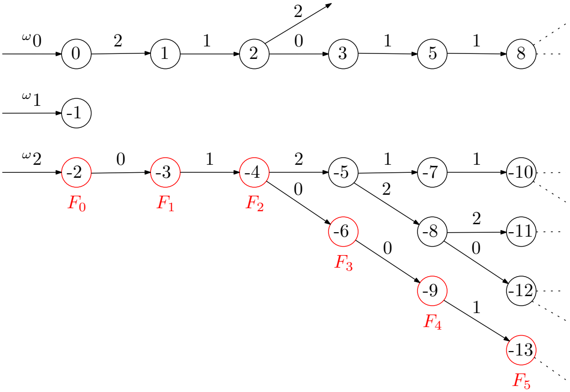
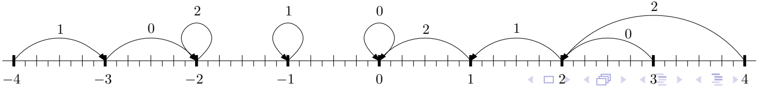
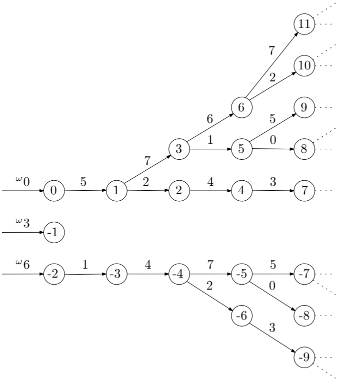


$$F_0 = -B = -\left\lfloor \frac{p-1}{p-q} \right\rfloor, \quad F_{i+1} = \left\lceil \frac{p}{q} F_i \right\rceil$$



Trees $\overline{T}_{\frac{1}{q}\frac{p}{q}}$ and $T_{\frac{1}{q}\frac{p}{q}}$

$p = 8, q = 5$



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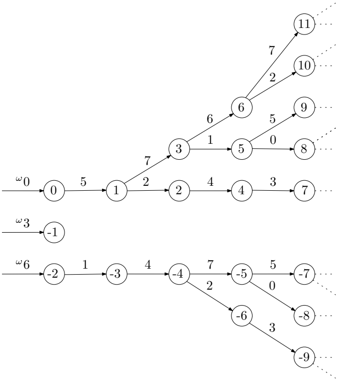


The trees $\overline{T}_{\frac{1}{q}\frac{p}{q}}$ and $T_{\frac{1}{q}\frac{p}{q}}$ are isomorphic if and only if

$$\frac{p-1}{p-q} \in \mathbb{Z}.$$

$\frac{1}{q}\frac{p}{q}$-expansions of negative integers – properties

- for $q = 1$ and p prime we get the standard p-adic representation,

$$\overline{L}_{\frac{1}{q}\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid {}^{\omega}bw \text{ is } \frac{1}{q}\frac{p}{q}\text{-expansion of } s \leqslant -B, b \notin Pref(w)\}$$

- $\overline{L}_{\frac{1}{q}\frac{p}{q}}$ is prefix-closed,
- any $u \in \mathcal{A}_p^+$ is a suffix of some $w \in \overline{L}_{\frac{1}{q}\frac{p}{q}}$,
- $\overline{L}_{\frac{1}{q}\frac{p}{q}}$ is not context-free (if $q \neq 1$).

MD algorithm for rationals

Algorithm (MD algorithm)

Let $x = \frac{s}{t}$, $s, t \in \mathbb{Z}$ co-prime, $s \neq 0$, and $t > 0$ co-prime to p. Put $s_0 = s$ and for all $i \in \mathbb{N}$ define s_{i+1} and $a_i \in \mathcal{A}_p$ by

$$q\frac{s_i}{t} = p\frac{s_{i+1}}{t} + a_i.$$

Return the $\frac{1}{q}\frac{p}{q}$-expansion of x: $< x >_{\frac{1}{q}\frac{p}{q}} = \cdots a_2 a_1 a_0$.

$$s = \sum_{i=0}^{\infty} \frac{a_i}{q} \left(\frac{p}{q}\right)^i$$

Examples

x	$< x >_{\frac{1}{q}\frac{p}{q}}$	$(s_i)_{i\geqslant 0})$	abs. values
		$p = 3, q = 2$	
5	2101	5, 3, 2, 1, 0, 0, ...	all
-5	${}^{\omega}2012$	-5,-3,-2,-2,-2, ...	$\mid\ \mid_3$
11/4	201	11,6,4,0,0, ...	all
11/8	${}^{\omega}1222$	11,2,-4,-8,-8,-8, ...	$\mid\ \mid_3$
11/5	${}^{\omega}(02)2112$	11,4,1,-1,-4,-6,-4,-6, ...	$\mid\ \mid_3$
		$p = 30, q = 11$	
5	11 25	5, 1, 0, 0, ...	all
-5	${}^{\omega}19\ 8\ 5$	-5,-2,-1,-1,...	$\mid\ \mid_2, \mid\ \mid_3, \mid\ \mid_5$
11/7	${}^{\omega}(12\ 21\ 5)\ 23\ 13$	11, 1, -5, -3, -6, -5, ...	$\mid\ \mid_2, \mid\ \mid_3, \mid\ \mid_5$

MD algorithm – properties

For the sequence $(s_i)_{i\geqslant 1}$ from the MD algorithm we have:

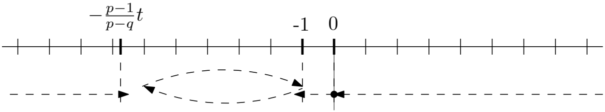
(i) if $s > 0$ and $t = 1$, $(s_i)_{i\geqslant 1}$ is eventually zero,

(ii) if $s > 0$ and $t > 1$, $(s_i)_{i\geqslant 1}$ is either eventually zero or eventually negative,

(iii) if $s < 0$, $(s_i)_{i\geqslant 1}$ is negative,

(iv) if $s_i < -\frac{p-1}{p-q}t$, then $s_i < s_{i+1}$,

(v) if $-\frac{p-1}{p-q}t \leqslant s_i < 0$, then $-\frac{p-1}{p-q}t \leqslant s_{i+1} < 0$.



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(iv) if $s_i < -\frac{p-1}{p-q} t$, then $s_i < s_{i+1}$,

(v) if $-\frac{p-1}{p-q} t \leqslant s_i < 0$, then $-\frac{p-1}{p-q} t \leqslant s_{i+1} < 0$.

Proposition

Let $x = \frac{s}{t} \in \mathbb{Q}$. Then $< x >_{\frac{1}{q} \frac{p}{q}}$ is eventually periodic with period less than $\frac{p-1}{p-q} t$.

$\frac{1}{q}\frac{p}{q}$-representation of r-adic numbers

$r, r_1, r_2, \ldots$ prime numbers. p an integer > 1.

Definition

A left infinite word $\cdots a_{-k_0+1}a_{-k_0}$, $k_0 \in \mathbb{N}$, over $\mathcal{A}_p$ is a $\frac{1}{q}\frac{p}{q}$-representation of $x \in \mathbb{Q}_r$ if $a_{-k_0} > 0$ or $k_0 = 0$ and

$$x = \sum_{i=-k_0}^{\infty} \frac{a_i}{q} \left(\frac{p}{q}\right)^i$$

with respect to $| \ |_r$.

Let $p = r_1^{j_1} r_2^{j_2} \cdots r_k^{j_k}$.

Theorem

Let $x \in \mathbb{Q}_{r_i}$ for some $i \in \{1, \ldots, k\}$.

(i) *If $k = 1$ (i.e. p is a power of a prime), there exists a unique $\frac{1}{q}\frac{p}{q}$-representation of x in $\mathbb{Q}_{r_1}$.*

(ii) *If $k > 1$, there exist uncountably many $\frac{1}{q}\frac{p}{q}$-representations of x in $\mathbb{Q}_{r_i}$.*

Let $p = r_1^{j_1} r_2^{j_2} \cdots r_k^{j_k}$.

Theorem

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(i) *If $k = 1$ (i.e. p is a power of a prime), there exists a unique $\frac{1}{q}\frac{p}{q}$-representation of x in $\mathbb{Q}_{r_1}$.*

(ii) *If $k > 1$, there exist uncountably many $\frac{1}{q}\frac{p}{q}$-representations of x in $\mathbb{Q}_{r_i}$.*

Let $x \in \mathbb{Q}$ and let $i_1, \ldots, i_\ell \in \{1, \ldots, k\}, \ell < k$, be distinct.

(i) *There exist uncountably many $\frac{1}{q}\frac{p}{q}$-representations which work in all fields $\mathbb{Q}_{r_{i_1}}, \ldots, \mathbb{Q}_{r_{i_\ell}}$.*

(ii) *There exists a unique $\frac{1}{q}\frac{p}{q}$-representation which works in all fields $\mathbb{Q}_{r_1}, \ldots, \mathbb{Q}_{r_k}$; namely, the $\frac{1}{q}\frac{p}{q}$-expansion $< x >_{\frac{1}{q}\frac{p}{q}}$.*

(iii) *The $\frac{1}{q}\frac{p}{q}$-expansion $< x >_{\frac{1}{q}\frac{p}{q}}$ is the only $\frac{1}{q}\frac{p}{q}$-representation which is eventually periodic.*

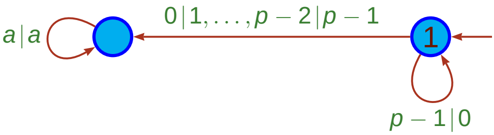
$p = 30, q = 11$

The following are aperiodic $\frac{1}{q}\frac{p}{q}$-representations of 1 in both fields $\mathbb{Q}_2$ and $\mathbb{Q}_3$:

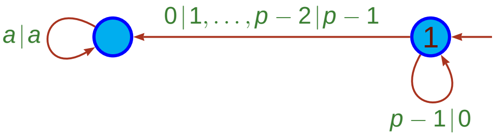
$$\cdots 27\ 24\ 24\ 29\ 26\ 29\ 27\ 25\ 25\ 24\ 28\ 24\ 28\ 27\ 29$$

$$\cdots 20\ 22\ 21\ 22\ 22\ 22\ 19\ 18\ 18\ 19\ 23\ 18\ 22\ 22\ 23$$

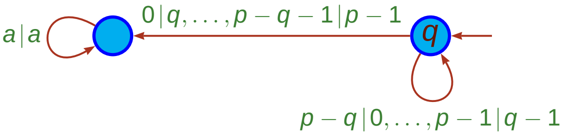
p-odometer



p-odometer



$\frac{1}{q}\frac{p}{q}$-odometer



Conversion from p-expansions to $\frac{1}{q}\frac{p}{q}$-expansions

Right on-line denumerable transducer:
$z_0 = 0, i = 0$ and, for $i \geqslant 0$

$$(z_i, i) \xrightarrow{a|b} (z_{i+1}, i+1),$$

with $a, b \in \mathcal{A}_p$ such that

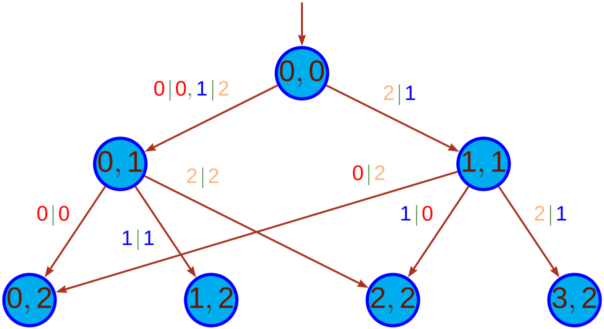
$$aq^i + z_i = \frac{b}{q} + \frac{p}{q} z_{i+1}.$$

Input: $\cdots a_1 a_0 \in {}^{\mathbb{N}}\mathcal{A}_p$ such that $x = \sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_r$ with r prime factor of p

Output: $\cdots b_1 b_0 \in {}^{\mathbb{N}}\mathcal{A}_p$ such that $x = \sum_{i=0}^{\infty} \frac{b_i}{q} \left(\frac{p}{q}\right)^i$.

The converter cannot be finite.

Base 3 to $\frac{1}{2}\frac{3}{2}$-system convertor



On ${}^{\mathbb{N}}\mathcal{A}_p$ define a distance δ:
for $\mathbf{a}, \mathbf{b} \in {}^{\mathbb{N}}\mathcal{A}_p$, $\delta(\mathbf{a}, \mathbf{b}) = 2^{-i}$ with $i = \min\{j \in \mathbb{N} \mid a_j \neq b_j\}$.

Proposition

The conversion from p-expansions to $\frac{1}{q}\frac{p}{q}$-expansions

$$\begin{array}{rcl} \chi : {}^{\mathbb{N}}\mathcal{A}_p & \to & {}^{\mathbb{N}}\mathcal{A}_p \\ \mathbf{a} & \mapsto & \mathbf{b} \end{array}$$

such that $x = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{\infty} \frac{b_i}{q}\left(\frac{p}{q}\right)^i \in \mathbb{Z}_r$ realized by the converter is Lipchitz, and thus uniformly continuous for the δ-topology.

Remark: the inverse conversion is also realizable by a right on-line transducer.

Corollary

The digits in a $\frac{1}{q}\frac{p}{q}$-expansion are uniformly distributed.

$\frac{p}{q}$-representations

Algorithm (MDbis algorithm)

Let $x = \frac{s}{t}$, $s, t \in \mathbb{Z}$ co-prime, $s \neq 0$, and $t > 0$ co-prime to p. Put $s_0 = s$ and for all $i \in \mathbb{N}$ define s_{i+1} and $a_i \in \mathcal{A}_p$ by

$$q\frac{s_i}{t} = p\frac{s_{i+1}}{t} + qa_i.$$

Return the $\frac{p}{q}$-expansion of x: $< x >_{\frac{p}{q}} = \cdots a_2a_1a_0$.

$$s = \sum_{i=0}^{\infty} a_i \left(\frac{p}{q}\right)^i$$

All the results on finiteness and periodicity for $\frac{1}{q}\frac{p}{q}$-representations are similar for $\frac{p}{q}$-representations.

The main difference is on the tree of representations of the integers.

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Theorem

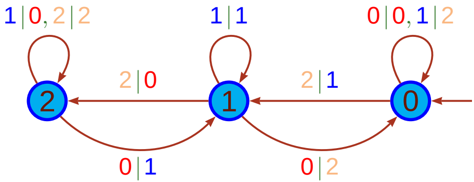
There exists a finite right sequential transducer $\mathcal{C}$ converting the $\frac{p}{q}$-representation of any $x \in \mathbb{Z}_r$, r prime factor of p, to its $\frac{1}{q}\frac{p}{q}$-representation; the inverse of $\mathcal{C}$ is also a finite right sequential transducer.

There is a one-to-one mapping between the sets of all $\frac{p}{q}$- and $\frac{1}{q}\frac{p}{q}$-representations of elements of $\mathbb{Z}_r$ which preserves eventual periodicity.

One can say that the $\frac{p}{q}$- and $\frac{1}{q}\frac{p}{q}$- numeration systems are isomorphic.

Right sequential transducer from $\frac{p}{q}$ to $\frac{1}{q}\frac{p}{q}$

$p = 3, q = 2$



Negative rational base

One can also define representations of the form

$$\sum \frac{a_i}{q}\left(-\frac{p}{q}\right)^i, \quad \text{and} \quad \sum a_i \left(-\frac{p}{q}\right)^i$$

with $a_i \in \mathcal{A}_p$, by modifying the MD and MDbis algorithms by

$$q\frac{s_i}{t} = -p\frac{s_{i+1}}{t} + a_i$$

and

$$q\frac{s_i}{t} = -p\frac{s_{i+1}}{t} + qa_i$$

Negative rational base

Definition (Kátai and Szabó 1975)

A Canonical Number System is a positional numeration system in which every integer (positive or negative) has a unique finite expansion of the form $a_n \cdots a_0$.

Proposition

The $\frac{1}{q}(-\frac{p}{q})$*- and the* $(-\frac{p}{q})$*- numeration systems are CNS.*

The $(-\frac{p}{q})$- numeration system has been previously considered by Gilbert (1991).

Conversions

Theorem

The conversion from the $\frac{1}{q}\frac{p}{q}$ *(resp. the* $\frac{p}{q}$*) -representation of any* $x \in \mathbb{Z}_r$, *r prime factor of p, to its* $\frac{1}{q}(-\frac{p}{q})$ *(resp. its* $(-\frac{p}{q})$*) representation is realizable by a finite right sequential transducer ; the inverse conversion as well.*