Density of Symbols in Discretized Rotation Configurations.

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Abstract. The aim of this paper is to study local configurations for discrete rotations. The algorithm of discrete rotation we consider is the following: a discretized rotation is defined as the composition of a Euclidean rotation with a rounding operation, as studied in [NR05] and [NR04]. It is possible to encode all the information concerning a discrete rotation as two multidimensional words C_α and C'_α that we call configurations. We introduce here two discrete dynamical systems defined by a $\mathbb{Z}^2$-action on the two-dimensional torus that allow us via a suitable symbolic coding to describe the configurations C_α and C'_α and to deduce the densities of occurrence of the symbols in the configurations.

1 Introduction

Symbolic dynamics and more generally, discrete dynamical systems have natural and deep interactions with combinatorics on words. This interaction is particularly well-illustrated in the Sturmian case, see e.g. [Lot02,Fog02]. The combinatorial objects involved are the Sturmian words, while the dynamical systems are the irrational rotations of the torus $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. A Sturmian word is indeed a coding with respect to a particular two-interval partition of the one-dimensional torus $\mathbb{T}^1$ of the orbit of a point under the action of an irrational rotation. This point of view allows one to deduce many combinatorial properties of Sturmian words, such as for instance the densities of occurrences of factors that can be computed thanks to the equidistribution properties of irrational rotations.

Several attempts of generalization of this fruitful interaction have been proposed. One of the first idea which comes to mind is a rotation of $\mathbb{T}^2$. As an example, the Tribonacci word, that is, the fixed point of the substitution $1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ codes the orbit of a point of the torus $\mathbb{T}^2$ under the action of a translation in $\mathbb{T}^2$ with respect to a partition of $\mathbb{T}^2$ into three pieces with fractal boundary [Rau82,Lot05].

A second approach, which is dual to the previous one, consists in working with two rotations of $\mathbb{T}^1$. It is indeed convenient to describe discrete planes by use of the coding with respect to a three-interval partition of a $\mathbb{Z}^2$-action

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by two irrational rotations on $\mathbb{T}^1$. One thus gets two-dimensional words over a three-letter alphabet that can be considered as two-dimensional Sturmian words [BV00].

We consider here a further generalization. Indeed, we study configurations associated with a discrete rotation, defined as the composition of a Euclidean rotation with a rounding operation. It is possible to encode all the information concerning a discrete rotation as two multidimensional words C_α and C'_α that we call configurations. The main purpose of the present paper is to prove that both configurations are codings of a $\mathbb{Z}^2$-action by two rotations on $\mathbb{T}^2$ with respect to a partition into a finite number of rectangles. We then deduce results concerning the density of each symbol in C_α and C'_α. As a motivation for this study, let note that we plan to use these results in a next future for an algorithm of randomization of discrete rotations.

2 Conventions

We work in the *discrete plane* $\mathbb{Z}^2$. For each point $\mathbf{v}$, $x_\mathbf{v}$ denotes its horizontal coordinate and $y_\mathbf{v}$ its vertical coordinate.

Let x be a real number. We recall that the floor function $x \mapsto \lfloor x \rfloor$ is defined as the greatest integer less or equal to x. The *rounding function* is defined as $[x] := \lfloor x + 0.5 \rfloor$ and $\{x\} := x - [x]$. These applications can be extended to vectors, by independent application on each component of the vector.

The *discretization cell* of the point $\mathbf{v} \in \mathbb{Z}^2$ is defined as the set of elements $\mathbf{w}$ in $\mathbb{R}^2$ which have the same image by discretization as $\mathbf{v}$, i.e., $[\mathbf{v}] = [\mathbf{w}]$. Hence the discretization cell of $\mathbf{v}$ is defined as the half-opened unit square centered in $[\mathbf{v}]$.

We use the canonical bijection between the torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the square $\{\mathbf{v} \in \mathbb{R}^2; x_\mathbf{v} \in [-\frac{1}{2}, \frac{1}{2}[$ and $y_\mathbf{v} \in [-\frac{1}{2}, \frac{1}{2}[\}$, i.e., the discretization cell of 0. By abuse of notation, we also denote by $\{\mathbf{v}\}$ the image under the canonical projection from $\mathbb{R}^2$ onto $\mathbb{T}^2$ of a point $\mathbf{v} \in \mathbb{R}^2$. Hence let us stress the fact that the map $x \mapsto \{x\}$ is an additive morphism from $\mathbb{R}^2$ onto $\mathbb{T}^2$.

Without loss of generality, we assume throughout this paper that $\alpha \in [0, \pi/4]$: the arguments used here can be easily extended to the case of any other octant. We denote by r_α the Euclidean rotation of angle α:

$$r_\alpha : \mathbb{R}^2 \to \mathbb{R}^2, \ \mathbf{v} \mapsto \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \mathbf{v}.$$

The discrete rotation $[r_\alpha]$ is defined as

$$[r_\alpha] : \mathbb{Z}^2 \to \mathbb{Z}^2, \ \mathbf{v} \mapsto [r_\alpha(\mathbf{v})].$$

By $\{r_\alpha\}$ we mean the map $\{r_\alpha\} : \ \mathbb{Z}^2 \to \mathbb{T}^2, \ \mathbf{v} \mapsto \{r_\alpha(\mathbf{v})\}$.

We denote by $(\mathbf{i}, \mathbf{j})$ the canonical basis of the Euclidean space $\mathbb{R}^2$. We similarly use the notation $\mathbf{i}_\alpha := r_\alpha(\mathbf{i})$ and $\mathbf{j}_\alpha := r_\alpha(\mathbf{j})$.

Let Q be a finite set called alphabet. A two-dimensional word in $Q^{\mathbb{Z}^2}$ is called a *configuration* over Q. An application from $\{0,1,\cdots,n-1\} \times \{0,1,\cdots,m-1\}$ to Q is called a *pattern* of size $[m,n]$. Let C be a configuration in $Q^{\mathbb{Z}^2}$. A pattern χ of size $[m,n]$ occurs at position $\mathbf{p}$ in C if $C(\mathbf{p}+\mathbf{v}) = \chi(\mathbf{v})$, for all $\mathbf{v}$ with $x_{\mathbf{v}}, y_{\mathbf{v}} \in \{0,1,\cdots,n-1\}\times\{0,1,\cdots,m-1\}$. We define $C^{[m,n]}$ as the configuration with values in the finite alphabet consisting of the patterns of size $[m,n]$ over Q, that is defined as the application that returns the pattern of size $[m,n]$ that occurs at the specified position in the configuration.

The *density* of the symbol $p \in Q$ in the configuration $C \in Q^{\mathbb{Z}^2}$ is defined as the following limit (if it exists):

$$\eta_C(p) = \lim_{n\to\infty} \frac{\#\{\mathbf{v} \in \mathbb{Z}^2,\ x_v, y_v \in \{-n,\cdots,n\} \text{ and } C(\mathbf{v}) = p\}}{(2n+1)^2}.$$

A *dynamical system* (X,T) is defined as the action of a continuous and onto map T on a compact space X. Given two continuous and onto maps T_1 and T_2 acting on X and satisfying $T_1 \circ T_2 = T_2 \circ T_1$, the $\mathbb{Z}^2$*-action* by T_1 and T_2 on X, that we denote (X, T_1, T_2), is defined by

$$\forall (m,n) \in \mathbb{Z}^2,\ \forall x \in X,\quad (m,n)\cdot x = T_1^m \circ T_2^n(x).$$

It is natural to associate a two-dimensional symbolic dynamical system to the triple (X,T_1,T_2) by coding the orbits of the points of X under the $\mathbb{Z}^2$-action as follows: given $x_0 \in X$ and given a *labelling function* l defined on X with values in a finite set Q that takes constant values on the atoms of a finite partition of X, the configuration C defined by

$$\forall (m,n) \in \mathbb{Z}^2,\ C(m,n) = l(T_1^m \circ T_2^n(x_0))$$

is called the coding of the orbit of x_0 under the $\mathbb{Z}^2$-action (X,T_1,T_2) with respect to the labelling function l.

3 Dynamical System Associated to C_α

According to [NR05], we associate a first configuration C_α to the discrete rotation $[r_\alpha]$ that encodes all the information concerning the discrete rotation (there exists indeed a planar transducer that uses the configuration C_α as input and gradually computes the action of the discrete rotation). For a given $\mathbf{v} \in \mathbb{Z}^2$, let $\mathcal{V}_4$ denote the set of 4-neighbours of $\mathbf{v}$, that is, $\mathcal{V}_4 = \{\mathbf{v}+\mathbf{i},\ \mathbf{v}+\mathbf{j},\ \mathbf{v}-\mathbf{i},\ \mathbf{v}-\mathbf{j}\}$. The configuration C_α maps each point $\mathbf{v}$ of $\mathbb{Z}^2$ to the set $[r_\alpha](\mathcal{V}_4) - [r_\alpha][\mathbf{v}]$, that is,

$$C_\alpha(\mathbf{v}) := \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \text{ with } (a_k = [r_\alpha(\mathbf{v}+r^k_{\pi/2}(\mathbf{i}))] - [r_\alpha(\mathbf{v})] \text{ for } k = 0,\cdots,3).$$

Let us note that C_α contains 3 or 4 non-zero elements, according to [NR03]. Let Q_α denote the finite set of values taken by C_α.

We define a *frame* of the torus $\mathbb{T}^2 \equiv [-\frac{1}{2},\frac{1}{2}[\times [-\frac{1}{2},\frac{1}{2}[$ as a rectangle of the form $[a,b[\times[c,d[$, with $-\frac{1}{2} \le a \le b < \frac{1}{2}$ and $-\frac{1}{2} \le c \le b < \frac{1}{2}$. The interpretation of C_α as a coding a $\mathbb{Z}^2$-action is based on the following result:

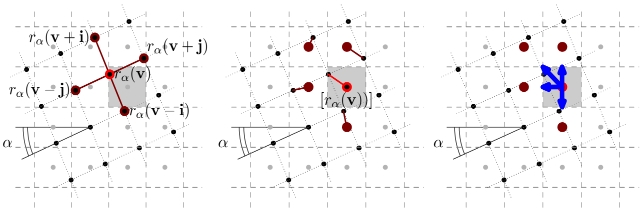


Fig. 1. A progressive construction of the configuration C_α: we represent the set of vectors that leads to the relative position of the 4-neighbors of $\mathbf{v}$ after the action of the discrete rotation.

$(0,0)\mapsto$	$(1,0)\mapsto$	$(2,0)\mapsto$	$(3,0)\mapsto$	$(4,0)\mapsto$	$(5,0)\mapsto$
$(0,1)\mapsto$	$(1,1)\mapsto$	$(2,1)\mapsto$	$(3,1)\mapsto$	$(4,1)\mapsto$	$(5,1)\mapsto$
$(0,2)\mapsto$	$(1,2)\mapsto$	$(2,2)\mapsto$	$(3,2)\mapsto$	$(4,2)\mapsto$	
$(0,3)\mapsto$	$(1,3)\mapsto$	$(2,3)\mapsto$	$(3,3)\mapsto$	$(4,3)\mapsto$	$(5,3)\mapsto$
$(0,4)\mapsto$	$(1,4)\mapsto$	$(2,4)\mapsto$	$(3,4)\mapsto$	$(4,4)\mapsto$	$(5,4)\mapsto$
$(0,5)\mapsto$	$(1,5)\mapsto$		$(3,5)\mapsto$	$(4,5)\mapsto$	$(5,5)\mapsto$

Fig. 2. Table describing the action of ϕ_c. The symbols represent the all the vectors of the set.

Theorem 1 ([NR05]). *There exists a partition P_α of the torus $\mathbb{T}^2$ into a finite number of frames such that for each $p \in Q_\alpha$, there exists a frame I_p such that for all $\mathbf{v} \in \mathbb{Z}^2$, then $C_\alpha(\mathbf{v}) = p$ if and only if $\{r_\alpha(\mathbf{v})\} \in I_p$.*

Consider the following two actions $T_{\mathbf{i}_\alpha} : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + \{\mathbf{i}_\alpha\}$, $T_{\mathbf{j}_\alpha} : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + \{\mathbf{j}_\alpha\}$. One has for every $\mathbf{v} \in ZZ^2$, $\{r_\alpha(\mathbf{v})\} = T_{\mathbf{i}_\alpha}^{x_\mathbf{v}} \circ T_{\mathbf{i}_\alpha}^{y_\mathbf{v}}(\mathbf{0})$. Let us define l_{C_α} as the labelling function associated to the partition P_α defined by $l_{C_\alpha} : \mathbb{T}^2 \to Q_\alpha,\ \mathbf{v} \mapsto \phi_c(f_{C_\alpha}(\mathbf{v}_x), f_{C_\alpha}(\mathbf{v}_y))$ with f_{C_α} defined as follows:

if $\alpha \in [0, \pi/6]$:

$$\left|\begin{array}{ll} [-\frac{1}{2}, \frac{1}{2} - \cos(\alpha)[& \mapsto 0 \\ [\frac{1}{2} - \cos(\alpha), \sin(\alpha) - \frac{1}{2}[& \mapsto 1 \\ [\sin(\alpha) - \frac{1}{2}, \frac{1}{2} - \sin(\alpha)[& \mapsto 2 \\ [\frac{1}{2} - \sin(\alpha), \cos(\alpha) - \frac{1}{2}[& \mapsto 3 \\ [\cos(\alpha) - \frac{1}{2}, \frac{1}{2}[& \mapsto 4 \end{array}\right|$$

if $\alpha \in [\pi/6, \pi/4]$:

$$\left|\begin{array}{ll} [-\frac{1}{2}, \frac{1}{2} - \cos(\alpha)[& \mapsto 0 \\ [\frac{1}{2} - \cos(\alpha), \frac{1}{2} - \sin(\alpha)[& \mapsto 1 \\ [\frac{1}{2} - \sin(\alpha), \sin(\alpha) - \frac{1}{2}[& \mapsto 5 \\ [\sin(\alpha) - \frac{1}{2}, \cos(\alpha) - \frac{1}{2}[& \mapsto 3 \\ [\cos(\alpha) - \frac{1}{2}, \frac{1}{2}[& \mapsto 4 \end{array}\right|$$

where ϕ_c is described in Figure 2. The values taken by C_α, that is, the elements of Q_α are represented in Figure 2 as sets of vectors.

Theorem 1 can then be reformulated as follows: C_α is the coding of the orbit of $\mathbf{0}$ under the $\mathbb{Z}^2$-action $(\mathbb{T}^2, T_{\mathbf{i}_\alpha}, T_{\mathbf{i}_\alpha})$ with respect to the labelling function l_{C_α}.

4 Distribution of Symbols in C_α

We can now deduce from the $\mathbb{Z}^2$-action introduced in Section 3 results concerning the densities of symbols in C_α by using classical tools from symbolic dynamics and ergodic theory.

Let $G_\alpha \subseteq \mathbb{T}^2$ denote the orbit of $\mathbf{0}$ under the $\mathbb{Z}^2$-action $(\mathbb{T}^2, T_{\mathbf{i}_\alpha}, T_{\mathbf{i}_\alpha})$ with respect to the labelling function l_{C_α}: this very orbit is the orbit coded by the configuration C_α. In other words, G_α is the image by the canonical projection $x \mapsto \{x\}$ onto $\mathbb{T}^2$ of the lattice $L_\alpha := \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$; G_α has a group structure, and is invariant by rotation by $\pi/2$.

Let us recall that an angle α is said *Pythagorean* if $\cos\alpha$ and $\sin\alpha$ are both rational. Let us distinguish two cases according to the fact that α is Pythagorean or not, that is, according to the density of G_α in $\mathbb{T}^2$.

The Dense Case

Lemma 1. *We assume that α is not Pythagorean. For every symbol $p \in Q_\alpha$, its density $\eta_{C_\alpha}(p)$ exists and is equal to the area of the frame I_p defined in Theorem 1.*

Proof (Sketch). If either $\cos(\alpha)$ or $\sin(\alpha)$ is irrational, then one cannot have simultaneously $p\cos(\alpha) + q\sin(\alpha) \in \mathbb{Z}$ and $-p\sin(\alpha) + q\cos(\alpha) \in \mathbb{Z}$, for any $(p, q) \in \mathbb{Z}^2$. Hence one concludes by using a classical argument on Weyl sums.□

The Pythagorean Case

If α is a Pythagorean angle then G_α is not dense in the torus $\mathbb{T}^2$: indeed, G_α is a finite cyclic group. It has order c where $(a, b, c) \in \mathbb{N}^3$ is the prime Pythagorean triple satisfying $1 \le b \le a \le c$, $a^2+b^2 = c^2$, $gcd(a, b, c) = 1$ and $c\exp(i\alpha) = a+ib$ that generates the angle α. More information on Pythagorean angles can be found in [NR04].

Lemma 2. *Let $\alpha \in [0, ...\pi/4[$ be a Pythagorean angle. Let c denote the order of the cyclic group G_α. The density $\eta_{C_\alpha}(p)$ of the symbol p in C_α satisfies*

$$\forall p \in Q_\alpha, \eta_{C_\alpha}(p) = \frac{Card\ (G'_\alpha \cap I_p)}{c}.$$

Proof (Sketch). By definition,

$$\eta_{C_\alpha}(p) = \lim_{n\to\infty} (\{r_\alpha\}(\{-n, \cdots, n\}^2) \cap I_p)/(2n+1)^2.$$

One first checks that

$$\eta_{C_\alpha}(p) = \lim_{n\to\infty} (\{r_\alpha\}(\{-c\lfloor n/c\rfloor, ..., c\lfloor n/c\rfloor\}^2) \cap I_p)/(2n+1)^2).$$

But as G_α is cyclic and of order c, then

$$\eta_{C_\alpha}(p) = \frac{\{r_\alpha\}(\{0, ..., c-1\}^2) \cap I_p}{c^2} = \frac{\text{Card}\ (G'_\alpha \cap I_p)}{c}.$$

□

5 Distribution of Symbols in C'_α

Let us define now C'_α:

$$\forall \mathbf{v} \in \mathbb{Z}^2,\ C'_\alpha(\mathbf{v}) := \bigcup_{\mathbf{w} \text{ such that } [r_\alpha(\mathbf{w})] = \mathbf{v}} C_\alpha(\mathbf{w}).$$

Let Q'_α denote the set of values taken by C'_α. We want to state a result analogous to Theorem 1 in order, first, to interpret the configuration C'_α as a coding of a symbolic dynamical system, and second, to compute the densities of the symbols in C'_α. Let us note that Corollary 1 in [NR05] does not directly yield a dynamical interpretation of C'_α.

Our strategy in order to describe C'_α as a coding of a $\mathbb{Z}^2$-action is the following. We first create a "block configuration" by working with patterns of size $[2,2]$ that occur in C'_α. We then introduce a particular domain of $\mathbb{R}^2$ that is a fundamental domain for the lattice $\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$, such that if we know the projection of a point $\mathbf{p} \in \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ in that domain, then we can recover the symbols that appear in the block configuration; therefore we find out what are the symbols that appear in C'_α. We thus deduce a symbolic dynamical system for the block configuration. Finally, we use this dynamical system, in order to get the density of the symbols both in the block configuration and in C'_α.

5.1 Dynamical System for C'_{B_α}

Let $C'_{B_\alpha}(\mathbf{v})$ be defined as the following 2×2-block configuration:

$$\forall \mathbf{v} \in \mathbb{Z}^2,\ C'_{B_\alpha}(\mathbf{v}) = C'^{[2,2]}_\alpha(2\mathbf{v}).$$

Since $C'_{B_\alpha}(\mathbf{v})$ is an application that returns patterns of size $[2,2]$, then $C'_\alpha(\mathbf{v}) = \left(C'_{B_\alpha}(\lfloor x_\mathbf{v}/2 \rfloor, \lfloor y_\mathbf{v}/2 \rfloor)\right)(\mathbf{v}_x \bmod 2, \mathbf{v}_y \bmod 2)$. For any $\mathbf{v} \in \mathbb{Z}^2$, one sets

$$F_B(\mathbf{v}) = [x_\mathbf{v} - \frac{1}{2}, x_\mathbf{v} + \frac{3}{2}[\times [y_\mathbf{v} - \frac{1}{2}, y_\mathbf{v} + \frac{3}{2}[.$$

The introduction of this block configuration is natural, since the intersection between $F_B(\mathbf{v})$ and $r_\alpha(\mathbb{Z}^2) = \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ is nonempty for every $\mathbf{v} \in \mathbb{Z}^2$; this is a direct consequence of the fact that two holes (a hole is an element $\mathbf{v} \in \mathbb{Z}^2$ that has no antecedent by $[r_\alpha]$) can never be adjacent (see [NR04]). An example of a hole is depicted in Figure 4 below. Let

$$F_{D_\alpha} := \left([-\frac{1}{2}, \cos\alpha - \frac{1}{2}[\right)^2 \cup \left([\cos\alpha - \frac{1}{2}, \cos\alpha + \sin\alpha - \frac{1}{2}[\times [-\frac{1}{2}, \sin\alpha - \frac{1}{2}[\right).$$

The set F_{D_α} is a fundamental domain for the lattice $L_\alpha = \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ (see Figure 3). Hence for any $\mathbf{v} \in \mathbb{Z}^2$, there exists a unique $\mathbf{w} \in \mathbb{Z}^2$ such that $r_\alpha(\mathbf{w}) \in \mathbf{v} + F_{D_\alpha}$. Therefore for all $\mathbf{v} \in \mathbb{Z}^2$, we first define

$$\theta' : \mathbb{Z}^2 \to L_\alpha,\ \mathbf{v} \mapsto r_\alpha(\mathbf{w}),$$

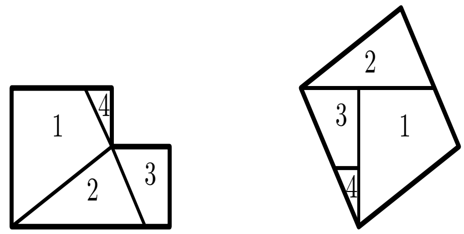


Fig. 3. An exchange of pieces between F_{D_α} and the canonical representation of $\mathbb{R}^2/L_\alpha$. This exchange of pieces only requires translations of the form $k\mathbf{i}_\alpha + k'\mathbf{j}_\alpha$, with $k, k' \in \mathbb{Z}$.

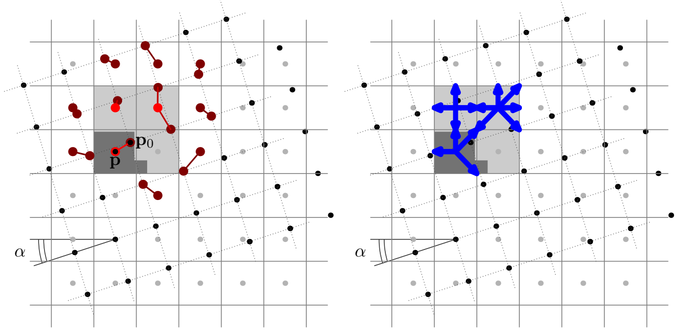


Fig. 4. From a point $\mathbf{p}_0 = \theta'(2\mathbf{v}) \in \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ that falls into the domain $F_{D_\alpha}(2\mathbf{v})$ (in dark gray), we can recover all the symbols of C'_α that contribute to the block of size $[2, 2]$ whose associated domain is $F_B(2\mathbf{v})$ in light gray.

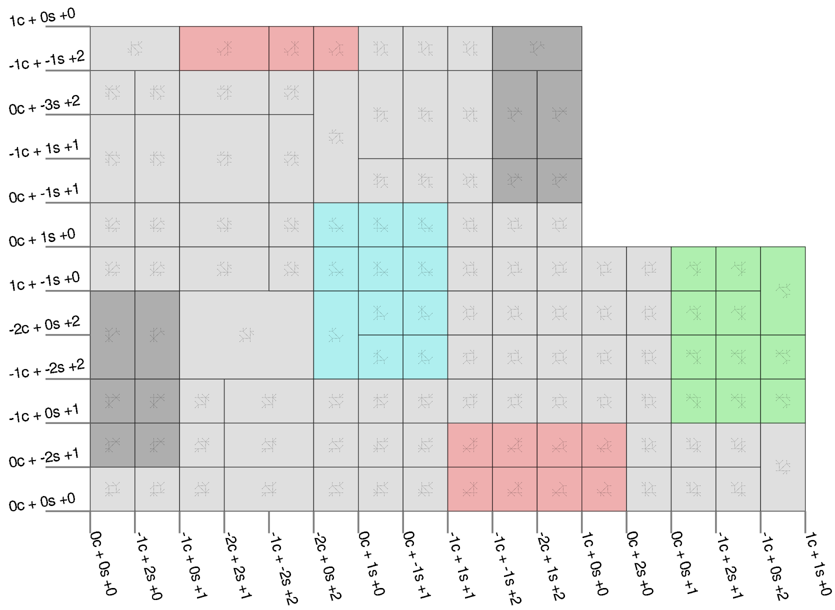


Fig. 5. A partition of the domain F_{D_α}, for $\alpha \approx 0.464705$ rad. This partition gives according to the position of $\theta(2\mathbf{v})$ inside that domain the pattern of size $[2,2]$ that appears in $C'_{B_\alpha}(\mathbf{v})$. On the axis the positions are labeled by expressions of the form $kc + k's + k''$, meaning that the corresponding line is located at $k\cos(\alpha) + k'\sin(\alpha) + k'' - \frac{1}{2}$ in F_{D_α}. For readability reasons, the scale is monotone but not linear.

where $\mathbf{w}$ is the unique point such that $r_\alpha(\mathbf{w}) \in \mathbf{v} + F_{D_\alpha}$, and then

$$\theta : \mathbb{Z}^2 \to F_{D_\alpha}, \ \mathbf{v} \mapsto \theta'(\mathbf{v}) - \mathbf{v}.$$

Theorem 2. *There exists a partition of F_{D_α} into a finite number of frames $J_{p'}$, for p' pattern of size $[2,2]$ that occurs in C'_α, such that for all $\mathbf{v} \in \mathbb{Z}^2$, $\theta(2\mathbf{v}) \in J_{p'}$ if and only if $C'_{B_\alpha}(\mathbf{v}) = p'$.*

Proof (Sketch). The proof is based on the following idea: from the location of $\theta(2\mathbf{v})$ in F_{D_α}, it is possible to deduce the value of $C'_{B_\alpha}(\mathbf{v})$. We notice that, for all the points $\mathbf{w}$ of $\mathbb{Z}^2$ that have their image by r_α in $F_B(2\mathbf{v})$ we can compute $C_\alpha(\mathbf{w})$. Indeed we show that if $x_{\theta(2\mathbf{v})} < \frac{1}{2}$, $[\theta(2\mathbf{v})] = 0$, else $[\theta(2\mathbf{v})] = 1$; we thus deduce $C_\alpha(\mathbf{w})$ from $\{\theta'(2\mathbf{v})\}$, according to Theorem 1. The same argument applies for all the points $\mathbf{w}' = r_\alpha(\mathbf{w})$ of $\mathbb{Z}_{\mathbf{i}_\alpha} + \mathbb{Z}_{\mathbf{j}_\alpha}$ that are inside $F_B(2\mathbf{v})$; note that $\mathbf{w}' = \theta(2\mathbf{v}) + k\mathbf{i}_\alpha + k'\mathbf{j}_\alpha$, with $k, k' \in \mathbb{Z}$. We thus similarly localize the position in $(2\mathbf{v} + \{0,1\}^2)$ of all the images of points in $\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha \cap F_B(2\mathbf{v})$. This is sufficient to conclude that we can infer the pattern $C'_B(\mathbf{v})$ from $\theta(2\mathbf{v})$. □

Let $l_{C'_{B_\alpha}}$ be the labeling function given by the partition of Theorem 2 that associates to a frame in F_{D_α} the corresponding pattern of size $[2,2]$.

From Theorem 2, we deduce that

$$\forall \mathbf{v} \in \mathbb{Z}^2, C'_{B_\alpha}(\mathbf{v}) = l_{C'_{B_\alpha}}(\theta(2\mathbf{v}))$$

Now, let $\mathbb{T}^2_\alpha = \mathbb{R}^2/(\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha)$; we denote as $\mathbf{v} \mapsto \{\mathbf{v}\}_\alpha$ the canonical projection on $\mathbb{T}^2_\alpha$, that is in one-to-correspondence with F_{D_α}. One has

$$\forall \mathbf{v} \in \mathbb{Z}^2,\ \theta(\mathbf{v}) \equiv -\{\mathbf{v}\}_\alpha \text{ modulo } L_\alpha.$$

Finally, the configuration C'_{B_α} is a coding of the orbit 0 under the $\mathbb{Z}^2$-action $(\mathbb{T}^2_\alpha, \mathbf{v} \mapsto \mathbf{v} + \{\mathbf{i}\}_\alpha, \mathbf{v} \mapsto \mathbf{v} + \{\mathbf{j}\}_\alpha)$ with respect to the labelling function $l_{C'_{B_\alpha}}$.

5.2 Application

We assume that α is not a Pythagorean angle. Similarly as in the study of C_α, the orbit of 0 under the $\mathbb{Z}^2$-action is dense and uniformly distributed in $\mathbb{T}^2_\alpha$. We thus deduce that

$$\forall p \in Q'_\alpha,\ \eta_{C'_\alpha}(p) = \sum_{p' \in Q^{[2,2]}_\alpha} n(p', p)\, \mu(f_{p'}),$$

where $Q^{[2,2]}_\alpha$ is the set of patterns of size $[2,2]$ that occur in C'_α, $n(p', p)$ is the function that returns the number of occurrences of p in the pattern p' of size $[2,2]$, and $\mu(J_{p'})$ denotes the area of frame $J_{p'}$ associated to the symbol p' according to Theorem 2.

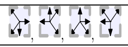
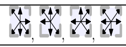
However practically, the computations for these symbolic maps are quite tedious. For each symbol p, there exist 40 patterns p' of size $[2,2]$ to compute. This leads to approximatively 360 inequations... and there are approximatively 25 symbols p to consider! See [BN05] for a program that handles these symbolical expressions. The results describing the densities of the symbols in C'_α have been summarized in Figure 6.

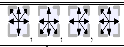
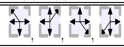
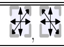
Let us note that in the Pythagorean case, the theory is also similar to the one developed for C_α.

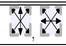
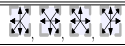
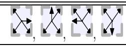
Remark Let us observe that all the results we have given here for symbols are extendable without major difficulty to patterns of a given size $[m,n]$. Actually, a frame is associated to each pattern, and the same theory can be used.

References

[BN05] V. Berthé and B. Nouvel. Dynamical systems and distribution of symbols in rotations configurations. `http://perso.ens-lyon.fr/bertrand.nouvel/pub/DSDSRC.pdf`, 2005.

α			
$[0..\arctan(\sqrt{2}/4)]$	$2\cos(\alpha)sin(\alpha)-2\cos(\alpha)-2\sin(\alpha)+2$	$2\cos(\alpha)sin(\alpha)-2\cos(\alpha)-2\sin(\alpha)+2$	$-(\cos(\alpha))^2-\cos(\alpha)sin(\alpha)+2\cos(\alpha)+\sin(\alpha)-1$
$[\arctan(\sqrt{2}/4)..\arctan(1/2)]$	$2\cos(\alpha)sin(\alpha)-2\cos(\alpha)-2\sin(\alpha)+2$	$2\cos(\alpha)sin(\alpha)-2\cos(\alpha)-2\sin(\alpha)+2$	$-(\cos(\alpha))^2+2\cos(\alpha)sin(\alpha)+\cos(\alpha)-2\sin(\alpha)$
$[\arctan(1/2)..\pi/6]$	$2\cos(\alpha)sin(\alpha)-2\cos(\alpha)-2\sin(\alpha)+2$	$2\cos(\alpha)sin(\alpha)-2\cos(\alpha)-2\sin(\alpha)+2$	0
$[\pi/6..\arctan(3/4)]$	$2\cos(\alpha)sin(\alpha)-2\cos(\alpha)-2\sin(\alpha)+2$	$-2\cos(\alpha)sin(\alpha)+1$	0
$[\arctan(3/4)..\pi/4]$	$2\cos(\alpha)sin(\alpha)-2\cos(\alpha)-2\sin(\alpha)+2$	$-2\cos(\alpha)sin(\alpha)+1$	0

α			
$[0..\arctan(\sqrt{2}/4)]$	$(\cos(\alpha))^2-2\cos(\alpha)+1$	$-2(\sin(\alpha))^2-2\cos(\alpha)sin(\alpha)+\cos(\alpha)+3\sin(\alpha)-1$	$3\cos(\alpha)sin(\alpha)-\cos(\alpha)-3\sin(\alpha)+1$
$[\arctan(\sqrt{2}/4)..\arctan(1/2)]$	$(\cos(\alpha))^2-2\cos(\alpha)+1$	$-2(\sin(\alpha))^2-2\cos(\alpha)sin(\alpha)+\cos(\alpha)+3\sin(\alpha)-1$	0
$[\arctan(1/2)..\pi/6]$	$2\cos(\alpha)sin(\alpha)-\cos(\alpha)-2\sin(\alpha)+1$	$-2(\sin(\alpha))^2-2\cos(\alpha)sin(\alpha)+\cos(\alpha)+3\sin(\alpha)-1$	0
$[\pi/6..\arctan(3/4)]$	0	0	0
$[\arctan(3/4)..\pi/4]$	0	0	$2(\cos(\alpha))^2-\cos(\alpha)sin(\alpha)-3\cos(\alpha)+\sin(\alpha)+1$

α			
$[0..\arctan(\sqrt{2}/4)]$	0	0	0
$[\arctan(\sqrt{2}/4)..\arctan(1/2)]$	$-3\cos(\alpha)sin(\alpha)+\cos(\alpha)+3\sin(\alpha)-1$	0	0
$[\arctan(1/2)..\pi/6]$	$-2(\cos(\alpha))^2+\cos(\alpha)sin(\alpha)+3\cos(\alpha)-\sin(\alpha)-1$	$(\cos(\alpha))^2-2\cos(\alpha)sin(\alpha)-\cos(\alpha)+2\sin(\alpha)$	0
$[\pi/6..\arctan(3/4)]$	$-2(\cos(\alpha))^2+\cos(\alpha)sin(\alpha)+3\cos(\alpha)-\sin(\alpha)-1$	$(\cos(\alpha))^2-2\cos(\alpha)+1$	$-2(\sin(\alpha))^2+2\cos(\alpha)sin(\alpha)-\cos(\alpha)+\sin(\alpha)$
$[\arctan(3/4)..\pi/4]$	0	$-(\cos(\alpha))^2+\cos(\alpha)sin(\alpha)+\cos(\alpha)-\sin(\alpha)$	$-2(\sin(\alpha))^2+2\cos(\alpha)sin(\alpha)-\cos(\alpha)+\sin(\alpha)$

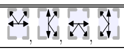
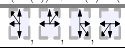
α				
$[0..\arctan(\sqrt{2}/4)]$	0	$-(\cos(\alpha))^2-\cos(\alpha)sin(\alpha)+2\cos(\alpha)+\sin(\alpha)-1$	0	$4(\sin(\alpha))^2-4\sin(\alpha)+1$
$[\arctan(\sqrt{2}/4)..\arctan(1/2)]$	0	$-(\cos(\alpha))^2-\cos(\alpha)sin(\alpha)+2\cos(\alpha)+\sin(\alpha)-1$	0	$4(\sin(\alpha))^2-4\sin(\alpha)+1$
$[\arctan(1/2)..\pi/6]$	0	$-(\cos(\alpha))^2-\cos(\alpha)sin(\alpha)+2\cos(\alpha)+\sin(\alpha)-1$	0	$4(\sin(\alpha))^2-4\sin(\alpha)+1$
$[\pi/6..\arctan(3/4)]$	$-2\cos(\alpha)sin(\alpha)+\cos(\alpha)+2\sin(\alpha)-1$	$-(\cos(\alpha))^2+\cos(\alpha)sin(\alpha)+\cos(\alpha)-\sin(\alpha)$	$4(\sin(\alpha))^2-4\sin(\alpha)+1$	0
$[\arctan(3/4)..\pi/4]$	$-2\cos(\alpha)sin(\alpha)+\cos(\alpha)+2\sin(\alpha)-1$	$-(\cos(\alpha))^2+\cos(\alpha)sin(\alpha)+\cos(\alpha)-\sin(\alpha)$	$4(\sin(\alpha))^2-4\sin(\alpha)+1$	0

Fig. 6. Table describing $\eta_{C'_\alpha}(p)$ for each symbol p that appears in C'_α, with respect to the value of α.

[BV00] V. Berthé and L. Vuillon. Tilings and rotations on the torus: a two-dimensional generalization of Sturmian sequences. *Discrete Math.*, 223:27–53, 2000.

[Fog02] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[Lot02] M. Lothaire. *Algebraic combinatorics on words.* Cambridge University Press, 2002.

[Lot05] M. Lothaire. *Applied combinatorics on words.* Cambridge University Press, 2005.

[NR03] B. Nouvel and E. Rémila. On colorations induced by discrete rotations. In *DGCI*, number 2886 in LNCS, pages 174–183, 2003.

[NR04] B. Nouvel and E. Rémila. Characterization of bijective discretized rotations. In *International Workshop on Combinatorial Images Analysis, 10th International Conference, IWCIA 2004, Auckland, New Zealand, December 1-4, 2004*, number 3322 in LNCS, 2004.

[NR05] B. Nouvel and E. Rémila. Configurations induced by discrete rotations: Periodicity and quasiperiodicity properties. *Discrete Applied Mathematics*, 2-3(147):325–343, 2005.

[Rau82] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.