Decidability of some classes of rational relations

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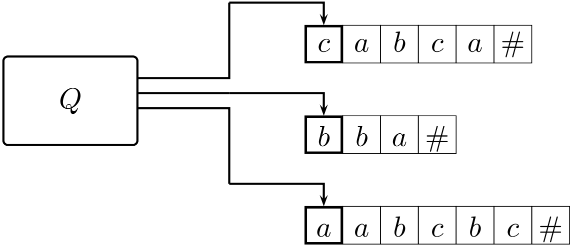
Szeged 2006

n-tapes automata (as Turing machines)

Run on the input $(cabca, bba, aabcbc) \in A_1^* \times A_2^* \times A_3^*$



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- read-only heads
- No backwards move of the heads
- stop on the end-markers

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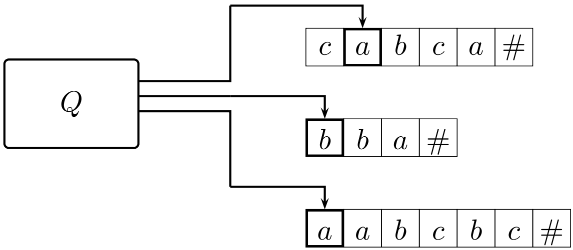

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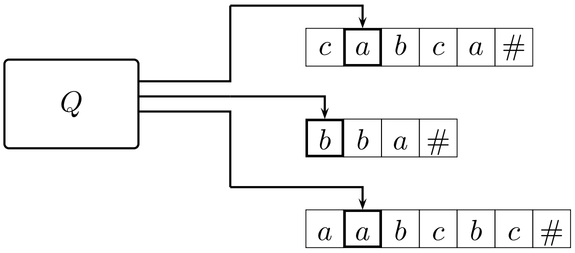

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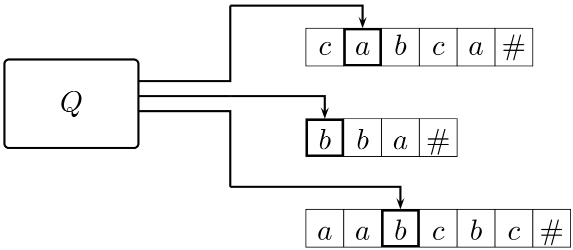

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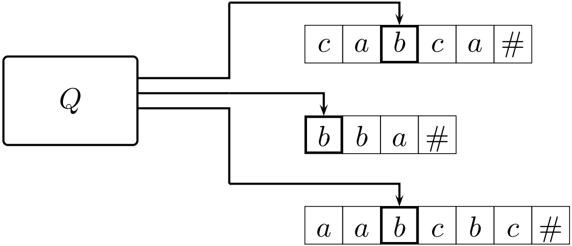

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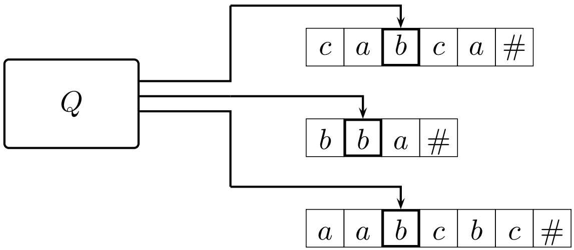

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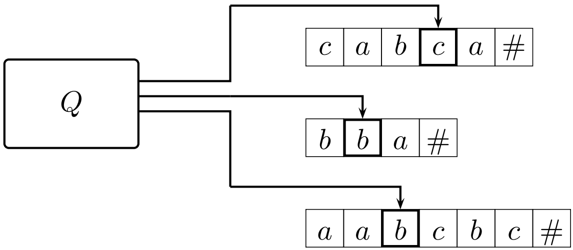

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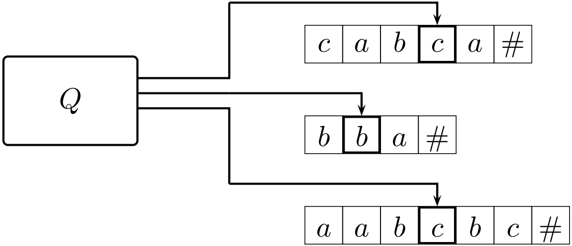

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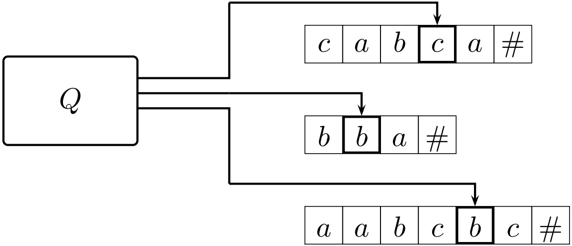

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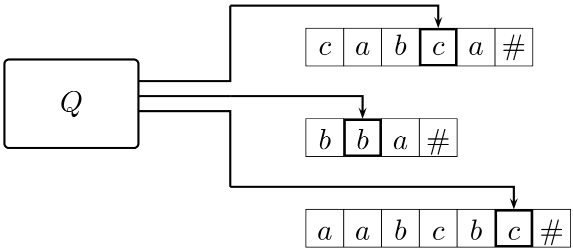

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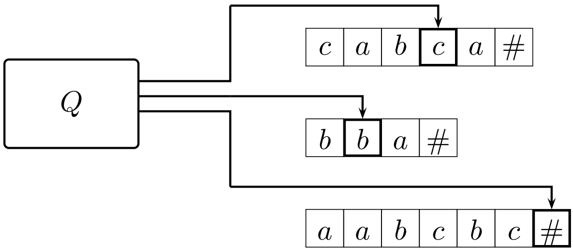

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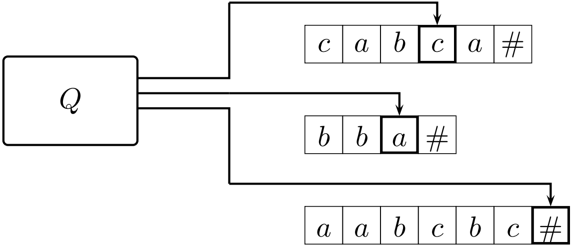

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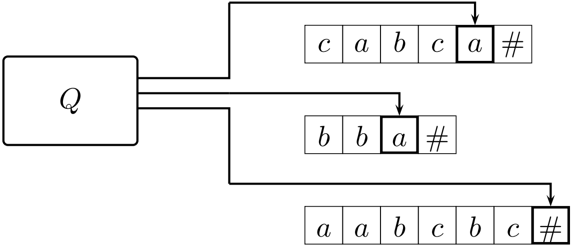

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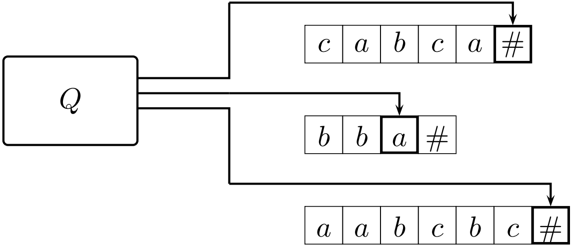

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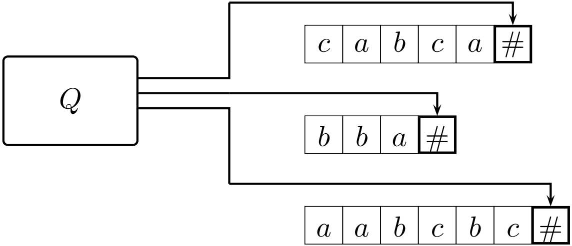

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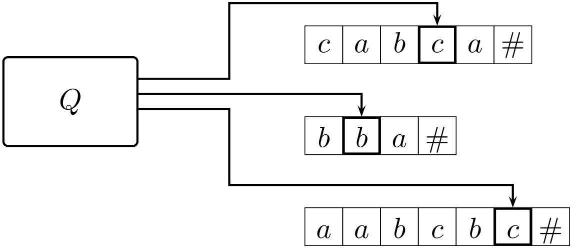
Deterministic n-tapes automata



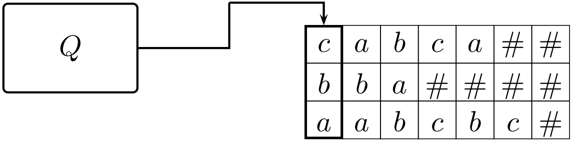
- deterministic control
- end-markers (as automata)

Examples

- $\{(a^n, b^n) \mid n \geq 0\}$ et $\{(a^n, b^{2n}) \mid n \geq 0\}$ are deterministic
- $\{(a^n, b^n) \mid n \geq 0\} \cup \{(a^n, b^{2n}) \mid n \geq 0\}$ is not deterministic

The set of relations accepted by deterministic automata is denoted $\mathrm{DRat}(M)$.

Synchronous n-tapes automata



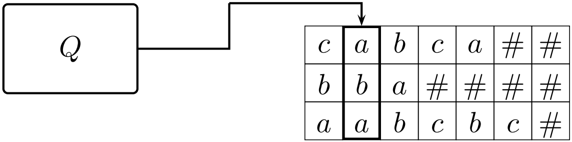
- Synchronous moves of the heads
- padding of the shorter words

Examples

- $\{(a^n, b^n) \mid n \geq 0\}$ et $\{(u, v) \mid u \text{ prefix of } v\}$ are synchronous
- $\{(a^n, b^{2n}) \mid n \geq 0\}$ is not synchronous
- $((a, ab) + (b, b))^*$ is not synchronous

The set of relations accepted by synchronous automata is denoted $\mathrm{Sync}(M)$.

Synchronous n-tapes automata



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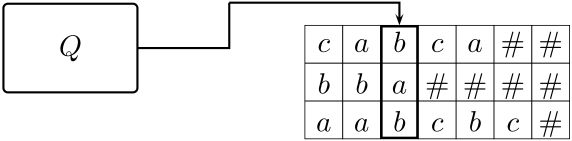

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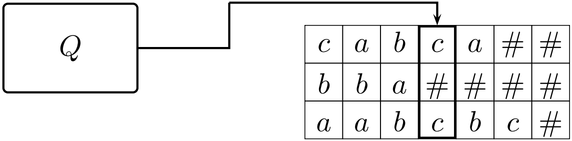

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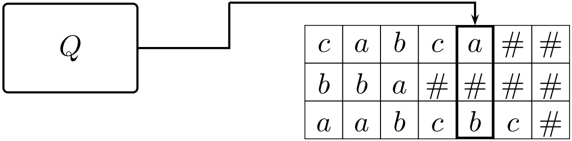

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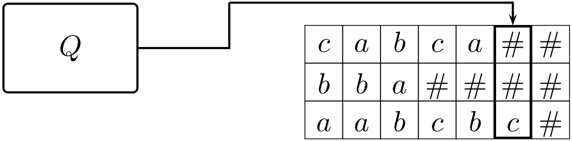
Synchronous n-tapes automata



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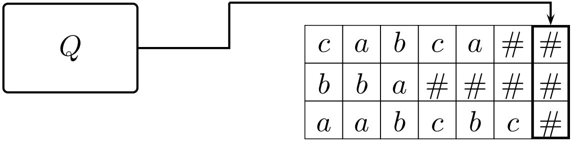

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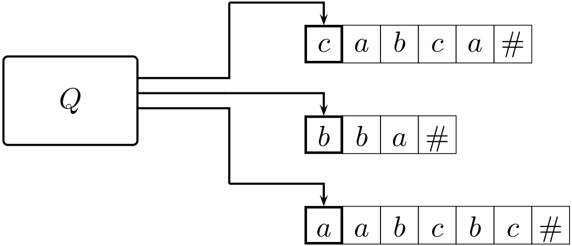

- sequential moves of the heads

Examples

- $R = (aA^* \times A^*a) \cup (bA^* \times A^*b)$ is recognizable
- $R = \bigcup_{i=1}^{n} K_i \times L_i$ is recognizable if the sets K_i and L_i are rational.
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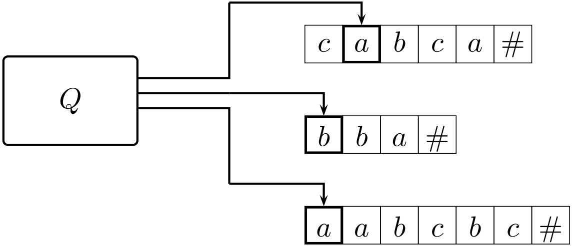
Asynchronous n-tapes automata



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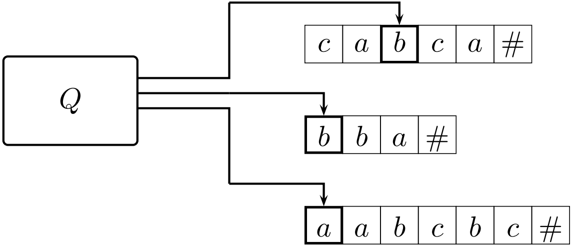

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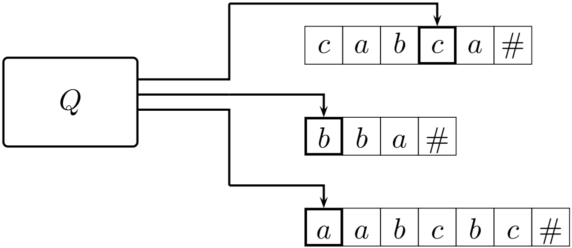

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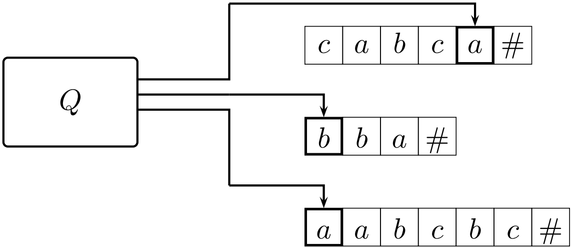

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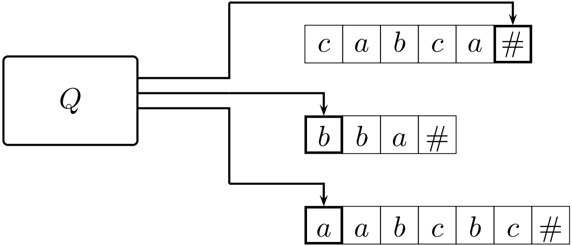

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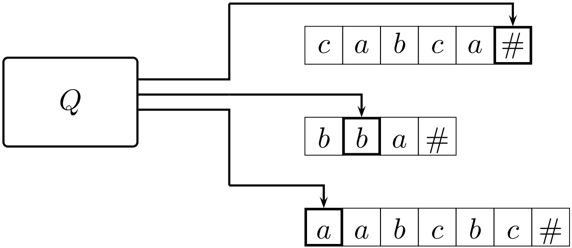

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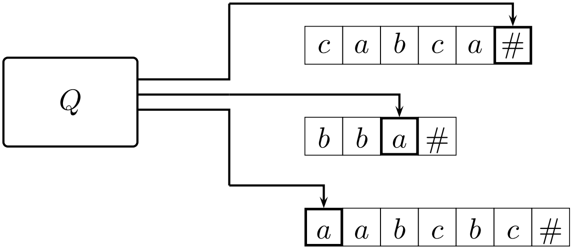

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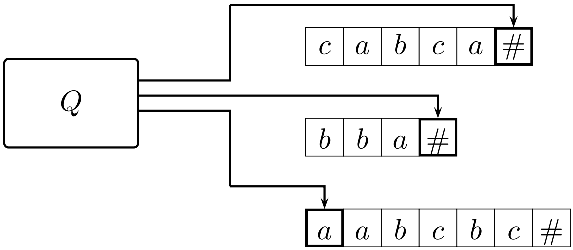

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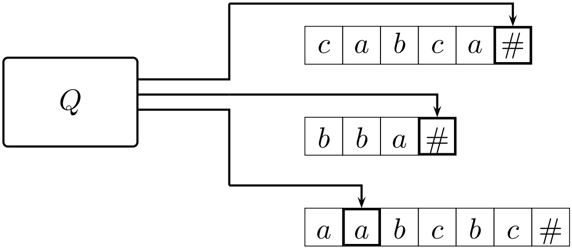

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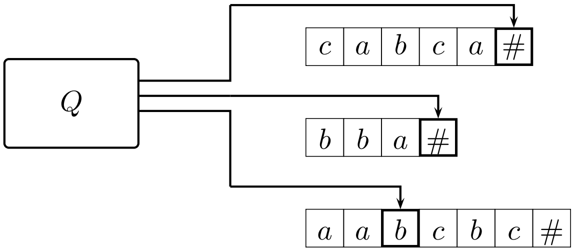

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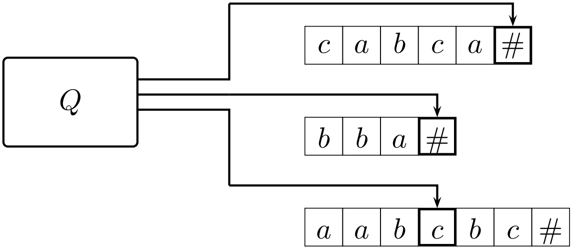

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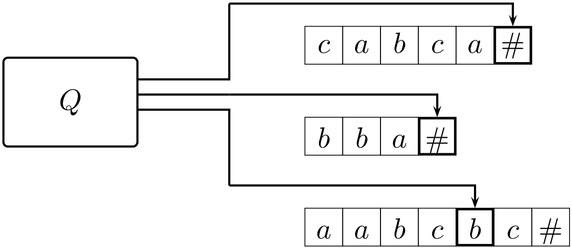

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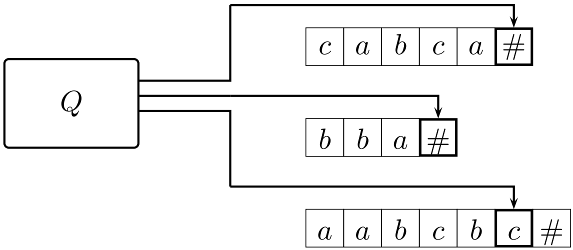

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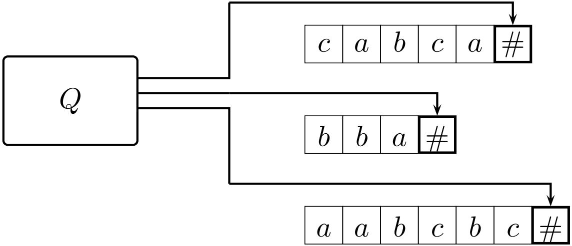

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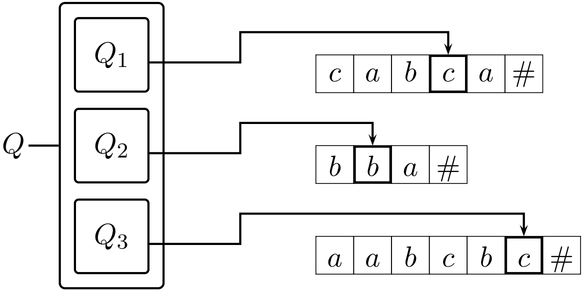
Asynchronous n-tapes automata (other view)



- one control state for each tape : $Q = Q_1 \times \cdots \times Q_n$

The set of relations accepted by asynchronous automata is denoted Rec(M).

The strict hierarchy

$$M = A_1^* \times \cdots \times A_n^*$$

$$\begin{array}{ccccccc} \mathrm{Rec}(M) & \subset & \mathrm{Sync}(M) & \subset & \mathrm{DRat}(M) & \subset & \mathrm{Rat}(M) \\ \mathcal{F}_0 & & \mathcal{F}_1 & & \mathcal{F}_2 & & \mathcal{F}_3 \end{array}$$

INCLUSION-I-IN-J
Input: $R \in \mathcal{F}_j$
Output: $R \in \mathcal{F}_i$?

The different cases

To avoid trivial cases, it is assumed that

- $n \geq 2$
- $|A_i| \geq 1$ for each $1 \leq i \leq n$.

There are two distinct main cases.

- $|A_1| = \cdots = |A_n| = 1$: all alphabets are of size 1.
  The monoid $M = A_1^* \times \cdots \times A_n^*$ is commutative : $M \approx \mathbb{N}^n$.
- $|A_1| \geq 2$: M is not commutative
  - $|A_2| = \cdots = |A_n| = 1$: exactly one the alphabets is of size greater or equal to 2.
  - $|A_2| \geq 2$: at least two of the alphabets are of size greater or equal to 2.

Known results (non commutative case)

	$\mathrm{Rat}(M)$	$\mathrm{DRat}(M)$	$\mathrm{Sync}(M)$
$\mathrm{DRat}(M)$	undecidable Fischer, Rosenberg 1967 Lisovik 1979		
$\mathrm{Sync}(M)$	undecidable idem	open	
$\mathrm{Rec}(M)$	undecidable idem	decidable Carton, Choffrut Grigorieff 2006	decidable in $2^{0(n)}$ Carton, Choffrut Grigorieff 2006

Relations as languages (binary case $A^* \times B^*$)

Correspondence relation $\leftrightarrow$ language.

$$R \subseteq A^* \times B^* \mapsto L_R \in A^* \# B^*$$

where

$$L_R = \{\tilde{u} \# v \in A^* \# B^* \mid (u, v) \in R\}$$

- R is a rational relation iff L_R is a linear language with a unique final production $X \to \#$.
- If R is a deterministic rational, then L_R is a deterministic pushdown language.
- R is a recognizable relation iff L_R is a rational language.

Stearns' results

Theorem (Stearns 1967)

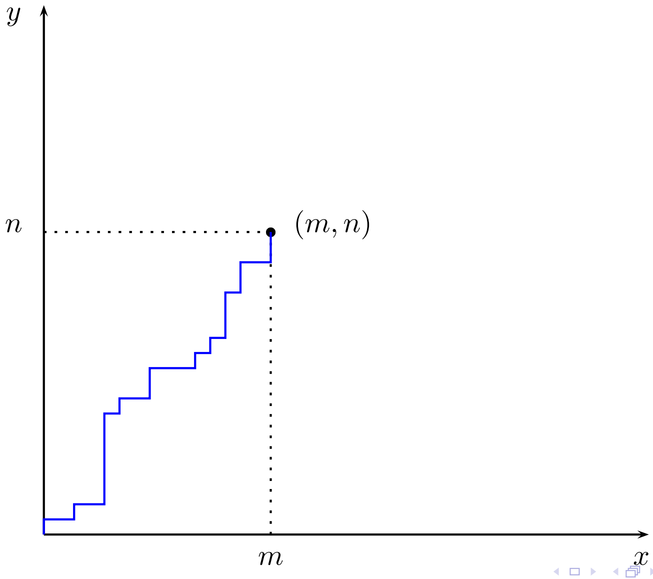
If can be decided whether a language given by a deterministic pushdown automaton is rational.

- Very nice proof (majoration of the stack height)
- complexity of the procedure: 3 exponentials
- The complexity has been lowered to 2 exponentials by Valiant in 1976 (optimal, see Meyer et Fischer 1971)
- Therefore, it can be decided if a binary deterministic relation is recognizable.

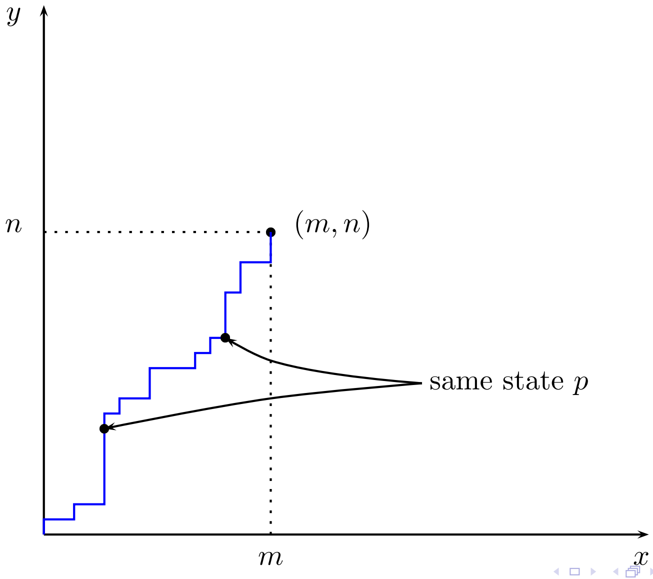
Known results (commutative case)

	$\mathrm{Rat}(M)$
$\mathrm{DRat}(M)$	decidable Carton, Choffrut Grigorieff 2006
$\mathrm{Sync}(M)$	decidable idem
$\mathrm{Rec}(M)$	decidable Ginsburg, Spanier 67

Run in $\mathbb{N}^2$

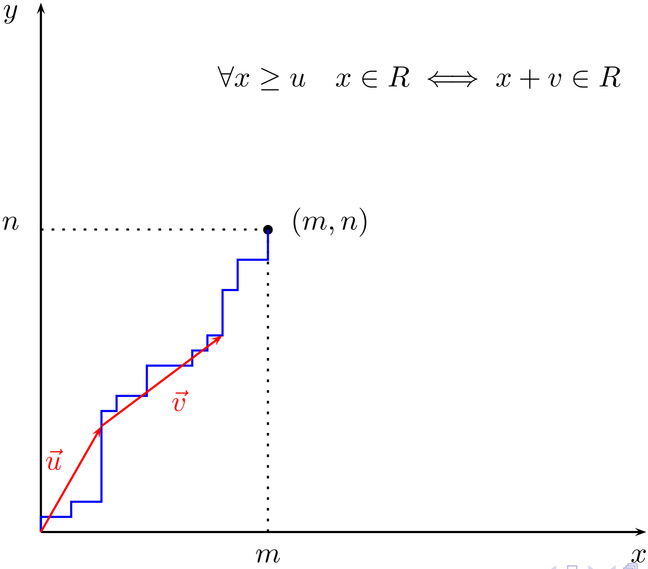


Run in $\mathbb{N}^2$



Run in $\mathbb{N}^2$

$$\forall x \geq u \quad x \in R \iff x + v \in R$$



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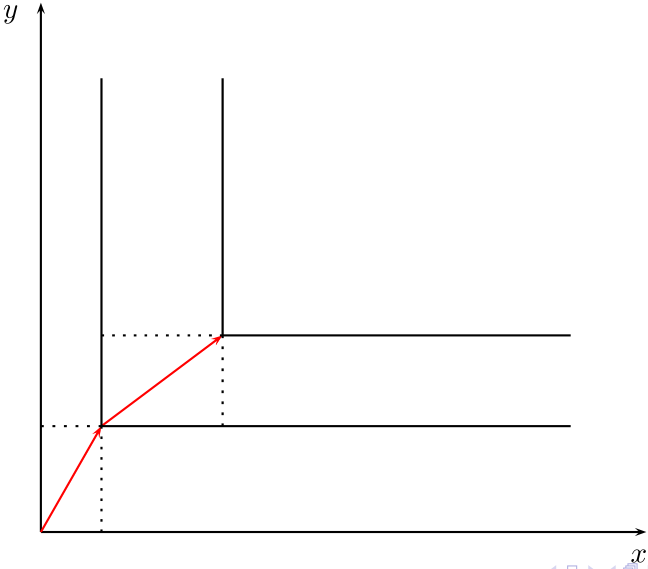
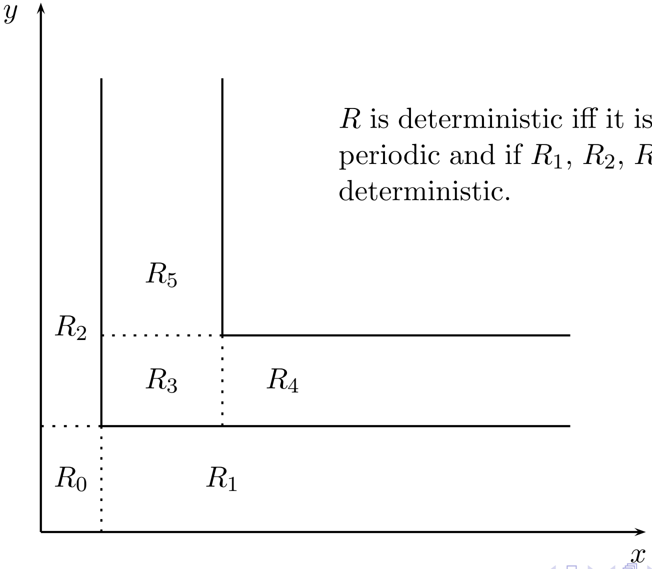


Run in $\mathbb{N}^2$



R is deterministic iff it is ultimately periodic and if R_1, R_2, R_4 et R_5 are deterministic.

Reduction to Presburger logic

Let R be given by the formula $\theta(x; b)$

$$R = \{x \in \mathbb{N}^k \mid \theta(x; b)\}$$

One constructs the formula $\Psi_\theta(b)$

$$\begin{aligned}\Psi_\theta(b) = \exists\mu\ \exists\pi\ \ (\forall x \geq 0 \quad \theta(x+\mu; b) \iff \theta(x+\mu+\pi; b) \wedge \\ \forall u < \mu\ \ \forall v < \pi \quad \bigwedge_{\varnothing \neq I \subseteq \{1,\ldots,k\}} \\ (\mathrm{Null}(\mu - u) = I \implies \Psi_{\theta'_I}(b, \mu, u)) \\ (\mathrm{Null}(\pi - v) = I \implies \Psi_{\theta'_I}(b, \mu+\pi, \mu+v)))\end{aligned}$$

which is satisfiable iff R is deterministic.