Numbers as words

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Representation of numbers

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Elements of a subset of $\mathbb{C}$ represented by strings (words) of digits

- Positional numeration systems
  - base β in $\mathbb{C}$, $|\beta| > 1$
  - basis $U = (u_n)_{n \geqslant 0}$
- Continued fractions
- Residue number system
- Logarithmic number system
- Abstract numeration systems
- ...

Representations can be finite, or right infinite, or left infinite words of digits.

Positional numeration systems

Basis $U = (u_n)_{n \geqslant 0}$, $u_n \in \mathbb{C}$
A *U*-representation of $x \in \mathbb{C}$ on a set D of complex digits can be

- a finite word $d_k \cdots d_0$, with $d_i \in D$, such that $x = \sum_{i=0}^{k} d_i u_i$
- or a right infinite word $d_1 d_2 \cdots$, with $d_i \in D$, such that $x = \sum_{i=1}^{\infty} d_i u_i$
- or a left infinite word $\cdots d_1 d_0$, with $d_i \in D$, $x = \sum_{i=0}^{\infty} d_i u_i$

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Most significant digit on the left side.

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Given a base or a basis, an algorithm $\mathcal{A}$ and a set $S \subset \mathbb{C}$:

- Does any element of S has an expansion by algorithm $\mathcal{A}$?

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Algorithm $\mathcal{A}$: (in general)

- greedy algorithm $\mathcal{G}$, produces most significant digit first
- (modified) Euclidean Division algorithm $\mathcal{D}$, produces least significant digit first.

$U = (u_n)_{n \geqslant 0}$ with $u_n = \beta^n$

Integer base $\beta > 1$ in $\mathbb{N}$

Greedy algorithm $\mathcal{G}$

N positive integer. $\exists ! k$ such that $\beta^k \leqslant N < \beta^{k+1}$.

$N_k := N$;

`for` $i := k$ `downto 0 do`

$\quad d_i := \lfloor \frac{N_i}{\beta^i} \rfloor; \quad N_{i-1} = N_i - d_i \beta^i$;

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Euclidean division algorithm $\mathcal{D}$

N positive integer.

$N_0 := N$;

$i := 0$;

`repeat`

  $d_i := N_i \mod \beta$; $N_{i+1} := N_i \operatorname{div} \beta$;

  $i := i + 1$;

`until` $N_i = 0$;

$N = \sum_{i=0}^{i=k} d_i \beta^i$, $d_i \in A = \{0, 1, \dots, \beta - 1\}$.

- By algorithms $\mathcal{G}$ and $\mathcal{D}$, every element of $\mathbb{N}$ has a unique finite expansion $d_k \cdots d_0$, with $d_k \neq 0$, d_i in the canonical digit set $A = \{0, 1, \ldots, \beta - 1\}$, $0 \leqslant i \leqslant k$.
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- By algorithm $\mathcal{D}$ every element of $\mathbb{Q}$ has an eventually periodic left infinite expansion on A (p-adic expansion).

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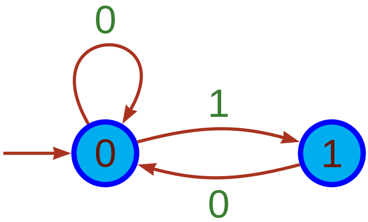
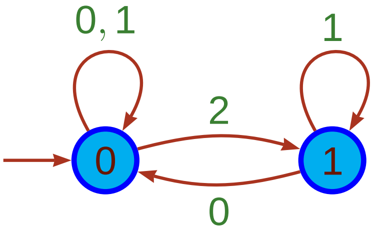
- When β is a Pisot number, every element of $\mathbb{Q}(\beta) \cap \mathbb{R}_+$ has an eventually periodic expansion (Schmidt).
- For some Pisot numbers, for instance the golden mean, every element of $\mathbb{Z}(\beta) \cap \mathbb{R}_+$ has a finite expansion.

Example The golden mean shift: system of finite type. Local automaton.



Example The β-shift for $\beta = \frac{3+\sqrt{5}}{2}$: sofic system not of finite type. Non-local automaton.



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Example Factorial numeration system: $u_n = n!$, digit set $= \mathbb{N}$.
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Basis $U = (u_n)_{n \geqslant 0}$ with $u_n \in \mathbb{R}_+$, summable and strictly decreasing (Muller), by a greedy algorithm.
Example $u_n = \log(1 + 2^{-n})$, $A = \{0, 1\}$: representation of positive reals by an infinite word on A.

Negative base

Negative integer base: Grünwald 1885.

When β is a real number, $(-\beta)$-expansions were introduced by Ito and Sadahiro.

Complex base

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- base $\imath\sqrt{2}$ and digit set $\{0, 1\}$.

Rational base $\frac{p}{q} > 1$

- By $\mathcal{G}$ every positive real has an aperiodic expansion on $\{0, 1, \ldots, \lceil \frac{p}{q} \rceil - 1\}$.
- By $\mathcal{D}$ every positive integer has a unique finite expansion on $\{0, 1, \ldots, p - 1\}$ (Akiyama, Frougny, Sakarovitch). The set of expansions of elements of $\mathbb{N}$ is not context-free.
- By $\mathcal{D}$ every element of $\mathbb{Q}$ has an eventually periodic left infinite expansion.

Same thing for base $-\frac{p}{q}$ (Frougny, Klouda).

Computations

Distance on the set of infinite words

Prefix distance on $A^{\mathbb{N}}$:

$$\rho(v,w) = \begin{cases} 2^{-r} & \text{where } r = \min\{i \mid v_i \neq w_i\} \\ 0 & \text{if } v = w \end{cases}$$

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$$\begin{aligned} 0(1)^k 0^\omega + 0^\omega &= 0(1)^k 0^\omega \\ 0(1)^k 0^\omega + (0)^k 1 0^\omega &= 1 0^\omega \end{aligned}$$

On-line functions

A function $\varphi : A^{\mathbb{N}} \to B^{\mathbb{N}}$ is on-line computable with delay δ if $\exists \delta \in \mathbb{N}$ such that $(b_n)_{n \geqslant 1} = \varphi((a_n)_{n \geqslant 1})$ iff $\forall n \geqslant 1$ there exists $\Phi_n : A^{n+\delta} \to B$ with $b_n = \Phi_n(a_1 \cdots a_{n+\delta})$.

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On-line denumerable transducer:

$$\underbrace{q_0 \xrightarrow{a_1|\varepsilon} q_1 \xrightarrow{a_2|\varepsilon} q_2 \cdots \xrightarrow{a_\delta|\varepsilon} q_\delta}_{\text{transient part}} \underbrace{\xrightarrow{a_{\delta+1}|b_1} q_{\delta+1} \xrightarrow{a_{\delta+2}|b_2} q_{\delta+2} \cdots}_{\text{synchronous part}}$$

Proposition

A function on-line computable with delay δ is 2^δ-Lipschitz, and thus uniformly continuous.

Numerical value in base $\beta > 1$:
$\pi_\beta : A^{\mathbb{N}} \to \mathbb{R}$ with $\pi_\beta((a_n)_{n\geqslant 1}) = \sum_{n\geqslant 1} a_n \beta^{-n}$.

If the following diagram commutes, $\varphi_{\mathbb{R}}$ is the real realization of φ in base β

$$\begin{array}{ccc} A^{\mathbb{N}} & \xrightarrow{\ \varphi\ } & B^{\mathbb{N}} \\ \downarrow{\scriptstyle \pi_\beta} & & \downarrow{\scriptstyle \pi_\beta} \\ \pi_\beta(A^{\mathbb{N}}) & \xrightarrow{\ \varphi_{\mathbb{R}}\ } & \pi_\beta(B^{\mathbb{N}}) \end{array}$$

Proposition (Eilenberg)

If φ is continuous than $\varphi_{\mathbb{R}}$ is continuous.

Finite transducers

A function $\varphi : A^{\mathbb{N}} \to B^{\mathbb{N}}$ is computable by a transducer $\mathcal{T} = (Q, A^* \times B^*, E, I, F)$ if the graph of φ is the set of labels of infinite paths starting in I and going infinitely often in F.
$\mathcal{T}$ is finite if E and Q are finite.

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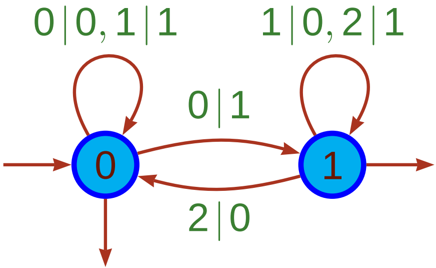
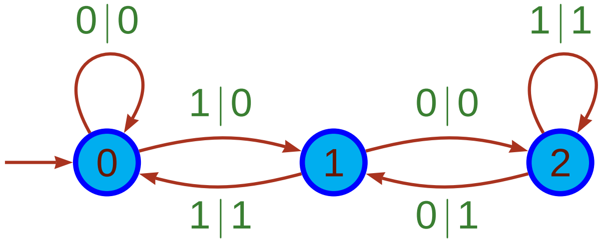
Example Addition of reals base 2 and digit set $\{0, 1\}$ = conversion from $\{0, 1, 2\}$ to $\{0, 1\}$



Sequentiality

$\mathcal{T} = (Q, A^* \times B^*, E, I, F)$ is sequential if $|I| = 1$, $F = Q$, and it is input deterministic.
Processing from left to right: left sequential.

Example Division by 3 in base 2 and digit set $\{0, 1\}$.



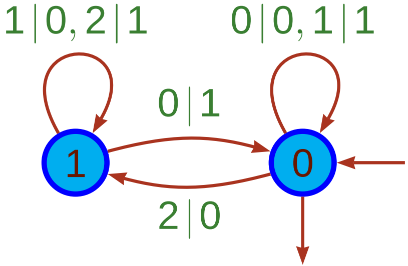
Proposition

A function computable by a finite sequential transducer is uniformly continuous.

Right sequentiality

Processing from right to left

Example Addition in base 2 and digit set $\{0, 1\}$.



Theorem

Any function computable by a finite transducer can be obtained by the composition of a finite right sequential transducer and a finite left sequential transducer.

Finite words: Elgot and Mezei

Infinite words: Carton

On-line finite transducer

Particular left sequential finite transducer.

Example Tent function in base 2 and digit set $\{0, 1\}$.

$$f(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant 1/2 \\ -2x + 2 & \text{if } 1/2 \leqslant x \leqslant 1 \end{cases}$$

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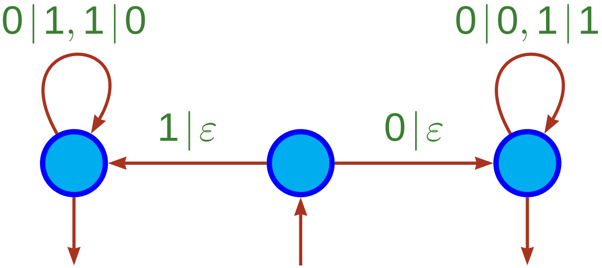

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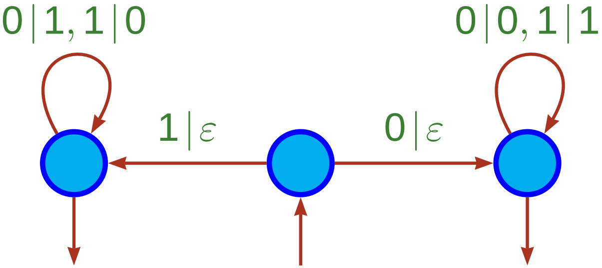

$$\frac{3}{4} = .110^{\omega} \mapsto \frac{1}{2} = .01^{\omega}$$

$$\frac{3}{4} = .101^{\omega} \mapsto \frac{1}{2} = .10^{\omega}$$

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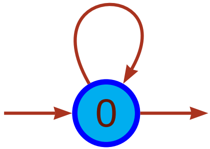
Theorem (Muller)

The real realization of a function computable by an on-line finite transducer (in integer positive base) is a piecewise affine function whose coefficients are rational numbers.

Conversion base 4 $\to$ base 2 is

1. left sequential and right sequential

$0\,|\,00,\ 1\,|\,01,\ 2\,|\,10,\ 3\,|\,11$



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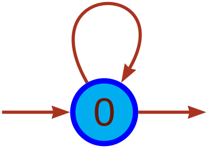

2. on-line computable with delay 0

   $a_n \in \{0, \ldots, 3\}$, $a_n = a_n^{(1)} a_n^{(2)}$,
   $a_n^{(1)}, a_n^{(2)}, b_n \in \{0, 1\}$,
   $\varphi((a_n)) = b_n$ with $b_{2n-1} = a_n^{(1)}$ and $b_{2n} = a_n^{(2)}$

   Uses a queue: $\varepsilon \xrightarrow{2|1} 0 \xrightarrow{3|0} 11 \xrightarrow{0|1} 100 \xrightarrow{1|1} 0001 \cdots$

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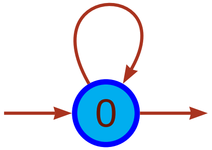

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3. not computable by a finite on-line transducer

Local functions

$\varphi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ is a p-local function if $\exists r, t > 0$, and $\exists \Phi : A^p \to B$, with $p = r + t + 1$, such that

$$(b_n)_{n\in\mathbb{Z}} = \varphi((a_n)_{n\in\mathbb{Z}}) \iff \forall n \in \mathbb{Z},\ \ b_n = \Phi(a_{n+t} \cdots a_{n-r}).$$

The image of $(a_n)_{n\in\mathbb{Z}}$ by φ is obtained through a sliding window of length p.

r is the memory and t is the anticipation of φ.

φ is called a sliding block code.

Local functions

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The image of $(a_n)_{n \in \mathbb{Z}}$ by φ is obtained through a sliding window of length p.
r is the memory and t is the anticipation of φ.
φ is called a sliding block code.
Locality ensures robustness: no propagation of errors.

A local function on finite words is computable by a parallel algorithm.

Proposition

A p-local function is computable by a finite on-line transducer with delay $p - 1$. The input automaton is local.

Addition, multiplication, etc...

Signed-digit representations

Base 10 and digit-set $\{-5, \ldots, 0, \ldots, 5\}$ Cauchy 1840

Base 10 and digit-set $\{-6, \ldots, 0, \ldots, 6\}$ Avizienis 1961

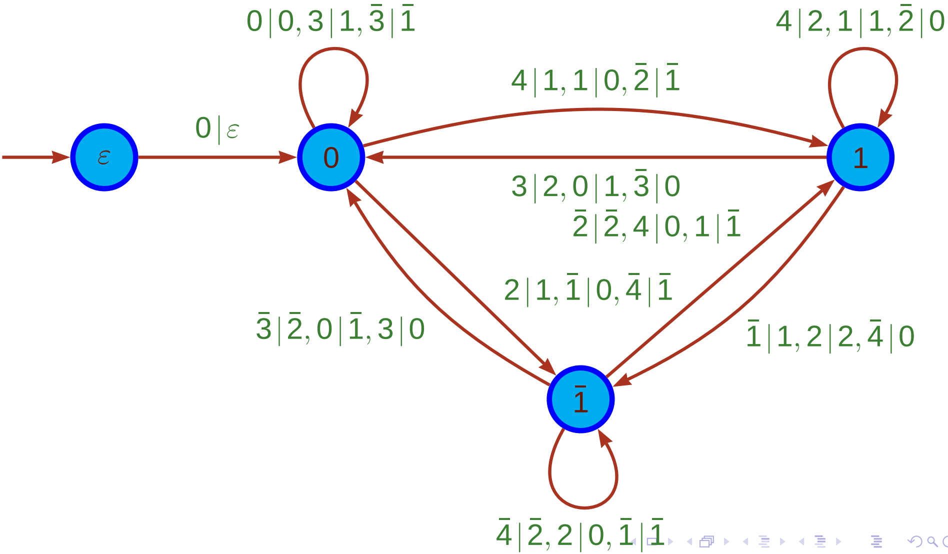
Base 2 and digit-set $\{-1, 0, 1\}$ Chow and Robertson 1978

In integer base b, $b \geqslant 3$, parallel addition on alphabet $\{-a, \ldots, 0, \ldots, a\}$, $b/2 < a \leqslant b-1$ is possible by Avizienis algorithm. It is a 2-local function.

In integer base $b = 2a$, $b \geqslant 2$, parallel addition on alphabet $\{-a, \ldots, 0, \ldots, a\}$ is possible by Chow and Robertson algorithm. It is a 3-local function.

Redundancy
No propagation of the carry.

On-line finite transducer with delay 1 realizing addition in base 3 on $\{\bar{2}, \dots, 2\}$: $p \xrightarrow{x|y} q \Leftrightarrow 3p + x = 3y + q$



Parallel addition

Theorem (Frougny, Pelantová, Svobodová)

Let $\beta \in \mathbb{C}$ with $|\beta| > 1$ be an algebraic number. If all its algebraic conjugates have modulus $\neq 1$ one can find an alphabet of contiguous integer digits on which addition can be done in parallel.

Redundancy is necessary.
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The result is not necessarily admissible.

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Example Addition in base the golden mean:

- on the minimal alphabet $\{-1, 0, 1\}$ is a 21-local function
- on $\{-3, \ldots, 3\}$ is 13-local
- on $\{-5, \ldots, 5\}$ is 9-local.

On-line addition

Suitable for real numbers.

- In real base $\pm\beta$, $\beta > 1$, addition is on-line computable on $\{0, \dots, \lfloor \beta \rfloor\}$ (the result is not admissible).

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- In real base $\pm\beta$, $\beta > 1$, addition is on-line computable on $\{0, \ldots, \lfloor\beta\rfloor\}$ (the result is not admissible).
- If β is a Pisot number, the on-line transducer is finite.
- To get an admissible result, normalization is necessary: If β is a non-integer Pisot number, normalization is computable by a finite transducer, which is neither left nor right sequential.

Real base on-line addition algorithm $\beta > 1$, $A = \{0, \ldots, \lfloor\beta\rfloor\}$

On-line algorithm with delay δ, where δ is the smallest positive integer such that

$$\beta^{\delta+1} + 2\lfloor\beta\rfloor \leqslant \beta^{\delta}(\lfloor\beta\rfloor + 1)$$

Input: $(d_j)_{j\geqslant 1} \in (A+A)^{\mathbb{N}}$.
Output: $(a_j)_{j\geqslant 1} \in A^{\mathbb{N}}$ such that $\sum_{j\geqslant 1} a_j\beta^{-j} = \sum_{j\geqslant 1} d_j\beta^{-j}$.

begin

$\quad q_0 := 0$

$\quad$ **for** $j := 1$ to δ **do**

$\quad\quad q_j := \beta q_{j-1} + d_j$

$\quad j := 1$

$\quad$ **while** $j \geqslant 1$ **do**

$\quad\quad z_{\delta+j} := \beta q_{\delta+j-1} + d_{\delta+j}$

$\quad\quad$ **if** $z_{\delta+j} < \beta^{\delta+1}$

$\quad\quad\quad$ **then** $a_j := \lfloor z_{\delta+j}/\beta^{\delta}\rfloor$

$\quad\quad\quad$ **else** $a_j := \lfloor\beta\rfloor$

$\quad\quad q_{\delta+j} := z_{\delta+j} - \beta^{\delta} a_j$

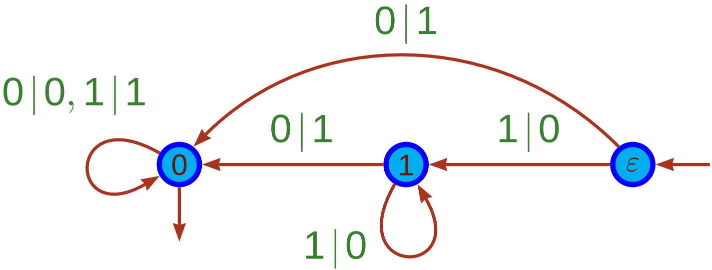
$\quad\quad j := j + 1$

end

Successor function = addition of 1

In integer base the successor function $\langle n \rangle \mapsto \langle n+1 \rangle$ is realizable by a right sequential letter-to letter finite transducer.

Successor function base 2:



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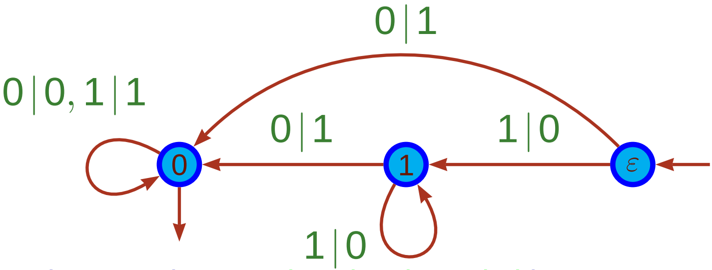

Theorem (Angrand and Sakarovitch)

Let L be a language ordered by the radix order and recognizable by a finite automaton. The successor function on L is realizable by a finite union of right sequential finite transducers with disjoints domains.

Application: L is the set of expansions of $\mathbb{N}$ in a given numeration system.

Multiplication

- Multiplication is not computable by a finite transducer
- Multiplication is on-line computable
  - in positive integer base $b \geqslant 2$ on $\{-a, \ldots, 0, \ldots, a\}$, $b/2 \leqslant a \leqslant b-1$ (Ercegovac and Trivedi)
  - in negative integer base $(-b)$ on $\{-a, \ldots, 0, \ldots, a\}$, $b/2 \leqslant a \leqslant b-1$
  - in real base $\beta > 1$ on $\{0, \ldots, \lfloor \beta \rfloor\}$. (The result is not admissible) (Frougny and Surarerks)
  - in the Knuth number system of base $i\sqrt{b}$, $b \geqslant 2$ integer, on $\{-a, \ldots, 0, \ldots, a\}$, $b/2 \leqslant a \leqslant b-1$
  - in the Penney numeration system of base $-1+i$ on $\{-1, 0, 1\}$ (Surarerks)

Redundancy is necessary

Real base on-line multiplication algorithm $\beta > 1$,
$A = \{0, \ldots, \lfloor\beta\rfloor\}$
On-line algorithm with delay δ, where δ is the smallest positive integer such that

$$\beta + \frac{2\lfloor\beta\rfloor^2}{\beta^\delta(\beta - 1)} \leqslant \lfloor\beta\rfloor + 1.$$

Input: $x = (x_j)_{j\geqslant 1}$ and $y = (y_j)_{j\geqslant 1}$ in $A^{\mathbb{N}}$ such that $x_1 = \cdots = x_\delta = 0$ and $y_1 = \cdots = y_\delta = 0$. $X_j = \sum_{1\leqslant i\leqslant j} x_i\beta^{-i}$
Output: $p = (p_j)_{j\geqslant 1}$ in $A^{\mathbb{N}}$ such that $\sum_{j\geqslant 1} p_j\beta^{-j} = \sum_{j\geqslant 1} x_j\beta^{-j} \times \sum_{j\geqslant 1} y_j\beta^{-j}$.

begin

$p_1 := 0, \ldots, p_\delta := 0$; $W_\delta := 0$
$j := \delta + 1$
while $j \geqslant \delta + 1$ **do**
$W_j := \beta(W_{j-1} - p_{j-1}) + y_jX_j + x_jY_{j-1}$
$p_j := \lfloor W_j \rfloor$
$j := j + 1$

end

On-line multiplication in base $\varphi = (1+\sqrt{5})/2$ on $\{0,1\}$

Delay $\delta = 5$. $x = y = .0^5 10101$. Result $p = .0^{10} 101000100001$.

j	$(W_j)_\varphi$	p_j
6	.000001	0
7	.00001	0
8	.0010001001	0
9	.010001001	0
10	.101000100001	0
11	1.01000100001	1
12	.1000100001	0
13	1.000100001	1
14	.00100001	0
15	.0100001	0
16	.100001	0
17	1.00001	1
18	.0001	0
19	.001	0
20	.01	0
21	.1	0
22	1.0	1
23	.0	0

Concluding remarks

Numbers can be seen as strings or flows of digits, either one by one most significant digit first, or looked at through a sliding window.

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For application to algorithms, sequentiality and synchronicity are important, as well as finite memory when it is possible.

- On-line functions: most significant digit first, well adapted to real numbers with infinite expansions. Pipelining with addition, multiplication and division...
  Division and square-root, some elementary functions are on-line computable in positive integer base (Ercegovac, Muller)
- Local functions: sliding window, parallel algorithms, adapted to arithmetical circuits. Internal additions in on-line algorithm.

Both need redundancy.

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- Find a “good" model for multiplication
- What are the functions computable by an on-line transducer with queue memory?
- Compromise between the size of the digit set and the size of the window for local functions?