Non-standard number representation: computer arithmetic, symbolic dynamics and quasicrystals

Christiane Frougny

LIAFA and University Paris 8

`http://www.liafa.jussieu.fr/~cf/`

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Numeration system

Positional numeration system

Base β: an integer, or a real, or a complex number of modulus > 1
or a basis $U = (u_n)_{n \geqslant 0}$ where u_n is an integer or a real number

Digits: integer, or real, or complex numbers

Part I: Computer arithmetic

Standard numeration

Base $\beta > 1$ integer

Canonical alphabet $A = \{0, \ldots, \beta - 1\}$

β-representation of N integer $\geqslant 0$: word $d_k \cdots d_0 = \langle N \rangle_\beta$ of A^* such that

$$N = \sum_{i=0}^{k} d_i \beta^i$$

Unique if $d_k \neq 0$ (called canonical)

β-representation of x in $[0, 1]$: infinite word $(x_i)_{i \geqslant 1}$ of $A^{\mathbb{N}}$ such that

$$x = \sum_{i \geqslant 1} x_i \beta^{-i}$$

Unique if does not end in $(\beta - 1)^\omega$ (canonical).

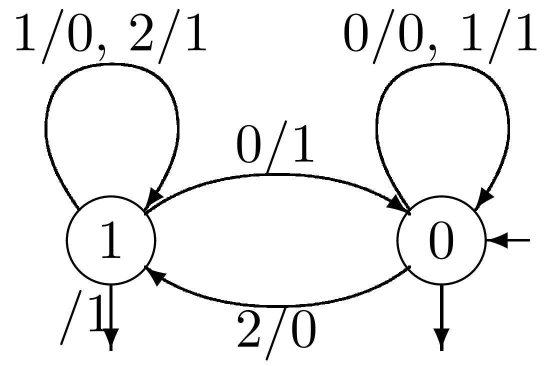
Notation

$$\langle x \rangle_\beta = \bullet x_1 x_2 \cdots$$

Addition of integers

Addition of integers in base 2 (from right to left)

$$
\begin{array}{rccccc}
 & & 1 & 1 & 1 & 1 \\
+ & & 0 & 1 & 1 & 1 \\
\hline
 & & 1 & 2 & 2 & 2 \\
\hline\hline
 & 1 & 0 & 1 & 1 & 0
\end{array}
$$



Addition base 2 is right subsequential.

Digit-set conversion

C alphabet of positive or negative digits containing $A = \{0, \ldots, \beta - 1\}$

Numerical value $\pi_\beta : C^* \to \mathbb{Z}$ such that $\pi_\beta(c_k \cdots c_0) = \sum_{i=0}^{k} c_i \beta^i$.

Conversion on C: $\chi_\beta : C^* \to A^*$ such that $\pi_\beta(c_k \cdots c_0) = \pi_\beta(a_n \cdots a_0)$.

Addition = conversion on $\{0, \ldots, 2(\beta - 1)\}$
Subtraction = conversion on $\{-(\beta - 1), \ldots, (\beta - 1)\}$
Multiplication by a fixed integer $m > 0$ = conversion on $\{0, \ldots, m(\beta - 1)\}$

Conversion on C^* is right subsequential for any C.

Division by a fixed integer is left subsequential.

Addition base β on $\{0, \ldots, \beta - 1\}$ is not left sequential.

Base 2 on $\{0, 1\}$

$$01^n0^\omega + 0^n10^\omega = 10^\omega$$

$$01^n0^\omega + 0^\omega = 01^n0^\omega$$

Multiplication is not computable by a finite automaton.

Redondant representations

Base β integer, digits $B = \{m, \ldots, M\}$, $m < M$ in $\mathbb{Z}$, n positions.

$I = [m\frac{\beta^n - 1}{\beta - 1}, M\frac{\beta^n - 1}{\beta - 1}]$.

If $|B| < \beta$, some integers in I have no representation in base β with n positions.

If $|B| = \beta$ every integer in I has a unique representation.

If $|B| > \beta$, every integer in I has a representation, non necessarily unique.

When $|B| > \beta$, there is redondancy.

Avizienis representations

Base β integer, digits in $B = \{\bar{a}, \ldots, a\}$.

Redondancy: $|B| \geqslant \beta + 1$ *i.e.* $2a \geqslant \beta$.

Sign: $N = \pi_\beta(d_k \cdots d_0)$ with $d_k \neq 0$ has same sign as d_k iff $a \leqslant \beta - 1$.

Choice $\beta/2 < a \leqslant \beta - 1$

Example $\beta = 10$, $a = 6$, $B = \{\bar{6}, \ldots, 6\}$.

Redondancy : $46 =_{10} 5\bar{4}$.

Addition with no carry propagation :

$$
\begin{array}{cccccc}
 & 0 & 2 & 5 & \bar{4} & 6 \\
+ & 0 & 5 & 0 & 1 & 6 \\
\hline
 & 0 & 7 & 5 & \bar{3} & 12 \\
\hline
 & 1 & 0 & 0 & 1 & \\
 & & \bar{3} & 5 & \bar{3} & 2 \\
\hline\hline
 & 1 & \bar{3} & 5 & \bar{2} & 2
\end{array}
$$

Rewriting rules

$$12 \rightarrow \begin{array}{c|c} 1 & \\ & 2 \end{array} \quad 11 \rightarrow \begin{array}{c|c} 1 & \\ & 1 \end{array}$$

$$10 \rightarrow \begin{array}{c|c} 1 & \\ & 0 \end{array} \quad 9 \rightarrow \begin{array}{c|c} 1 & \\ & \bar{1} \end{array} \quad 8 \rightarrow \begin{array}{c|c} 1 & \\ & \bar{2} \end{array}$$

$$7 \rightarrow \begin{array}{c|c} 1 & \\ & \bar{3} \end{array} \quad 6 \rightarrow \begin{array}{c|c} 1 & \\ & \bar{4} \end{array}$$

Idem for negative digits.

Works for $\beta \geqslant 3$ and $\beta/2 < a \leqslant \beta - 1$. Rewrite between $2a$ and a (resp. $-2a$ and $-a$).

Addition in constant time in parallel. Addition is 2-local.

$\beta = 2$, $B = \{\bar{1}, 0, 1\}$ ([Chow and Robertson]).

Rewriting rules

$$2 \quad \rightarrow \quad \begin{array}{c|c} 1 & \\ & 0 \end{array}$$

For digit 1 one uses a window

$$1 \quad \rightarrow \quad \begin{array}{c|c} 1 & \\ & \bar{1} \end{array} \qquad \text{if the right neighbour of 1 is} \geqslant 0$$

otherwise nothing.

Idem for negative digits.

Similar algorithm for addition in even base β with $a = \beta/2$.

Addition in constant time in parallel.
Addition is 3-local.

Carry-Save representations

Redondant representations in base 2 with digits in $B = \{0, 1, 2\}$

Addition in base 2 of a representation over B and a representation over $A = \{0, 1\}$ with result on B can be done in constant time in parallel.

Used for internal additions in multipliers.

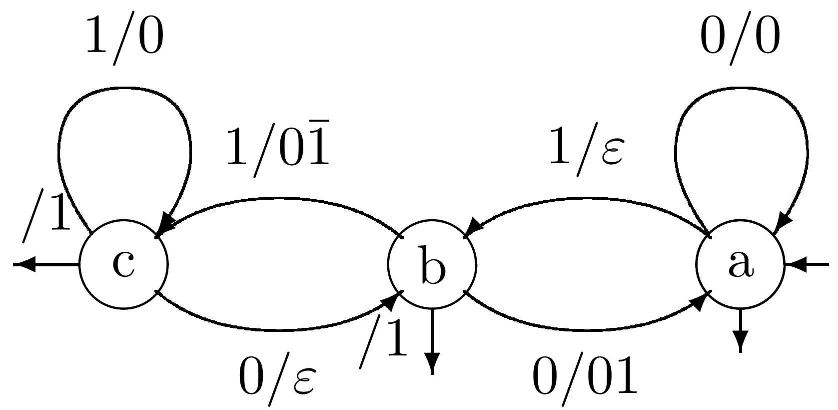
Booth canonical representations

Base 2, $A = \{0, 1\}$, $B = \{\bar{1}, 0, 1\}$.

Find a representation on B with the minimum number of non-zero digits.

Right-to-left recoding: every factor of form 01^n, with $n \geqslant 2$, is transformed into $10^{n-1}\bar{1}$.

The Booth recoding is a right subsequential function from A^* to B^*.



Applications of the Booth normal form

- multiplication
- internal representation for division: base 4 with digits in $\{\bar{3}, \ldots, 3\}$
- computations on elliptic curves.

On-line computability

To pipe-line additions/subtractions, multiplications and divisions, computations are to be done Most Sigificant Digit First, *i.e.* from left to right.

Additional requirement: deterministic processing and, after a certain delay δ of latency, for one input digit there is one output digit. [Ercegovac and Trivedi, 77]

$$\begin{aligned} \varphi : A^{\mathbb{N}} &\rightarrow B^{\mathbb{N}} \\ (a_j)_{j\geqslant 1} &\mapsto (b_j)_{j\geqslant 1} \end{aligned}$$

φ is on-line computable with delay δ if there exists δ such that, for each $j \geqslant 1$ there exists

$$\Phi_j : A^{j+\delta} \rightarrow B$$

such that

$$b_j = \Phi_j(a_1 \cdots a_{j+\delta})$$

$A^{j+\delta}$ is the set of sequences of length $j+\delta$ of elements of A.

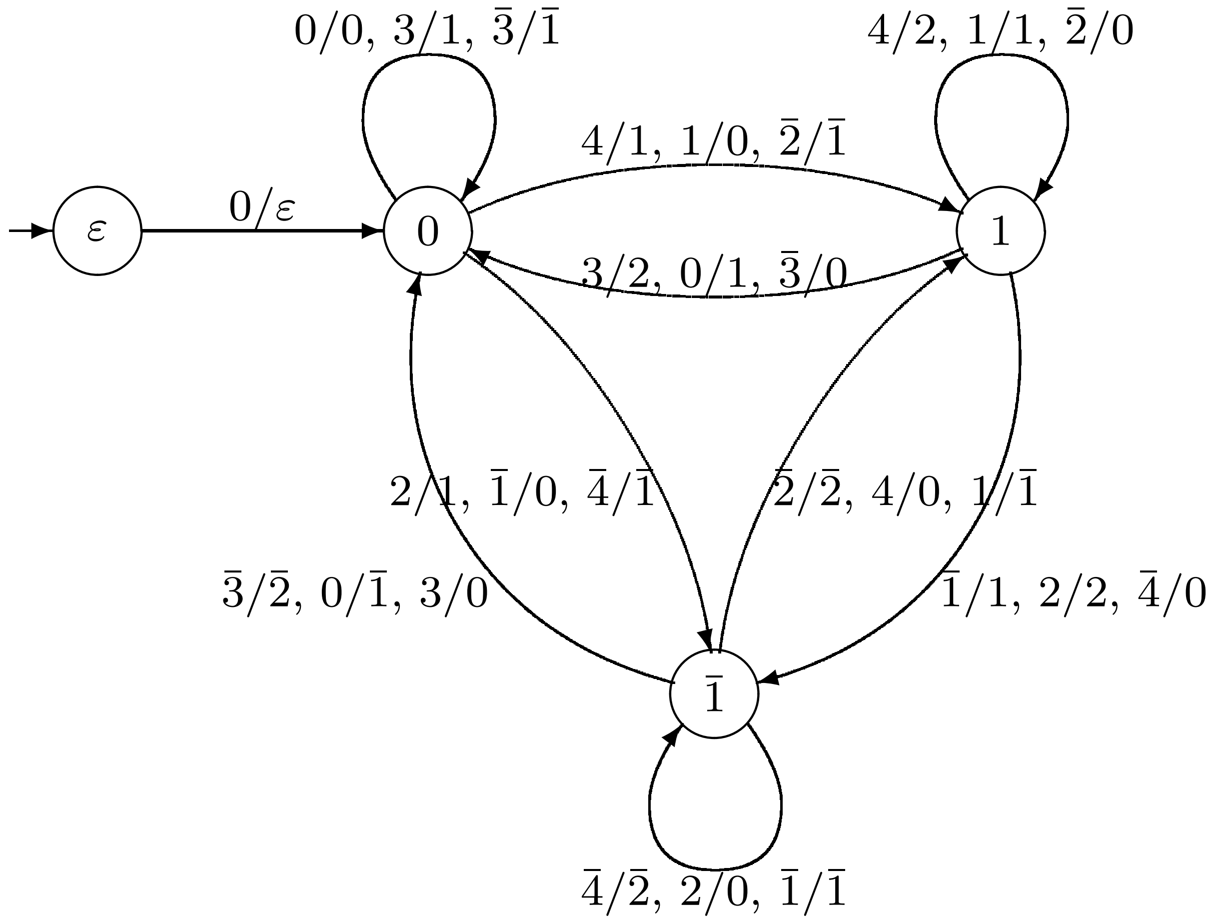
An on-line computable function is continuous. [J.-M. Muller]

On-line finite automata

On-line finite automaton = particular (sub)-sequential automaton:

- transient part: during a time δ (delay) the automaton reads without writing
- synchronous part: then the transitions are letter-to-letter

On-line finite automaton with delay 1 for addition base 3 on $B = \{\bar{2}, \ldots, 2\}$



$$p \xrightarrow{x/y} q \Leftrightarrow 3p + x = 3y + q$$

$$\omega(q) = q$$

Every affine function with rational coefficients is computable by an on-line finite automaton in integer base β on $B = \{\bar{a}, \ldots, a\}$, with $\beta/2 \leqslant a \leqslant \beta - 1$.

Conversely, let $D = \{\bar{d}, \ldots, d\}$ with $d \geqslant a$,
$I = [-a/(\beta - 1), a/(\beta - 1)]$,
$J = [-d/(\beta - 1), d/(\beta - 1)]$.

$$\begin{array}{ccc} D^{\mathbb{N}} & \xrightarrow{\chi} & B^{\mathbb{N}} \\ \pi_\beta \downarrow & & \downarrow \pi_\beta \\ J & \xrightarrow[\chi_{\mathbb{R}}]{} & I \end{array}$$

such that χ is computed by an on-line finite automaton. If $\chi_{\mathbb{R}}''$ is piecewise continuous, then, in each interval where $\chi_{\mathbb{R}}''$ is continuous, $\chi_{\mathbb{R}}$ is affine with rational coefficients. [J.-M. Muller]

On-line multiplication

[Trivedi and Ercegovac 1977]

Multiplication of two numbers represented in integer base $\beta > 1$ with digits in $S = \{-a, \dots, a\}$, $\beta/2 \leqslant a \leqslant \beta - 1$, is computable by an on-line algorithm with delay δ, where δ is the smallest positive integer such that

$$\frac{\beta}{2} + \frac{2a^2}{\beta^\delta(\beta - 1)} \leqslant a + \frac{1}{2}.$$

If $\beta = 2$ and $a = 1$, $\delta = 2$.
If $\beta = 3$ and $a = 2$, $\delta = 2$.
If $\beta = 2a \geqslant 4$ then $\delta = 2$.
If $\beta \geqslant 4$ and if $a \geqslant \lfloor \beta/2 \rfloor + 1$, $\delta = 1$.

Complex base, integer digits

Base $\beta = i\sqrt{b}$, with b integer $\geqslant 2$, digits $A = \{0, \ldots, b-1\}$ Knuth

Every complex number has a representation.

If $b = c^2$, every Gaussian integer has a unique representation $d_k \cdots d_0 \bullet d_{-1}$.

Example $\beta = 2i$, $A = \{0, \ldots, 3\}$, $z = 4 + i$ is represented by $10310 \bullet 2$.

On A addition in base $\beta = i\sqrt{b}$ is right subsequential.
On $B = \{\bar{a}, \ldots, a\}$ with $b/2 \leqslant a \leqslant b-1\}$, addition is computable in constant time in parallel, and realizable by an on-line finite automaton.

Multiplication is on-line computable. [Nielsen and Muller, Frougny, Surarerks]

Base $\beta = -b + i$, with b integer $\geqslant 1$, digits $A = \{0, \ldots, b^2\}$ [Case $b = 1$ Penney]

Every complex number has a representation.

Every Gaussian integer has a unique representation $d_k \cdots d_0 \in A^*$.

On A addition in base $\beta = -b + i$ is right subsequential. [Safer]

Case $\beta = -1 + i$, $A = \{0, 1\}$. $\beta^4 = -4$. On $B = \{\bar{a}, \cdots, a\}$, with $a = 1$, 2 or 3, addition in base $-1 + i$ is computable in constant time in parallel, and realizable by an on-line finite automaton. [Herreros, Nielsen and Muller, Frougny, Surarerks]

Real basis, integer digits

$U = (u_n)_{n \geqslant 0}$ a decreasing sequence of positive real numbers, summable.
A a finite alphabet of integers.
A real number x can be represented as

$$x = \sum_{n \geqslant 0} d_n u_n$$

with $d_n \in A$ under certain conditions by a greedy algorithm [Muller].

Example $u_n = \log(1 + 2^{-n})$, and $A = \{0, 1\}$.
Remark: if $x = \sum_{n \geqslant 0} d_n \log(1 + 2^{-n})$ then

$$e^x = \prod_{n \geqslant 0} \log(1 + 2^{-n})^{d_n}$$

Application CORDIC algorithms for computation of elementary functions.

Double base representation

Finite redundant representations of the form

$$N = \sum_{i,j} d_{i,j} 2^i 3^j$$

with $d_{i,j} \in \{0, 1\}$. [Dimitrov, Jullien and Miller]

Application Modular exponentiation, and digital signal processing.

Part II: Symbolic dynamics

Beta-numeration (Rényi, Parry)

$\beta > 1$, $x > 0$

Beta-expansion of x:

$\beta^k \leqslant x < \beta^{k+1}$

$x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$.

For $i < k$, let $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$.

$$\langle x \rangle_\beta = x_k x_{k-1} \dots x_1 x_0 \cdot x_{-1} x_{-2} \dots$$

$x_i \in A_\beta$ where $A_\beta = \{0, \dots, \beta - 1\}$ if β integer or $A_\beta = \{0, \dots, \lfloor \beta \rfloor\}$ if β is not an integer.

If the sequence (x_i) ends in 0^ω, it is said finite.

β-expansion of 1

Let $T_\beta(x) = \beta x \bmod 1$ and $d_\beta(1) = (t_i)_{i \geqslant 1}$, where $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$.

Redondancy A number may have several β-representations.

Example $\varphi = (1+\sqrt{5})/2$, $A_\varphi = \{0, 1\}$
$d_\varphi(1) = 11$.

$x = 3 - \sqrt{5}$, $\langle x \rangle_\varphi = 10010^\omega$.

The factor 11 is forbidden in $\langle x \rangle_\varphi$.

Other φ-representations de x
01110^ω, $100(01)^\omega$, $011(01)^\omega$, …

Beta-shift

σ shift on $A_\beta^{\mathbb{N}}$: $\sigma((x_i)_{i\geqslant 1}) = (x_{i+1})_{i\geqslant 1}$.

$D_\beta = \{\langle x\rangle_\beta \mid x \in [0,1[\,\}$ is a shift-invariant subset of $A_\beta^{\mathbb{N}}$.

β-shift S_β = topological closure of D_β.

Symbolic dynamical systems

$S \subseteq A^{\mathbb{N}}$ symbolic dynamical system = closed shift-invariant subset

$F(S)$ = set of finite factors of S
$X(S)$ set of minimal forbidden words.

S is of finite type if $X(S)$ is finite. Equivalent to $F(S)$ recognizable by a local finite automaton.

S is sofic if $X(S)$ is recognizable by a finite automaton. Equivalent to $F(S)$ recognizable by a finite automaton.

S is coded if there exists a prefix code $Y \subset A^*$ such that $F(S) = F(Y^*)$. Equivalent to $S = \overline{Y^\omega}$.

S_β is coded. [Blanchard and Hansel]

If $d_\beta(1) = (t_i)_{i \geqslant 1}$ is infinite,

$Y = \{t_1 \cdots t_{n-1} a \mid 0 \leqslant a < t_n,\ n \geqslant 1\}$

If $d_\beta(1) = t_1 \cdots t_m$,

$Y = \{t_1 \cdots t_{n-1} a \mid 0 \leqslant a < t_n,\ 1 \leqslant n \leqslant m\}$.

S_β is coded by Y.

Entropy

Topological entropy of S_β

$$h(S_\beta) = \lim_{n\to\infty} \frac{1}{n} \log B(n)$$

where $B(n)$ = number of words of S_β of length n.

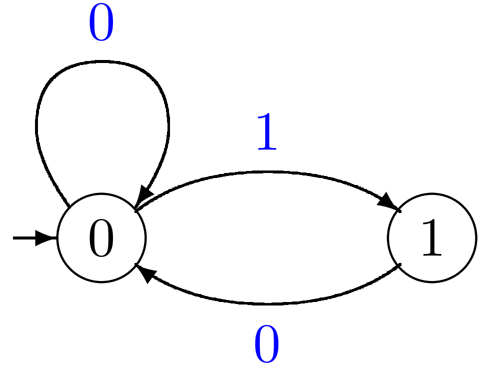
Entropy of β-shift S_β is $\log \beta$.

Characterisation

S_β is of finite type iff $d_\beta(1)$ is finite. [Ito and Takahashi]

Example $\varphi = (1+\sqrt{5})/2$, $d_\varphi(1) = 11$.

$\{11\}$ = minimal forbidden words.

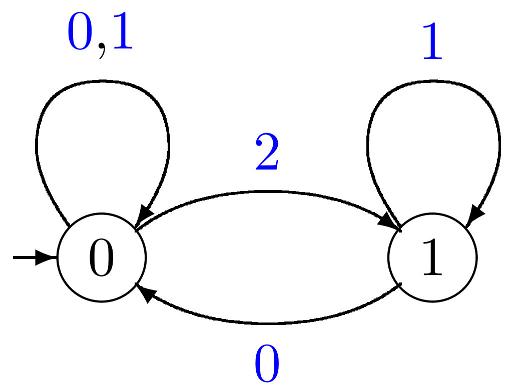


Local automaton for $F(S_\varphi)$

S_β is sofic iff $d_\beta(1)$ is finite. [Bertrand]

Example $\gamma = (3+\sqrt{5})/2$, then $d_\gamma(1) = 21^\omega$.

Minimal forbidden words $= 21^*2$.



Automaton for $F(S_\gamma)$

Numbers

Pisot number algebraic integer such that every conjugate is < 1 in modulus.
Salem number algebraic integer such that every conjugate is $\leqslant 1$ in modulus, and the equality is attained.
Perron number algebraic integer β such that every conjugate is $< \beta$ in modulus.

Example Integers, the golden ratio, $\gamma = (3 + \sqrt{5})/2$ are Pisot numbers.

If β is Pisot then $d_\beta(1)$ is eventually periodic and thus S_β is sofic. [A. Bertrand]

If $d_\beta(1)$ is eventually periodic, β is called a Parry number.
If $d_\beta(1)$ is finite, β is called a simple Parry number.

If S_β is sofic then β is Perron.

β is the dominant eigenvalue of the adjacency matrix of the finite automaton recognizing $F(S_\beta)$.

If β is Salem of degree 4 then $d_\beta(1)$ is eventually periodic. [D. Boyd]

Open problem for Salem of degree $\geqslant 6$.

Open problem: Characterize β such that the β-shift is sofic or of finite type.
Results for degree 3 [Bassino] and 4 [Akiyama and Njini].

Digit-set conversion and multiplication in base β are on-line computable.
If β is Pisot then digit-set conversion is computable by an on-line finite automaton.

U-representations

$U = (u_n)_{n \geqslant 0}$ a strictly increasing sequence of integers with $u_0 = 1$.

U-representation of $N \geqslant 0$ is a finite sequence of integers $(d_i)_{k \geqslant i \geqslant 0}$ such that $N = \sum_{i=0}^{k} d_i u_i$.

$(N)_U = d_k \cdots d_0$

Normal or greedy U-representation of N :
$q(m, p)$ and $r(m, p)$ = quotient and remainder of the Euclidean division of m by p.
$u_k \leqslant N < u_{k+1}$.
$d_k = q(N, u_k)$ and $r_k = r(N, u_k)$,
$d_i = q(r_{i+1}, u_i)$ and $r_i = r(r_{i+1}, u_i)$.
$N = d_k u_k + \cdots + d_0 u_0$. Normal U-representation of $N = \langle N \rangle_U = d_k \cdots d_0$

Example $U = \{1, 2, 3, 5, 8, \ldots\}$ set of Fibonacci numbers.

$G(U)$= set of greedy or normal U-representations of all the non-negative integers.

If U is linearly recurrent such that its characteristic polynomial is exactly the minimal polynomial of a Pisot number then $G(U)$ is recognizable by a finite automaton.

Everything works “well”.

U-recognizability

$S \subset \mathbb{N}$ is said to be U-recognizable if the set $\{< n >_U \mid n \in S\}$ is recognizable by a finite automaton.

Generalization of Cobham theorem

β and γ two multiplicatively independent Pisot numbers. U and Y two linear sequences with characteristic polynomial equal to the minimal polynomial of β and γ respectively. The only sets of integers that are both U-recognizable and Y-recognizable are unions of arithmetic progressions. (Bès)

Conversely

If β and γ are multiplicatively dependent, then a set which is U-recognizable is Y-recognizable. (Frougny)

Other generalizations

Abstract numeration system associated with a rational language L: n is represented by the $(n+1)$-th word of the ordered language L.
Extension for the representation of real numbers.
[Lecomte, Rigo]

Multi-dimensional numeration systems:
application to coding by finite type constraints [Frougny, Vuillon].
Open problem: computation of the entropy.

Representation in rational base: [Akiyama, Frougny, Sakarovitch].

Related domains

Substitutions: Berthé, Siegel, Canterini, Arnoux, Ito, Durand, Dumont

Dynamical properties: Liardet, Barat, Tichy, Grabner, Sidorov

Tilings: Akiyama

Logic: Bruyère, Hansel, Bès

Part III: Quasicrystals

Crystals and quasicrystals

Crystals: solids in dimension 2 or 3, with atoms arranged periodically.
Symmetry of order n.
n must satisfy

$$\rho = 2\cos\frac{2\pi}{n} \in \mathbb{Z}$$

hence $n = 1, 2, 3, 4, 6$.

Quasicrystal Alloy aluminium-manganese with order 5 symmetry Shechtman and al. 1984
Quasi-periodicity.

Geometrical modelization

$\Lambda \subset \mathbb{R}^d$ is uniformly discrete if exists $r > 0$ such that every ball of radius r contains at most a point of Λ.
Λ is relatively dense if exists $R > 0$ such that every ball of radius R contains at least a point of Λ.

If both conditions are satisfied, Λ is said to be a Delaunay set.

Model set (Y. Meyer 1970, 1972)

Cut and projection scheme

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times G & \xrightarrow{\pi_2} & G \\ & & \cap & & \\ & & D & & \end{array}$$

G loc. compact abelian group (internal space)
$\mathbb{R}^d$ physical space
D lattice *i.e.* discret sub-group of $\mathbb{R}^d \times G$ such that $(\mathbb{R}^d \times G)/D$ is compact
$\pi_1|_D$ 1-to-1
$\pi_2(D)$ dense in G.
$\Lambda \subset \mathbb{R}^d$ is a model set if there exist a cut and projection scheme and a relatively compact set $\Omega \subset G$ of non-empty interior such that

$$\Lambda = \{\pi_1(x,g) \mid (x,g) \in D,\ \pi_2(x,g) \in \Omega\}$$

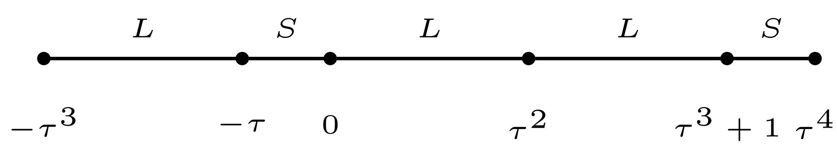
Example The Fibonacci chain: $\tau = \frac{1+\sqrt{5}}{2}$.

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R}\times\mathbb{R} & \xrightarrow{\pi_2} & \mathbb{R} \\ & & \cap & & \\ & & \mathbb{Z}^2 & & \end{array}$$

$\pi_1|_{\mathbb{Z}^2} \sim \mathbb{Z}[\tau] = \{a + b\tau \mid a, b \in \mathbb{Z}\}$

Fibonacci chain

$$\begin{aligned} \mathcal{F} &= \{x = a + b\tau \mid x' = a - \frac{b}{\tau} \in \Omega = [0,1)\} \\ &= \{\ldots, -\tau^3, -\tau, 0, \tau^2, \tau^3 + 1, \tau^4, \ldots\} \end{aligned}$$



Tiling of $\mathbb{R}$ with 2 tiles L and S

$$\begin{aligned} L &\mapsto LLS \\ S &\mapsto LS \end{aligned}$$

with $|L| = \tau^2$ and $|S| = \tau$

$$\begin{array}{rcl} S & | & L \\ LS & | & LLS \\ LLSLS & | & LLSLLSLS \end{array}$$

Meyer set

$\Lambda \subset \mathbb{R}^d$ is a Meyer set if it is Delaunay and if there exists a finite set F such that

$$\Lambda - \Lambda \subset \Lambda + F$$

Model set $\Rightarrow$ Meyer set.
Conversely if Λ is a Meyer set, there exist a finite set F and a model set M_0 such that $\Lambda \subset M_0 + F$.

If $\Lambda \subset \mathbb{R}^d$ is a Meyer set and if $\beta > 1$ is a real number such that $\beta\Lambda \subset \Lambda$ then β is a Pisot or a Salem number.

Conversely for each d and for each Pisot or Salem number β, there exists a Meyer set $\Lambda \subset \mathbb{R}^d$ such that $\beta\Lambda \subset \Lambda$.

Beta-integers

The set of beta-integers

$$\begin{aligned} \mathbb{Z}_\beta &= \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_0\} \\ &= \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \end{aligned}$$

Then

$$\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta \ , \ \mathbb{Z}_\beta = -\mathbb{Z}_\beta$$

If β is a Pisot number then $\mathbb{Z}_\beta$ is a Meyer set.
[Burdìk, Frougny, Gazeau].

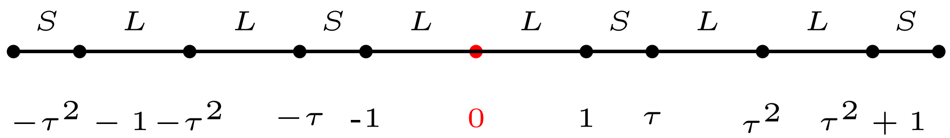
Open problem Characterize the finite set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$, see Bassino and Frougny, Ambrož, Bernat, Masakovà and Pelantovà.

Example Fibonacci

$$\begin{aligned}\mathbb{Z}_\tau &= \mathbb{Z}_\tau^+ \cup (-\mathbb{Z}_\tau^+) \\ &= \{0,\ 1,\ \tau,\ \tau^2,\ \tau^2+1,\ \ldots\} \\ &\cup\ \{-1,\ -\tau,\ -\tau^2,\ -\tau^2-1,\ \ldots\}\end{aligned}$$

generated by the Fibonacci substitution

$$\begin{aligned} L &\mapsto LS \\ S &\mapsto L \end{aligned}$$



$\mathbb{Z}_\tau$ is a Meyer set which is not a model set.

Cyclotomic Pisot numbers

If n is not crystallographic, $\rho = 2\cos\frac{\pi}{n}$ is an algebraic integer of degree $\leqslant \lfloor n-1 \rfloor/2$.

Cyclotomic Pisot number β such that

$$\mathbb{Z}[\rho] = \mathbb{Z}[\beta]$$

$\mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$ is a ring invariant under rotation of order n, with $\zeta = \exp(2i\pi/n)$.

Real quasicrystals

- $n = 5$ or $n = 10$: $\beta = \rho = \frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5}$,
$M_\beta(X) = X^2 - X - 1$
- $n = 8$: $\beta = 1 + \rho = 1 + \sqrt{2} = 1 + 2\cos\frac{\pi}{4}$,
$M_\beta(X) = X^2 - 2X - 1$
- $n = 12$: $\beta = 2 + \rho = 2 + \sqrt{3} = 2 + 2\cos\frac{\pi}{6}$,
$M_\beta(X) = X^2 - 4X + 1$.

Quadratic Pisot units

Other cyclotomic Pisot units.

• $n = 7$ or $n = 14$: $\beta = 1 + \rho = 1 + 2\cos\frac{\pi}{7}$,
$M_\beta(X) = X^3 - 2X^2 - X + 1$

• $n = 9$ or $n = 18$: $\beta = 1 + \rho = 1 + 2\cos\frac{\pi}{9}$,
$M_\beta(X) = X^3 - 3X^2 + 1$.

Complete classification of cyclotomic Pisot numbers of degree $\leqslant 4$ given by Bell and Hare. See also D. Boyd for higher degree.

Beta-lattices

β is a cyclotomic Pisot number with order n symmetry, and $\zeta = \exp(2i\pi/n)$.
For $1 \leqslant q \leqslant n-1$, β-lattice

$$\Gamma_q = \mathbb{Z}_\beta + \mathbb{Z}_\beta \zeta^q$$

Beta-lattices are based on beta-integers as lattices are based on integers.

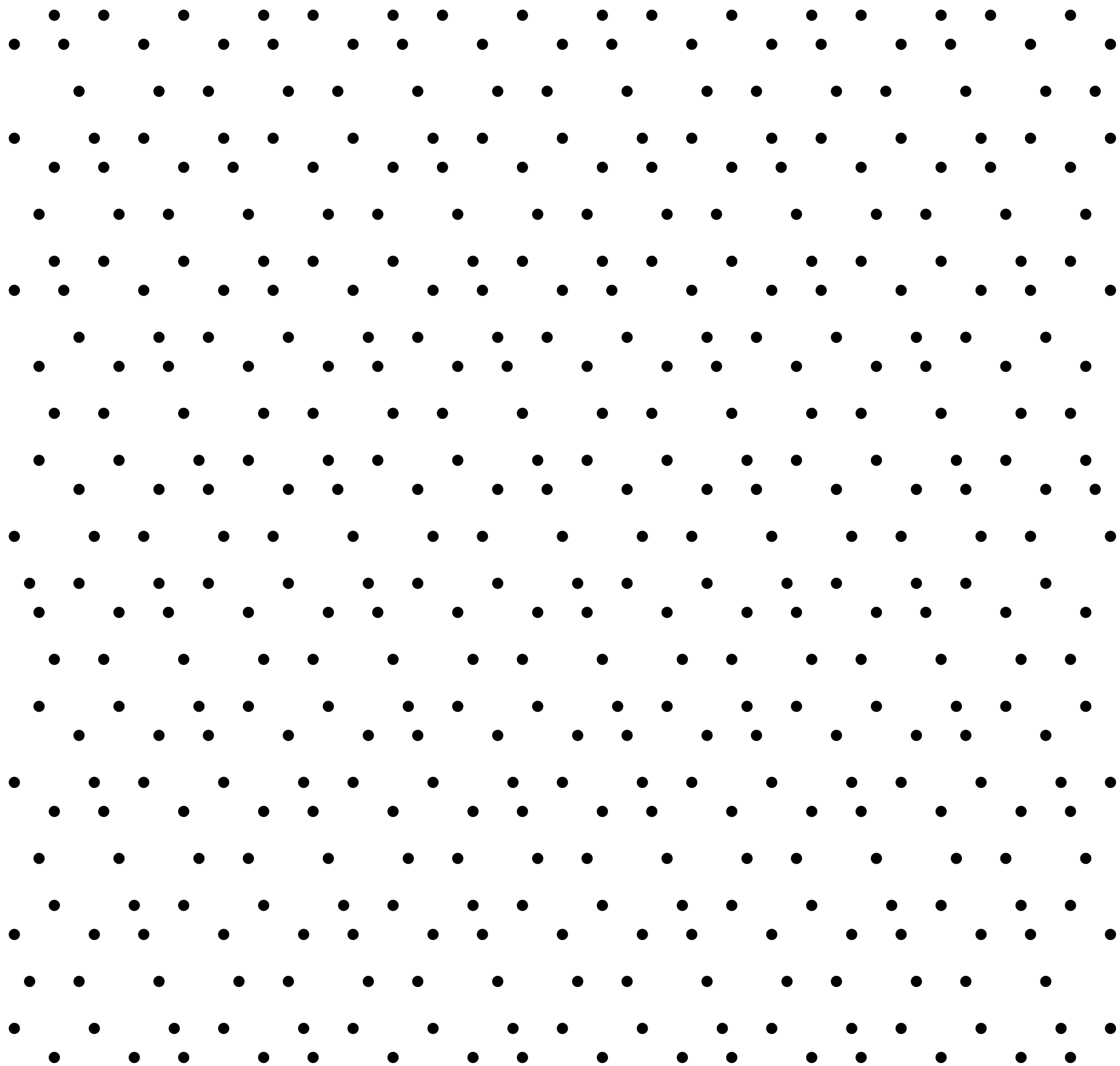
Beta-lattices = good frames for quasiperiodic point-sets and tilings.

[Elkharrat, Gazeau, Frougny, Verger-Gaugry]

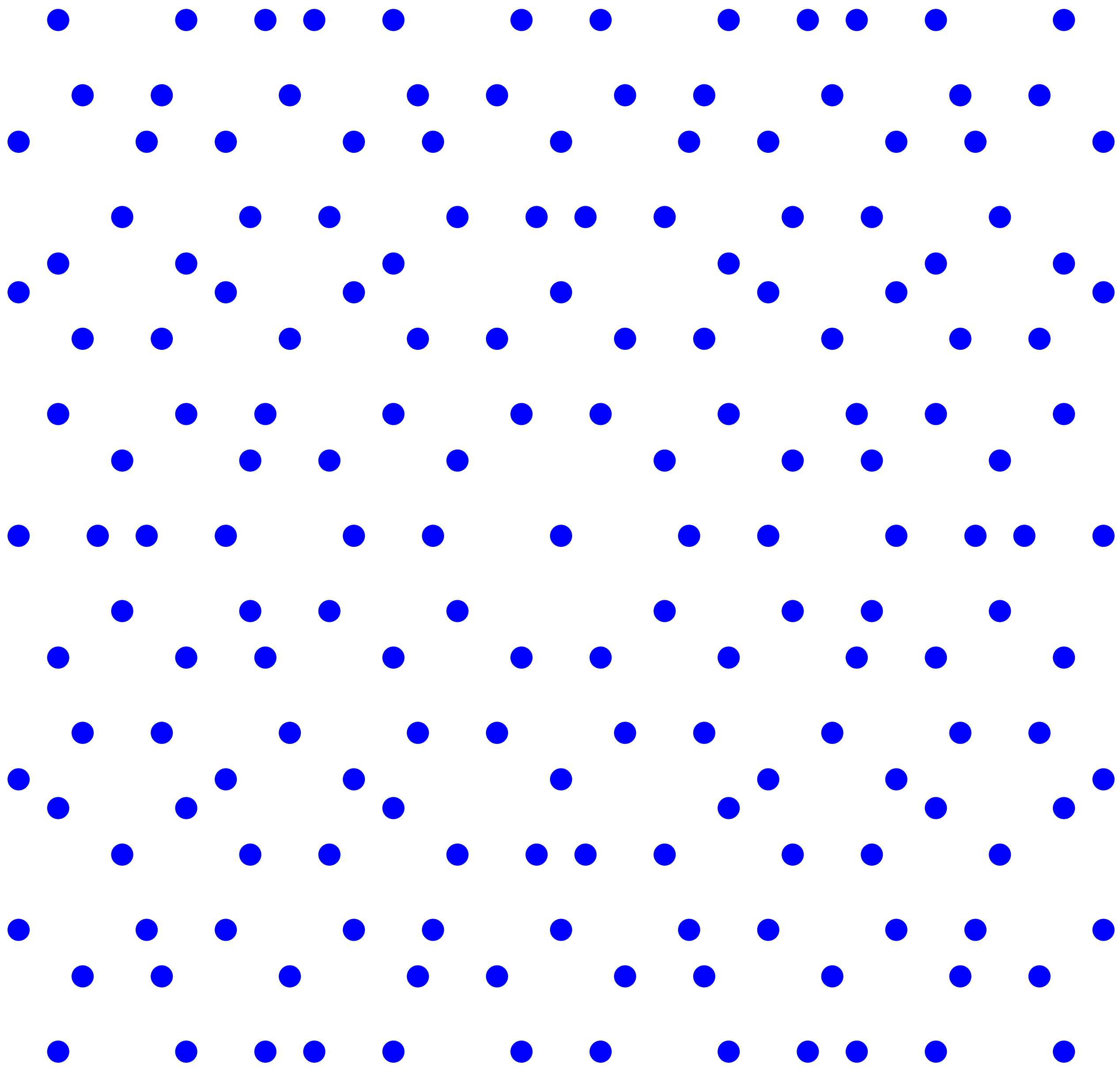
$$\Lambda_q = \bigcup_{j=0}^{n-1} \Gamma_q \zeta^j \text{ and } \mathbb{Z}_\beta[\zeta] = \sum_{j=0}^{n-1} \mathbb{Z}_\beta \zeta^j$$

$\mathbb{Z}_\beta[\zeta]$ and Λ_q for $1 \leqslant q \leqslant n-1$ are Meyer sets.

The τ-lattice $\Gamma_1(\tau) = \mathbb{Z}_\tau + e^{\frac{i\pi}{5}}\mathbb{Z}_\tau$



2D decagonal model set



Embedding of the 2D decagonal model set into $\Gamma_1(\tau)$