$A = \{0,1\}$

Mots Sturmiens

Proposition Pour $x \in A^{\mathbb{N}}$, les propositions suivantes sont équivalentes :

① x ult. périodique, i.e. $x = uv^{\mathbb{N}}$ pour $u,v \in A^*$

② $\exists K \geq 0 \quad \forall n \quad \#(F(x) \cap A^n) \leq K$

③ $\exists n \quad \#(F(x) \cap A^n) \leq n$

$1 \Rightarrow 2 \quad \#(F(uv^{\mathbb{N}}) \cap A^n) \leq |u| + |v|$

$2 \Rightarrow 3$ ok

$3 \Rightarrow 2 \quad \#(F(x) \cap A) = 2$ parce que $A = \{0,1\}$

donc $\exists k \; \#(F(x) \cap A^{k-1}) = \#(F(x) \cap A^k)$

$\rightarrow$ prolongement unique.

Definition $L \subseteq A^*$ L balanced

Si $\forall u,v \in A^*$

$$\left.\begin{array}{l} u,v \in L \\ |u| = |v| \end{array}\right\} \Rightarrow \big||u|_1 - |v|_1\big| \leq 1$$

Proposition $F \subseteq A^*$ Factorial, balanced, prolongeable à droite

alors $\#(F \cap A^n) \leq n+1 \quad \forall n \geq 1$

Preuve $n=1 \quad \#(F \cap A) \leq \# A = 2 \leq 2$

$n=2$ On ne peut pas avoir $00, 11 \in F$

donc $\#(F \cap A^2) \leq 3 \leq 2+1$

~~Proposition~~ Soit n le plus petit entier tel que $\#(F \cap A^n) \geq n+2$

et $\#(F \cap A^{n-1}) \leq n$

Il existe deux mot y,z tels que $|y| = |z| = n-1$

et $y0, y1, z0, z1 \in F$

Soit w le plus long suffixe commun de y,z

donc $0w$ et $1w$ sont suffixes de y ou z

$0w0$ et $1w1$ | facteurs de $y0$ ou $z0$ $\rightarrow$ F non balanced

| suffixe $y1$ ou $z1$

<u>Propositions</u> $F \subseteq A^*$ factoriel,

Si F non balanced, alors il existe un palindrome w tel que $0w0$ et $1w1 \in F$

Preuve On choisit u, v de longueur minimale tels que $|u| = |v|$ et $\big| |u|_1 - |v|_1 \big| \geq 2$.

Comme u, v de longueur minimale,
- u et v ne commencent pas par la même lettre
- u et v ne se terminent pas — — —

Suppose $u = 0u'1$
$v = 1v'0$

$$\big| |u'|_1 - |v'|_1 \big| = \big| |u|_1 - |v|_1 \big| \geq 2$$

contradiction avec la longueur minimale de u et v

Donc $u = 0u'0$
$v = 1v'1$

On montre que $u' = v'$ et $u' = v'$ palindrome

Soit w plus long préfixe commun de u' et v'

$u = 0wa\cancel{b}u''0$ avec $a \neq b$
$v = 1w\cancel{b}v''1$

Si $a = 1$ et $b = 0$ $\big| |u''0|_1 - |v''1|_1 \big| = \big| |u|_1 - |v|_1 \big| \geq 2$
$\rightarrow$ contradiction

donc $a = 0$ et $b = 1$ Par minimalité de u, v

on a $u = 0w0$ et $v = 1w1$

Montrons que w palindrome.

Soit x plus long préfixe de w tel que $\tilde{x}$ suffixe de w

On $|x| < |w|/2$ et $w = xaw'b\tilde{x}$ avec $a \neq b$

$u = 0xaw'b\tilde{x}0$ $a = 0$ contradiction avec $0x0$ et $1\tilde{x}1$

$v = 1xaw'b\tilde{x}1$ $a = 1$ — $1x1$ et $0\tilde{x}0$

Théorem $x \in A^{\mathbb{N}}$, the following conditions are equivalent.

① x is Sturmian

② $F(x)$ is balanced and x not ult. periodic

Proof $2 \Rightarrow 1$

$F(x)$ balanced $\rightarrow$ $\#(F(x) \cap A^n) \leq n+1$

x not ult. periodic $\rightarrow$ $\#(F(x) \cap A^n) \geq n+1$

$\Big\} = n+1$

$1 \Rightarrow 2$ Suppose that $F(x)$ not balanced.

There exists a palindrome w such that $0w0$ and $1w1 \in F(x)$.

w is a right special factor

Therefor the right special factor of length $|w|+1 = n$ is either $0w$ or $1w$.

Suppose that it is $0w$. $0w0$ and $0w1 \in F(x)$

$1w1 \in F(x)$ $1w0 \notin F(x)$

Let v such that $|v| = n-1$ $1w1v \in F(x)$

Claim. all factors of length n+1 of $1w1v$ are different from $0w$.

1	w		1	v	
1	s	0	t	1	·

In w after $|t|$ symbols there is 1

But since $\tilde{t}0$ prefix of w, there is also a 0

$\Big\}$ contradiction

In $1w1v$, there are $|1w1v| - (n+1)+1$ factors of length n+1

n+1 of length n

Since $0w$ does not occur, the same factor occur twice

$\rightarrow$ ult. periodic

Slope

$A = \{0,1\}$

Proposition $F \subseteq A^*$

F balanced iff $\forall u,v \in A^+$ $\left| \frac{|u|_1}{|u|} - \frac{|v|_1}{|v|} \right| < \frac{1}{|u|} + \frac{1}{|v|}$

Proof

($\Leftarrow$) Apply inequality for $|u| = |v|$ to obtain $||u|_1 - |v|_1| \leq 1$

($\Rightarrow$) F balanced If $|u| = |v|$

$||u|_1 - |v|_1| \leq 1$ and thus $\left| \frac{|u|_1}{|u|} - \frac{|v|_1}{|v|} \right| \leq \frac{1}{|u|} < \frac{2}{|u|}$

If $|u| > |v|$ $u = u'w$ with $|u'| = |v|$

$$\begin{aligned}
\frac{|u|_1}{|u|} - \frac{|v|_1}{|v|} &= \frac{|u'|_1}{|u|} + \frac{|w|_1}{|u|} - \frac{|v|_1}{|v|} \\
&= \frac{|u'|}{|u|}\frac{|u'|_1}{|u'|} + \frac{|w|}{|u|}\frac{|w|_1}{|w|} - \frac{|v|_1}{|v|} \\
&= \frac{|u'|}{|u|}\left(\frac{|u'|_1}{|u'|} - \frac{|v|_1}{|v|}\right) + \frac{|w|}{|u|}\left(\frac{|w|_1}{|w|} - \frac{|v|_1}{|v|}\right) \\
\left| \quad \right| &\leq \frac{|u'|}{|u|}\frac{1}{|u'|} + \frac{|w|}{|u|}\left(\frac{1}{|w|} + \frac{1}{|v|}\right) \\
&\not\leq \frac{1}{|u|} + \frac{1}{|v|}
\end{aligned}$$

Corollary $x \in A^{\mathbb{N}}$ $F(x)$ Balanced

$$\lim_{n \to +\infty} \frac{|x[1:n]|_1}{n} = \alpha \quad (\text{Slope of } x)$$

$\hookrightarrow$ Cauchy sequence

Corollary $x \in A^{\mathbb{N}}$ $F(x)$ balanced

$$\left| \frac{|u|_1}{|u|} - \alpha \right| \leq \frac{1}{|u|} \quad \text{For } u \in F(x)$$

Proof $\left| \frac{|u|_1}{|u|} - \frac{|v|_1}{|v|} \right| < \frac{1}{|u|} + \frac{1}{|v|}$ Make $|v| \to \infty$

$\lim \frac{|v|_1}{|v|} = \alpha$

Corollary $x \in A^{\mathbb{N}}$ $F(x)$ balance α slope of x

$$\alpha|u| - 1 < |u|_1 \leq \alpha|u| + 1 \quad \text{for all } u \in F(x)$$

$$\alpha|u| - 1 \leq |u|_1 < \alpha|u| + 1 \quad \text{———} \quad \text{——}$$

Proof From $\left|\frac{|u|_1}{|u|} - \alpha\right| \leq \frac{1}{|u|}$ follows

$$\alpha|u| - 1 \leq |u|_1 \leq \alpha|u| + 1$$

Suppose
$$|u|_1 = \alpha|u| + 1$$
$$|v|_1 = \alpha|v| - 1$$

$$\left|\frac{|u|_1}{|u|} - \frac{|v|_1}{|v|}\right| = \frac{1}{|u|} + \frac{1}{|v|} \rightarrow \text{contradiction}$$

Proposition $x \in A^{\mathbb{N}}$ $F(x)$ balanced

x ult. periodic $\Leftrightarrow$ α rational ($\in \mathbb{Q}$)

($\Rightarrow$)
$$x = uv^{\mathbb{N}} \qquad \alpha = \frac{|v|_1}{|v|} \in \mathbb{Q}$$

($\Leftarrow$) Suppose $\alpha = \frac{p}{q}$ with $p \wedge q = 1$

Suppose $\alpha|u| - 1 < |u|_1 \leq \alpha|u| + 1 \quad \forall u \in F(x)$

If $|u| = q$ $\qquad p \leq |u|_1 \leq p + 1$

then $|u|_1 = p+1$ or $|u|_1 = p$

Claim Factors u such that $|u|_1 = p+1$ and $|u| = q$ have finitely many occurrences. Otherwise there exists a factor uvu'

where $|u| = |u'| = q$ $\quad |u|_1 = |u'|_1 = p+1$

$$|uvu'|_1 = |v| + 2p + 2 \leq \alpha|uvu'| + 1 = \alpha|v| + 2p + 1$$

$$\rightarrow \quad |v| \leq \alpha|v| - 1 \rightarrow \text{contradiction}$$

All factors, after some pos have the same # of 1

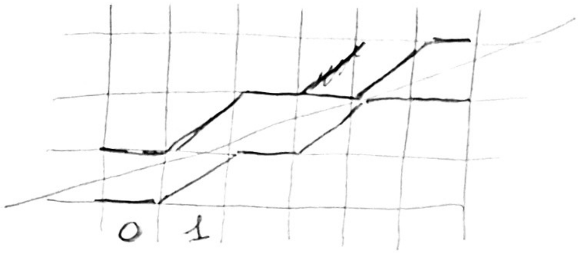
$|a|$ $\quad |b|$ $\quad \rightarrow a = b$

Mechanical words

Given α and ρ with $0 \leq \alpha \leq 1$, define the two infinite words $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ by

$$s_{\alpha,\rho}[n] = \lfloor \alpha(n+1)+\rho \rfloor - \lfloor \alpha n + \rho \rfloor$$

$$s'_{\alpha\rho}[n] = \lceil \alpha(n+1)+\rho \rceil - \lceil \alpha n + \rho \rceil$$



Theorem $x \in A^{\mathbb{N}}$. The following conditions are equivalent

1. x Sturmian
2. x balanced ($F(x)$ balanced) and aperiodic
3. x is irrationnal mechanical.

Key formula: $x'-x-1 < \lfloor x' \rfloor - \lfloor x \rfloor < x'-x+1$

<u>Lemma</u> s mechanical with slope α
then s balanced with slope α

Proof

Suppose $s = s_{\alpha,\rho}$

Let u factor of s

$u = s[n]\, s[n+1] \qquad s[n+p-1]$ of length p

$|u|_1 = \lfloor \alpha(n+p)+\rho \rfloor - \lfloor \alpha n + \rho \rfloor$

Thus $\alpha|u| - 1 < |u|_1 < \alpha|u| + 1$

$\lfloor \alpha|u| \rfloor \leqslant |u|_1 \leqslant \lfloor \alpha|u| \rfloor + 1$

↳ Two possible values.

→ s balanced $\left| \frac{|u|_1}{|u|} - \alpha \right| < \frac{1}{|u|}$

→ the slope is α.

If α irrational $s_{\alpha,\rho}$ aperiodic

If $\alpha = \frac{p}{q}$ $\quad s(n+q) = s(n)$

<u>Lemma</u> If s balanced, aperiodic then s is mechanical irrational.

<u>Proof</u>

Let $h_n = |s[0:n-1]|_1$

<u>claim</u>: for each $\tau \in \mathbb{R}$

either $h_n \leqslant \lfloor \alpha n + \tau \rfloor \quad \forall n \geqslant 0$

or $h_n \geqslant \lfloor \alpha n + \tau \rfloor \quad \forall n \geqslant 0$.

Suppose $h_n < \lfloor \alpha n + \tau \rfloor$ and $h_{n+k} >$ $\lfloor \alpha(n+k)+\tau \rfloor$

$h_{n+k} - h_n \geqslant 2 + \lfloor \alpha(n+k)+\tau \rfloor - \lfloor \alpha n + \tau \rfloor \geqslant \alpha k + 1$

→ contradiction $\left| \frac{|u|_1}{|u|} - \alpha \right| \leqslant \frac{1}{|u|}$

Set $\rho = \inf \{ \tau : h_n \leqslant \lfloor \alpha n + \tau \rfloor$ for all $n \}$

$\forall n \qquad h_n \leqslant \alpha n + \rho \leqslant h_n + 1$