On parallel addition in non-standard numeration systems

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Signed-digit representations

Base 10 and digit-set $\{-5, \dots, 0, \dots, 5\}$ Cauchy 1840

Base 10 and digit-set $\{-6, \dots, 0, \dots, 6\}$ Avizienis 1961

Base 2 and digit-set $\{-1, 0, 1\}$ Chow and Robertson 1978

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Redundancy

Algorithm of Avizienis 1961

Base $\beta = b$, $b \geqslant 3$ integer, parallel addition on alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$, $b/2 < a \leqslant b - 1$.

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, $m \leqslant n$, $x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: $z_{n+1} \cdots z_m$ in $\mathcal{A}^*$ such that

$$z = x + y = \sum_{i=m}^{n+1} z_i \beta^i.$$

for each i in parallel do

0. $z_i := x_i + y_i$
1. if $z_i \geqslant a$ then $q_i := 1, r_i := z_i - b$
   if $z_i \leqslant -a$ then $q_i := -1, r_i := z_i + b$
   if $-a + 1 \leqslant z_i \leqslant a - 1$ then $q_i := 0, r_i := z_i$
2. $z_i := q_{i-1} + r_i$

Avizienis

$\beta = 10$, digit-set $\{-6, \ldots, 0, \ldots, 6\}$

x	$\mapsto$		2	5	$\overline{2}$	5	$\overline{5}$	6	0	3
y	$\mapsto$	5	1	2	$\overline{2}$	5	$\overline{4}$	0	6	5
z	$\mapsto$	5	3	7	$\overline{4}$	10	$\overline{9}$	6	6	8
0	$\mapsto$		1	$\overline{10}$						
0	$\mapsto$				1	$\overline{10}$				
0	$\mapsto$					$\overline{1}$	10			
0	$\mapsto$						1	$\overline{10}$		
0	$\mapsto$							1	$\overline{10}$	
0	$\mapsto$								1	$\overline{10}$
z	$\mapsto$	5	4	$\overline{3}$	$\overline{3}$	$\overline{1}$	2	$\overline{3}$	$\overline{3}$	$\overline{2}$

Minimal polynomial of β is $X - 10$

$1\ \overline{(10)}$ is a (strong) representation of 0

Algorithm of Chow and Robertson 1978

Base $\beta = b = 2a$, $a \geqslant 1$, parallel addition on $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$.

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, $m \leqslant n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: $z_{n+1} \cdots z_m$ in $\mathcal{A}^*$ such that $z = x + y = \sum_{i=m}^{n+1} z_i \beta^i$.

`for each` i `in parallel do`

0. $z_i := x_i + y_i$
1. `if` $a + 1 \leqslant z_i \leqslant b$ `then` $q_i := 1$, $r_i := z_i - b$
   `if` $-b \leqslant z_i \leqslant -a - 1$ `then` $q_i := -1$, $r_i := z_i + b$
   `if` $-a + 1 \leqslant z_i \leqslant a - 1$ `then` $q_i := 0$, $r_i := z_i$
   `if` $z_i = a$ and $z_{i-1} > 0$ `then` $q_i := 1$, $r_i := -a$
   `if` $z_i = a$ and $z_{i-1} \leqslant 0$ `then` $q_i := 0$, $r_i := a$
   `if` $z_i = -a$ and $z_{i-1} < 0$ `then` $q_i := -1$, $r_i := a$
   `if` $z_i = -a$ and $z_{i-1} \geqslant 0$ `then` $q_i := 0$, $r_i := -a$
2. $z_i := q_{i-1} + r_i$

Chow and Robertson (Cauchy)

$\beta = 10$, digit-set $\{-5, \ldots, 0, \ldots, 5\}$

x	$\mapsto$		2	$\bar{5}$	$\bar{1}$	0	$\bar{3}$	2	0	3
y	$\mapsto$	1	3	$\bar{1}$	2	5	$\bar{5}$	3	5	5
z	$\mapsto$	1	5	$\bar{6}$	1	5	$\bar{8}$	5	5	8
0	$\mapsto$		$\bar{1}$	10						
0	$\mapsto$					$\bar{1}$	10			
0	$\mapsto$						1	$\overline{10}$		
0	$\mapsto$							1	$\overline{10}$	
0	$\mapsto$								1	$\overline{10}$
z	$\mapsto$	1	4	4	1	4	3	$\bar{4}$	$\bar{4}$	$\bar{2}$

Excursion into symbolic dynamics

A subset $S \subseteq \mathcal{A}^{\mathbb{Z}}$ is a symbolic dynamical system if it is closed and shift-invariant.

$S \subseteq \mathcal{A}^{\mathbb{Z}}$ and $T \subseteq \mathcal{B}^{\mathbb{Z}}$ symbolic dynamical systems.
$\varphi : S \to T$ is a *p-local function* if $\exists r, t > 0$, and $\exists \Phi : \mathcal{A}^p \to \mathcal{B}$, with $p = r + t + 1$, such that if $u = (u_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ and $v = (v_i)_{i \in \mathbb{Z}} \in \mathcal{B}^{\mathbb{Z}}$, then

$$v = \varphi(u) \iff \forall i \in \mathbb{Z}, \; v_i = \Phi(u_{i+t} \cdots u_{i-r}).$$

The image of u by φ is obtained through a sliding window of length p.
r is the *memory* and t is the *anticipation* of φ.
φ is called a *sliding block code*.

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A local function is computable by a finite sequential transducer.

Differences between the two algorithms

Decision (choice) in step 1:

- Avizienis algorithm is neighbour free.
- Chow and Robertson algorithm is neighbour sensitive.

Locality : Addition on $\mathcal{A}$ is a function from $(\mathcal{A}+\mathcal{A})^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$

- Avizienis addition is 2-local.
- Chow and Robertson addition is 3-local.

Strong representation of zero property

Base β algebraic number with $|\beta| > 1$.

Definition

β satisfies the *strong representation of zero property* (β *is SRZ*) if there exist integers $b_k, b_{k-1}, \ldots, b_1, b_0, b_{-1}, \ldots, b_{-h}$ such that β is a root of the polynomial

$$S(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$$

and

$$B = b_0 > 2 \sum_{i \neq 0} |b_i| = 2M \, .$$

The polynomial S is said to be a *strong polynomial* for β.

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and

$$B = b_0 > 2 \sum_{i \neq 0} |b_i| = 2M\,.$$

The polynomial S is said to be a *strong polynomial* for β.

$$(b_k b_{k-1} \cdots b_1 b_0 \centerdot b_{-1} \cdots b_{-h})_\beta = 0$$

Suppose that β is SRZ, i.e. $B > 2M$.
Working alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$
with

$$a = \left\lceil \frac{B-1}{2} \right\rceil + \left\lceil \frac{B-1}{2(B-2M)} \right\rceil M.$$

Let

$$a' = \left\lceil \frac{B-1}{2} \right\rceil \quad \text{and} \quad c = \left\lceil \frac{B-1}{2(B-2M)} \right\rceil.$$

Then $a = a' + cM$.
$\mathcal{A}' = \{-a', \ldots, 0, \ldots, a'\} \subset \mathcal{A}$ is the inner alphabet.

Parallel addition for base β SRZ on

$$\mathcal{A} = \{-a, \ldots, 0, \ldots, a\},\ a = \underbrace{\left\lceil \frac{B-1}{2} \right\rceil}_{a'} + \underbrace{\left\lceil \frac{B-1}{2(B-2M)} \right\rceil}_{c} M$$

Algorithm (S)

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, with $m \leqslant n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: $z_{n+k} \cdots z_{m-h}$ in $\mathcal{A}^*$ such that $z = x + y = \sum_{i=m-h}^{n+k} z_i \beta^i$.

`for each` i `in parallel do`

0. $z_i := x_i + y_i$
1. find $q_i \in \{-c, \ldots, 0, \ldots, c\}$ such that $z_i - q_i B \in \mathcal{A}'$
2. $z_i := z_i - \sum_{j=-h}^{k} q_{i-j} b_j$

Parallel addition for base β SRZ on

$$\mathcal{A} = \{-a, \ldots, 0, \ldots, a\},\ a = \underbrace{\left\lceil \frac{B-1}{2} \right\rceil}_{a'} + \underbrace{\left\lceil \frac{B-1}{2(B-2M)} \right\rceil}_{c} M$$

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Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, with $m \leqslant n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: $z_{n+k} \cdots z_{m-h}$ in $\mathcal{A}^*$ such that $z = x + y = \sum_{i=m-h}^{n+k} z_i \beta^i$.

`for each` i `in parallel do`

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1. find $q_i \in \{-c, \ldots, 0, \ldots, c\}$ such that $z_i - q_i B \in \mathcal{A}'$
2. $z_i := z_i - \sum_{j=-h}^{k} q_{i-j} b_j$

Algorithm (S) is neighbour free.

Integer base

$\beta = b$ integer $\geqslant 3$ is SRZ for the polynomial $-X + b$, and Algorithm (S) works with $c = 1$, $a' = \lceil \frac{b-1}{2} \rceil$, and $a = \lceil \frac{b+1}{2} \rceil$.

`for each` i `in parallel do`

0. $z_i := x_i + y_i$
1. find $q_i \in \{-1, 0, 1\}$ such that $z_i - q_i b \in \mathcal{A}'$
2. $z_i := z_i - q_i b + q_{i-1}$

Algorithm (S) is the algorithm of Avizienis with $a = \lceil \frac{b+1}{2} \rceil$.

For $\beta = 2$, $-X + 2$ is not a strong polynomial. But β satisfies the strong polynomial

$$-X^2 + 4.$$

So Algorithm (S) works for base 2 on $\{-3, \ldots, 0, \ldots, 3\}$.

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$$-X^2 + 4.$$

So Algorithm (S) works for base 2 on $\{-3, \ldots, 0, \ldots, 3\}$.

Remind that the Chow and Robertson algorithm works with smaller alphabet $\{-1, 0, 1\}$, but need to examine the right neighbour of current position.

The Golden Mean

$\beta = \frac{1+\sqrt{5}}{2}$, the Golden Mean.
Every real number $\geqslant 0$ has an expansion on alphabet $\{0,1\}$.

β is one root of $X^2 - X - 1$, the second root is $\beta' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\beta}$.
Since $\beta^4 + (\beta')^4 = 7$, β is a root of the strong polynomial

$$S(X) = -X^4 + 7 - \frac{1}{X^4}$$

with $B = 7$ and $M = 2$. Thus $c = 1$, $a' = 3$, and $a = 5$. The working alphabet of Algorithm (S) is $\mathcal{A} = \{-5, \dots, 0, \dots, 5\}$.

$\bar{1}0007000\bar{1}$ is a strong β-representation of 0.

$a' = 3$, $a = 5$

x	$\mapsto$						2	5	$\overline{2}$	5	$\overline{5}$	0	0	3				
y	$\mapsto$					5	1	2	$\overline{2}$	5	$\overline{4}$	0	0	5				
z	$\mapsto$					5	3	7	$\overline{4}$	10	$\overline{9}$	0	0	8				
0	$\mapsto$	1	0	0	0	$\overline{7}$	0	0	0	1								
0	$\mapsto$			1	0	0	0	$\overline{7}$	0	0	0	1						
0	$\mapsto$				$\overline{1}$	0	0	0	7	0	0	0	$\overline{1}$					
0	$\mapsto$					1	0	0	0	$\overline{7}$	0	0	0	1				
0	$\mapsto$						$\overline{1}$	0	0	0	7	0	0	0	$\overline{1}$			
0	$\mapsto$									1	0	0	0	$\overline{7}$	0	0	0	1
z	$\mapsto$	1	0	1	$\overline{1}$	$\overline{1}$	2	0	3	5	$\overline{2}$	1	$\overline{1}$	2	$\overline{1}$	0	0	1

Locality

Corollary

If β is SRZ with strong polynomial
$S(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$
then addition realized by Algorithm (S) is a $(h + k + 1)$-local function from $\{-2a, \ldots, 0, \ldots, 2a\}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$.

Reduction of the alphabet

Definition

β satisfies the *weak representation of zero property* (β *is WRZ*) if there exist integers $b_k, b_{k-1}, \ldots, b_1, b_0, b_{-1}, \ldots, b_{-h}$ such that β is a root of the polynomial

$$W(X) = b_k X^k + b_{k-1} X^{k-1} + \ldots + b_1 X + b_0 + b_{-1} X^{-1} + \ldots + b_{-h} X^{-h}$$

and

$$B = b_0 > \sum_{i \neq 0} |b_i| = M \,.$$

The polynomial W is said to be a *weak polynomial* for β.

β is WRZ, i.e. $B > M$. Working alphabet

$$\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}, \quad \text{where} \quad a = \left\lceil \frac{B-1}{2} \right\rceil + M\,.$$

Inner alphabet is $\mathcal{A}' = \{-a', \ldots, 0, \ldots, a'\}$ with $a' = \lceil \frac{B-1}{2} \rceil$.

Algorithm (W) works with $\left\lceil \frac{a}{B-M} \right\rceil$ iterations.

Parallel addition for base β WRZ on $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$, $a = \underbrace{\left\lceil \frac{B-1}{2} \right\rceil}_{a'} + M$

Algorithm (W)

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, with $m \leqslant n$, $x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.
Output: $z_{n+k} \cdots z_{m-h}$ in $\mathcal{A}^*$ such that

$$z = x + y = \sum_{i=m-h}^{n+k} z_i \beta^i .$$

for each i in parallel do
0. $z_i := x_i + y_i$
1. for $\ell := 1$ to $\left\lceil \frac{a}{B-M} \right\rceil$ do
if $z_i \in \mathcal{A}'$ then $q_i := 0$ else $q_i := \operatorname{sgn} z_i$
$z_i := z_i - \sum_{j=-h}^{k} q_{i-j} b_j$

Example

$\beta = \frac{1+\sqrt{5}}{2}$, the Golden Mean.

Since $-\beta^2 + 3 - \frac{1}{\beta^2} = 0$, β is a root of the weak polynomial

$$W(X) = -X^2 + 3 - \frac{1}{X^2}$$

with $B = 3$ and $M = 2$. Thus $a' = 1$, and $a = 3$.

Algorithm (W) works on $\mathcal{A} = \{-3, \ldots, 0, \ldots, 3\}$, with 3 iterations.

$\bar{1}030\bar{1}$ is a weak β-representation of 0.

Step 0.	x	$\mapsto$					3	$\bar{1}$	3	0	3				
	y	$\mapsto$					2	0	3	$\bar{2}$	3				
	z	$\mapsto$					5	$\bar{1}$	6	$\bar{2}$	6				
Step 1.	0	$\mapsto$			1	0	$\bar{3}$	0	1						
	0	$\mapsto$					1	0	$\bar{3}$	0	1				
	0	$\mapsto$						$\bar{1}$	0	3	0	$\bar{1}$			
	0	$\mapsto$							1	0	$\bar{3}$	0	1		
	z	$\mapsto$			1	0	3	$\bar{2}$	5	1	4	$\bar{1}$	1		
Step 2.	0	$\mapsto$			1	0	$\bar{3}$	0	1						
	0	$\mapsto$				$\bar{1}$	0	3	0	$\bar{1}$					
	0	$\mapsto$					1	0	$\bar{3}$	0	1				
	0	$\mapsto$							1	0	$\bar{3}$	0	1		
	z	$\mapsto$			2	$\bar{1}$	1	1	4	0	2	$\bar{1}$	2		
Step 3.	0	$\mapsto$	1	0	$\bar{3}$	0	1								
	0	$\mapsto$					1	0	$\bar{3}$	0	1				
	0	$\mapsto$							1	0	$\bar{3}$	0	1		
	0	$\mapsto$									1	0	$\bar{3}$	0	1
	z	$\mapsto$	1	0	$\bar{1}$	$\bar{1}$	3	1	2	0	1	$\bar{1}$	0	0	1

Corollary

If β is WRZ with weak polynomial $W(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$ then addition realized by Algorithm (W) is a $(h \left\lceil \frac{a}{B-M} \right\rceil + k \left\lceil \frac{a}{B-M} \right\rceil + 1)$-local function from $\{-2a, \ldots, 0, \ldots, 2a\}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$.

Corollary

If β is WRZ with weak polynomial $W(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$ *then addition realized by Algorithm (W) is a* $(h \left\lceil \frac{a}{B-M} \right\rceil + k \left\lceil \frac{a}{B-M} \right\rceil + 1)$*-local function from* $\{-2a, \ldots, 0, \ldots, 2a\}^{\mathbb{Z}}$ *to* $\mathcal{A}^{\mathbb{Z}}$*.*

Algorithm (W) is neighbour free.

Corollary

If β is WRZ with weak polynomial $W(X) = b_k X^k + b_{k-1} X^{k-1} + \cdots + b_1 X + b_0 + b_{-1} X^{-1} + \cdots + b_{-h} X^{-h}$ then addition realized by Algorithm (W) is a $(h \lceil \frac{a}{B-M} \rceil + k \lceil \frac{a}{B-M} \rceil + 1)$-local function from $\{-2a, \ldots, 0, \ldots, 2a\}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$.

Algorithm (W) is neighbour free.

Remark

Algorithm (S) and Algorithm (W) coincide if, and only if, $B \geqslant 4M - 1$.

Example

- If b integer $\geqslant 3$, $-X + b$ is a strong polynomial for b. Algorithm (S) and Algorithm (W) coincide with $\mathcal{A} = \{-a, \ldots, a\}$, $a = \lceil \frac{b+1}{2} \rceil$.
- For $b = 2$, $-X + 2$ is a weak polynomial. Algorithm (W) works with $\mathcal{A} = \{-2, -1, 0, 1, 2\}$ with 2 iterations.

What numbers are SRZ (or WRZ)?

Theorem

Let β with $|\beta| > 1$ be an algebraic number.
β is SRZ (or WRZ) $\iff$ all its algebraic conjugates have modulus $\neq 1$.

The proof gives a constructive method to obtain a strong (or weak) polynomial from the minimal polynomial of β.

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The proof gives a constructive method to obtain a strong (or weak) polynomial from the minimal polynomial of β.

Remark

Let β with $|\beta| > 1$ be an algebraic number of degree d.

- *If d is odd or*
- *if $d = 2$ or*
- *if d is even $\geqslant 4$ and the minimal polynomial of β is not reciprocal,*

then β has no conjugate of modulus 1.

The Golden Mean

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It is known that it is not possible to perform parallel addition in base the Golden Mean on $\{0, 1\}$.

We give Algorithm (G) for parallel addition in base the Golden Mean on $\{-1, 0, 1\}$. This algorithm is neighbour sensitive.

We use the weak representation of zero $-\beta^2 + 3 - \frac{1}{\beta^2} = 0$.

Algorithm A: Base $\beta = \frac{1+\sqrt{5}}{2}$, reduction from $\{-2, -1, 0, 1, 2\}$ to $\{-1, 0, 1, 2\}$.

Input: a finite sequence of digits (z_i) of $\{-2, -1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.
Output: a finite sequence of digits (z_i) of $\{-1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.

`for each` i `in parallel do`

1. `case` $\left\{ \begin{array}{l} z_i = -2 \\ z_i = -1 \\ z_i = 0 \text{ and } z_{i+2} < 0 \text{ and } z_{i-2} < 0 \end{array} \right\}$ `then`
$q_i := -1$

    `else` $q_i := 0$

2. $z_i := z_i - 3q_i + q_{i+2} + q_{i-2}$

Algorithm B: Base $\beta = \frac{1+\sqrt{5}}{2}$, reduction from $\{-1, 0, 1, 2\}$ to $\{-1, 0, 1\}$.

Input: a finite sequence of digits (z_i) of $\{-1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.
Output: a finite sequence of digits (z_i) of $\{-1, 0, 1\}$, with $z = \sum z_i \beta^i$.

for each i in parallel do
1. case

$$\left\{\begin{array}{l} z_i = 2 \\ z_i = 1 \text{ and } (z_{i+2} \geqslant 1 \text{ or } z_{i-2} \geqslant 1) \\ z_i = 0 \text{ and } z_{i+2} = z_{i-2} = 2 \\ z_i = 0 \text{ and } z_{i+2} = z_{i-2} = 1 \text{ and } z_{i+4} \geqslant 1 \text{ and } z_{i-4} \geqslant 1 \\ z_i = 0 \text{ and } z_{i+2} = 2 \text{ and } z_{i-2} = 1 \text{ and } z_{i-4} \geqslant 1 \\ z_i = 0 \text{ and } z_{i-2} = 2 \text{ and } z_{i+2} = 1 \text{ and } z_{i+4} \geqslant 1 \end{array}\right\}$$

then $q_i := 1$

else $q_i := 0$

2. $z_i := z_i - 3q_i + q_{i+2} + q_{i-2}$

Algorithm G: Base $\beta = \frac{1+\sqrt{5}}{2}$, parallel addition on $\mathcal{A} = \{-1, 0, 1\}$.

Input: two finite sequences of digits (x_i) and (y_i) of $\{-1, 0, 1\}$, with $x = \sum x_i \beta^i$ and $y = \sum y_i \beta^i$.
Output: a finite sequence of digits (z_i) of $\{-1, 0, 1\}$ such that

$$z = x + y = \sum z_i \beta^i.$$

`for each` i `in parallel do`

0. $v_i := x_i + y_i$
1. use Algorithm A with input (v_i) and output (w_i)
2. use Algorithm B with input (w_i) and output (z_i)

Addition in base the Golden Mean on $\{-1, 0, 1\}$ realized by Algorithm G is a 21-local function.

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Algorithm S on $\{-5, \ldots, 5\}$: 9-local

Algorithm W on $\{-3, \ldots, 3\}$: 13-local

Minimal alphabets for parallel addition

$\mathcal{A}$ alphabet of contiguous integer digits containing 0.
β algebraic number, $|\beta| > 1$.

- $\beta = b \geqslant 2$ integer, any alphabet of cardinality $b+1$ is minimal.
  Example: $\mathcal{A} = \{-1, 0, \ldots, b-1\}$ or $\{0, \ldots, b-1, b\}$
  Addition is a 3-local function.
- β is the Golden Mean: $\mathcal{A} = \{-1, 0, 1\}$ is minimal.

Lower bounds

$\mathcal{A}$ finite alphabet of contiguous integers containing 0 with at least two elements.
β algebraic number, $|\beta| > 1$

Theorem

1. *β a real algebraic number > 1. If addition on $\mathcal{A}$ is computable in parallel then*

$$\#\mathcal{A} \geqslant \lceil \beta \rceil$$

2. *β an algebraic integer with minimal polynomial $f(X)$. If addition on $\mathcal{A}$ is computable in parallel then*

$$\#\mathcal{A} \geqslant |f(1)|$$

If β is a real algebraic integer then

$$\#\mathcal{A} \geqslant |f(1)| + 2$$

In the previous theorem

1. “$\#\mathcal{A} \geqslant \lceil \beta \rceil$" can be replaced by

   $$\text{“}\#\mathcal{A} \geqslant \max\{\lceil \gamma \rceil \mid \gamma \text{ or } \gamma^{-1} \text{ is a positive conjugate of } \beta\}\text{”}.$$

2. “β is an algebraic integer" can be replaced by “β or $\frac{1}{\beta}$ is an algebraic integer"
   “β is an algebraic integer > 1" can be replaced by “β is an algebraic integer and one of its algebraic conjugates is > 1".

Addition versus conversion

$\mathcal{A} = \{m, \ldots, 0 \ldots, M\}$.

1. $m = 0$: Addition on $\mathcal{A}$ is parallelizable $\iff$

   $$\text{greatest digit elimination}: \ \mathcal{A} \cup \{M + 1\} \to \mathcal{A}$$

   is parallelizable.
2. $\{-1, 0, 1\} \subset \mathcal{A}$: Addition on $\mathcal{A}$ is parallelizable $\iff$

   $$\text{greatest digit elimination}: \mathcal{A} \cup \{M + 1\} \to \mathcal{A}$$

   and

   $$\text{smallest digit elimination}: \{m - 1\} \cup \mathcal{A} \to \mathcal{A}$$

   are parallelizable.

How to pass from one alphabet allowing parallel addition to another one of same size

Proposition

For $K, d \in \mathbb{Z}$, where $0 \leqslant d \leqslant K-1$, denote

$$\mathcal{A}_{-d} = \{-d, \dots, 0, \dots, K-1-d\}.$$

Let φ be a p-local function realizing conversion in base β from $\mathcal{A}_0 \cup \{K\}$ to $\mathcal{A}_0$. If

- $\varphi({}^{\omega}d \bullet d^{\omega}) = {}^{\omega}d \bullet d^{\omega}$ *and*
- $\varphi({}^{\omega}(K-1-d) \bullet (K-1-d)^{\omega}) = {}^{\omega}(K-1-d) \bullet (K-1-d)^{\omega}$

then addition is performable in parallel on $\mathcal{A}_{-d}$ as well.

Positive integer base

$\beta = b$, $b \geqslant 2$ integer. Minimal polynomial $f(X) = X - b$.
Lower bound $|f(1)| + 2 = b + 1$ is attained.

Parallel addition is feasible on any alphabet of cardinality $b + 1$ containing 0, in particular on alphabets $\mathcal{A} = \{0, 1, \ldots, b\}$ and $\mathcal{A} = \{-1, 0, 1, \ldots, b - 1\}$ (folklore).

If b is even, $b = 2a$, parallel addition is realizable on the alphabet $\mathcal{A} = \{-a, \ldots, a\}$ of cardinality $b + 1$ by the algorithm of Chow and Robertson (see Cauchy).

Negative integer base

$\beta = -b$, $b \geqslant 2$ integer.
Every integer has a unique finite representation with digits in $\{0, 1, \ldots, b-1\}$ (Grünwald 1885).

Minimal polynomial $f(X) = X + b$. Lower bound $|f(1)| = b + 1$ is attained.

Theorem

Let $\beta = -b \in \mathbb{Z}$, $b \geqslant 2$. Any alphabet $\mathcal{A}$ of contiguous integers containing 0 with cardinality $\#\mathcal{A} = b + 1$ allows parallel addition in base $\beta = -b$ and this alphabet is minimal in size.

Parallel addition on $\{0,\ldots,b\}$: It is enough to show that greatest digit elimination between $\{0,\ldots,b+1\}$ to $\{0,\ldots,b\}$ is performable in parallel.

Algorithm N: Base $\beta=-b$, greatest digit elimination from $\{0,\ldots,b+1\}$ to $\{0,\ldots,b\}$.

Input: a finite sequence of digits (z_i) of $\{0,\ldots,b+1\}$, with $z=\sum z_i\beta^i$.
Output: a finite sequence of digits (z_i) of $\{0,\ldots,b\}$, with $z=\sum z_i\beta^i$.

`for each` i `in parallel do`

1. `case` $\left\{\begin{array}{l} z_i=b+1 \\ z_i=b \ \texttt{ and } z_{i-1}=0 \end{array}\right\}$ `then` $q_i:=1$

   `if` $z_i=0$ `and` $z_{i-1}\geqslant b$ `then` $q_i:=-1$
   `else` $q_i:=0$;

2. $z_i:=z_i-bq_i-q_{i-1}$

Base $\sqrt[k]{b}$, b integer, $|b| \geqslant 2$

Proposition

Let $\beta = \sqrt[k]{b}$, b in $\mathbb{Z}$, $|b| \geqslant 2$ and $k \geqslant 1$ integer. Any alphabet $\mathcal{A}$ of contiguous integers containing 0 with cardinality $\#\mathcal{A} = b + 1$ allows parallel addition.

Use that $\gamma = \beta^k = b$.

Proposition

If b is in $\mathbb{N}$ the polynomial $X^k - b$ is minimal for β, thus the cardinality $b + 1$ is minimal.

Complex bases

Penney numeration system (1964): every integer has a unique finite expansion in base $\beta = -1 + \imath$ with digits in $\{0, 1\}$.

Example: $3 = 1101$.

Minimal polynomial $f(X) = X^2 + 2X + 2$, and lower bound $= |f(1)| = 5$.

$\beta^4 = -4$. Parallel addition is possible on any alphabet of minimal cardinality 5.

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Knuth numeration system (1955): $\beta = 2\imath$ with digits in $\{0, \ldots, 3\}$.
Minimal polynomial $f(X) = X^2 + 4$, and lower bound $=|f(1)| = 5$. Parallel addition is possible on any alphabet of minimal cardinality 5.

Complex bases

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Minimal polynomial $f(X) = X^2 + 2X + 2$, and lower bound $=|f(1)| = 5$.
$\beta^4 = -4$. Parallel addition is possible on any alphabet of minimal cardinality 5.

Knuth numeration system (1955): $\beta = 2\imath$ with digits in $\{0, \ldots, 3\}$.
Minimal polynomial $f(X) = X^2 + 4$, and lower bound $=|f(1)| = 5$. Parallel addition is possible on any alphabet of minimal cardinality 5.

$\beta = \imath\sqrt{2}$ with digits in $\{0, 1\}$.
Minimal polynomial $f(X) = X^2 + 2$, and lower bound $=|f(1)| = 3$. Parallel addition is possible on any alphabet of minimal cardinality 3.

β root of $X^2 = aX - 1$, $a \geqslant 3$

β is a quadratic Pisot unit.
By the greedy algorithm of Rényi 1957, every positive real has an expansion on the canonical alphabet $\mathcal{C} = \{0, \ldots, a-1\}$.
Uniqueness iff avoids any string of the form $(a-1)(a-2)^k(a-1)$.
If no admissibility condition, then redundancy, which is sufficient.

Minimal polynomial $f(X) = X^2 - aX + 1$ and lower bound = $|f(1)| + 2 = a$

Theorem

β root of $X^2 = aX - 1$, $a \geqslant 3$. Every alphabet of size a containing 0 allows parallel addition.

β root of $X^2 = aX + 1$, $a \geqslant 1$

β is a quadratic Pisot unit.
Every positive real has an expansion on the canonical alphabet $\mathcal{C} = \{0, \ldots, a\}$.
Uniqueness iff avoids any string of the form $a1$.
If no admissibility condition, then redundancy, but it's not sufficient.

$a = 1$: Golden Mean. Minimal alphabet has size 3.

Minimal polynomial $f(X) = X^2 - aX - 1$ and lower bound = $|f(1)| + 2 = a + 2$

Theorem
β root of $X^2 = aX + 1$, $a \geqslant 1$. Every alphabet of size $a + 2$ containing 0 allows parallel addition.

Rational base $\beta = a/b$

By a modification of the Euclidean division algorithm any natural integer has a unique and finite expansion on the alphabet $\{0, \ldots, a-1\}$ in base $\beta = a/b$ (Akiyama, Frougny and Sakarovitch 2008; Frougny and Klouda 2011).

Example: $\beta = 3/2$, then $4 = 21$

If $b \geqslant 2$, a/b is an algebraic number which is not an algebraic integer, so our lower bound is $\lceil a/b \rceil$, which is not attained.

Theorem

In base $\beta = a/b$, with a and b co-prime such that $a > b \geqslant 1$, the only alphabets of minimal cardinality $a+b$ allowing parallel addition are:

- $\{0, \ldots, a+b-1\}$ *and* $\{-a-b+1, \ldots, 0\}$
- *every alphabet of cardinality $a+b$ containing $\{-b, \ldots, 0, \ldots, b\}$.*

Negative rational base $\beta = -a/b$

By a modification of the Euclidean division algorithm any integer has a unique and finite expansion on the alphabet $\{0, \ldots, a-1\}$ in base $\beta = -a/b$ (F. and Klouda 2011).

If $b \geqslant 2$, $-a/b$ is a negative algebraic number which is not an algebraic integer, so we have no lower bound

Theorem

In base $\beta = -a/b$, with a and b co-prime such that $a > b \geqslant 1$, every alphabet of minimal cardinality $a + b$ containing 0 allows parallel addition.

Base	Canonical alphabet	Minimal alphabet for parallel addition
$b \geqslant 2$ in $\mathbb{N}$	$\{0, \ldots, b-1\}$	All alphabets of size $b+1$
$-b$, $b \geqslant 2$ in $\mathbb{N}$	$\{0, \ldots, b-1\}$	All alphabets of size $b+1$
$\sqrt[k]{b}$, $b \geqslant 2$ in $\mathbb{N}$		All alphabets of size $b+1$
$-1+\imath$	$\{0, 1\}$	All alphabets of size 5
$2\imath$	$\{0, \ldots, 3\}$	All alphabets of size 5
$\imath\sqrt{2}$	$\{0, 1\}$	All alphabets of size 3
$\beta^2 = a\beta - 1$	$\{0, \ldots, a-1\}$	All alphabets of size a
$\beta^2 = a\beta + 1$	$\{0, \ldots, a\}$	All alphabets of size $a+2$
a/b	$\{0, \ldots, a-1\}$	$\{0, \ldots, a+b-1\}$, $\{-a-b+1, \ldots, 0\}$, and all alphabets of size $a+b$ containing $\{-b, \ldots, 0, \ldots, b\}$
$-a/b$	$\{0, \ldots, a-1\}$	All alphabets of size $a+b$