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TILINGS ASSOCIATED WITH BETA-NUMERATION AND SUBSTITUTIONS

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Abstract

This paper surveys different constructions and properties of some multiple tilings (that is, finite-to-one coverings) of the space that can be associated with beta-numeration and substitutions. It is indeed possible, generalizing Rauzy's and Thurston's constructions, to associate in a natural way either with a Pisot number β (of degree d) or with a Pisot substitution σ (on d letters) some compact basic tiles that are the closure of their interior, that have non-zero measure and a fractal boundary; they are attractors of some graph-directed Iterated Function System. We know that some translates of these prototiles under a Delone set Γ (provided by β or σ) cover $\mathbb{R}^{d-1}$; it is conjectured that this multiple tiling is indeed a tiling (which might be either periodic or self-replicating according to the translation set Γ). This conjecture is known as the *Pisot conjecture* and can also be reformulated in spectral terms: the associated dynamical systems have pure discrete spectrum. We detail here the known constructions for these tilings, their main properties, some applications, and focus on some equivalent formulations of the Pisot conjecture, in the theory of quasicrystals for instance. We state in particular for Pisot substitutions a finiteness property analogous to the well-known (F) property in beta-numeration, which is a sufficient condition to get a tiling.

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Introduction

Substitutions in numeration Substitutions are combinatorial objects (one replaces a letter by a word) which produce sequences by iteration and generate simple symbolic dynamical systems with zero entropy. These systems, produced by this elementary algorithmic process, have a highly ordered self-similar structure. Substitutions occur in many mathematical fields (combinatorics on words [Lot02], ergodic theory and spectral theory [Que87, Fog02, Sol92, Sol97, DHS99], geometry of tilings [Ken96, Rob04], Diophantineapproximation and transcendence [ABF04, Roy04, AS02]), as well as in theoretical computer science [BP97, Lot05] or physics [BT86, LGJJ93, VM00, VM01].

The connections with numeration systems are numerous (see for instance [Dur98a, Dur98b, Fab95]) and natural: one can define a numeration system based on finite factors of an infinite word generated by a primitive substitution σ, known as the Dumont-Thomas numeration [DT89, DT93, Rau90]; this numeration system provides generalized radix expansions of real numbers with digits in a finite subset of the number field $\mathbb{Q}(\beta)$, β being the Perron-Frobenius eigenvalue of σ. The analogy between substitutions and beta-numeration is highlighted by the work of Thurston [Thu89], where tilings associated with beta-substitutions are introduced; a characteristic example is given by the Fibonacci substitution $1 \mapsto 12,\ 2 \mapsto 1$ and by the Fibonacci numeration, where nonnegative integers are represented thanks to the usual Fibonacci recurrence with digits in $\{0,1\}$ and no two one's in a row allowed; in this case, the Perron-Frobenius eigenvalue is equal to the golden ratio $\frac{1/\sqrt{5}}{2}$.

Tribonacci substitution In the so-called Tribonacci substitution case $1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$, nonnegative integers are expanded thanks to the Tribonacci recurrence $T_{n+3} = T_{n+2}+T_{n+1}+T_n$ with digits in $\{0,1\}$ and no three consecutive one's; this numeration called the Tribonacci numeration belongs to the family of beta-numerations [Rén57, Par60] (more details are to be found in Section 1.1).

Let β be the root larger than 1 of the polynomial $X^3 - X^2 - X - 1$, and let α be one of the two complex conjugate roots; one has $|\alpha| < 1$. The algebraic integer β is a Pisot number, that is, all its algebraic conjugates have modulus less than one. The set of complex numbers of the form

$$\mathcal{T}_\beta = \{\sum_{i\geq 0} w_i\alpha^i;\ \forall i,\ w_i \in \{0,1\},\ w_iw_{i+1}w_{i+2} = 0\} \tag{1}$$

is a compact subset of $\mathbb{C}$ called the central tile or the Rauzy fractal. This set was introduced in [Rau82, Rau88], see also [IK91, Mes98, Mes00]. It is shown in Fig. 1 with its division into three basic tiles $\mathcal{T}_\beta(i)$, $i = 1,2,3$, indicated by different shades. They correspond respectively to the sequences $(w_i)_{i\geq 0}$ such that either $w_0 = 0$, or $w_0w_1 = 10$, or $w_0w_1 = 11$. One interesting feature of the central tile is that it can tile the plane in

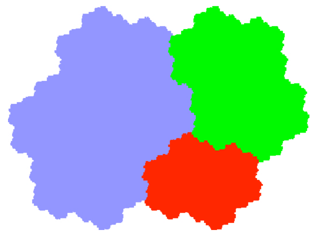

Figure 1: Tribonacci substitution: the central tile divided into its basic tiles.

two different ways. These two tilings are depicted in Fig. 2. The first one corresponds to a periodic tiling (a lattice tiling), and the second one to a self-replicating tiling.

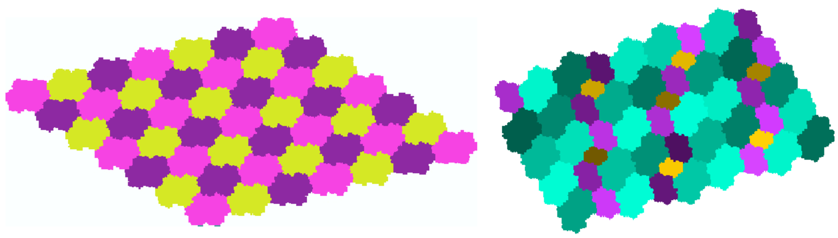

Figure 2: Lattice and self-replicating Tribonacci tilings.

By tiling, we mean here tilings by translation having finitely many tiles up to translation (a tile is assumed to be the closure of its interior): there exists a finite set of tiles $\mathcal{T}_i$ and a finite number of translation sets Γ_i such that $\mathbb{R}^d = \cup_i \cup_{\gamma_i \in \Gamma_i} \mathcal{T}_i + \gamma_i$, and distinct translates of tiles have non-intersecting interiors; we assume furthermore that each compact set in $\mathbb{R}^d$ intersects a finite number of tiles; the sets Γ_i of translation vectors are thus assumed to be Delaunay sets. See for instance [Ken90, Ken96, Ken99, Rad95, Rob96]. By *multiple tiling*, we mean according for instance to [LW03], arrangements of tiles in $\mathbb{R}^d$ such that almost all points in $\mathbb{R}^d$ are covered exactly p times for some positive integer p.

The basic tiles are attractors for a graph-directed Iterated Function System (IFS), in the flavor of [LW96, MW88, Vin00]: indeed one has

$$\begin{cases} \mathcal{T}_\beta(1) = \alpha(\mathcal{T}_\beta(1) \cup \mathcal{T}_\beta(2) \cup \mathcal{T}_\beta(3)) \\ \mathcal{T}_\beta(2) = \alpha(\mathcal{T}_\beta(1)) + 1 \\ \mathcal{T}_\beta(3) = \alpha(\mathcal{T}_\beta(2)) + 1. \end{cases}$$

Hence each basic tile can be mapped onto a finite union of translates of basic tiles, when multiplied by the parameter α^{-1}. The maps in the IFS are contractive, hence the compact non-empty sets satisfying this equation are uniquely determined [MW88]; they have non-zero measure and are the closure of their interior [SW02]. By using this IFS, that is, by expanding each basic tile by α^{-1} and subdividing the result, one generates the self-replicating multiple tiling (in the sense of [KV98]), that is proved to be aperiodic and repetitive, i.e., any finite collection of tiles up to translation reoccurs in the tiling at a bounded distance from any point of the tiling. Repetitivity, also called quasiperiodicity, or uniform recurrence, is equivalent with the minimality of the tiling dynamical system [Sol97]. For more details, we refer the reader to Section 1.4, 1.5 and 2.5. Let us note that the subdivision rule in the IFS is closely connected to the substitution σ: indeed the subdivision matrix is the transpose of the incidence matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, which counts the number of occurrences of letters in the images of the letters of the substitution.

The Tribonacci lattice tiling has been widely studied and presents many interesting features. One interpretation of this tiling is that the symbolic dynamical system generated by the Tribonacci substitution (or equivalently the β-shift endowed with the odometer map, with $\beta > 1$ root of $X^3 - X^2 - X - 1$) is measure-theoretically isomorphic to a translation of the torus $\mathbb{T}^2$, the isomorphism being a continuous onto map [Rau82]: the symbolic dynamical system has thus pure discrete spectrum. Furthermore, the Tribonacci central tile has a "nice" topological behavior (0 is an inner point and it is shown to be connected with simply connected interior [Rau82]), which leads to interesting applications in Diophantine approximation [CHM01]. For more details, see Chap. 10 in [Lot05].

Central tiles More generally, it is possible to associate a central tile with any Pisot unimodular substitution [AI01, CS01a] or to beta-shifts with β Pisot unit [Aki98, Aki99, AS98, Aki00, Aki02, AN04b]. A substitution is said Pisot unimodular if its incidence matrix admits as characteristic polynomial the minimal polynomial of a Pisot unit. Let us note that not all central tiles associated with Pisot numbers need to satisfy the same topological properties as the Tribonacci tile does: they might be not connected or not simply connected, and 0 is not always an inner point of the central tile; see for instance, the examples given in [Aki02].

There are mainly two methods of construction for central tiles, as illustrated above with the Tribonacci substitution. The first one is based on formal power series seen as digit expansions such as (1), and is inspired by the seminal paper [Rau82]; see e.g., [Aki99, Aki98, Aki02, CS01a, CS01b, Sie03, Mes98, Mes00]. The second approach via Iterated Function Systems and generalized substitutions has been developed following ideas from [Thu89, IK91], and [AI01, SAI01]: central tiles are described as attractors of some graph-directed IFS, as developed in [HZ98, Sir00a, Sir00b, SW02]. For more details on both approaches, see Chap. 7 and 8 in [Fog02]. We try as best as possible to give a combination of both approaches in the present paper.

The Pisot conjecture Several multiple tilings can be associated with the central tile. In the beta-numeration case, a first multiple tiling can be defined from which one can recover a self-replicating tiling, up to a division of its tiles into basic tiles (see Section 1). Both a self-replicating (see Section 2) and a lattice multiple tiling (see Section 3) with analogous tiles can also be introduced for Pisot substitution dynamical systems. These multiple tilings are in particular proved to be tilings in the Tribonacci case. Furthermore, as soon as one of those multiple tilings can be proved to be a tiling, then all the other multiple tilings are also indeed tilings [IR06]. Hence the Pisot conjecture states that as soon as β is a Pisot number, then all the multiple tilings are tilings.

Let us note that in each case sufficient conditions for tilings exist; for more details, see Section 1.6 and 5. The most simple one is the (F) condition, the so-called finiteness condition: all the nonnegative elements of $\mathbb{Z}[1/\beta]$ are assumed to have a finite β-expansion. One of the main purpose of this paper is to provide a property analogous to the (F) condition stated for Pisot unimodular substitutions.

Contents of the paper We have chosen to handle both the beta-numeration and the substitutive cases for the following reasons. First, the literature on both subjects is often scattered among several series of papers, some of them dealing with beta-numeration (or analogously with Canonical Number systems, and Shift Radix Systems), other ones with symbolic dynamics and substitutions; a third group also deals with the interplay between the Pisot conjecture and spectral properties of tilings and quasicrystals. Second, the beta-shift is a rather natural framework for the introduction and motivation of the required algebraic formalism which is somehow heavier in the substitutive case. Lastly, the methods and motivations are very close and bring mutually insight on the subject.

We introduce in Section 1 the tiles and the multiple tiling associated with the beta-numeration when β is assumed to be a unit Pisot number. The basic tiles are shown to satisfy an IFS, the subdivision matrix of which is equal to the transpose of the adjacency matrix of the minimal automaton which recognizes the language of the beta-numeration; this is the core of Theorem 2. We then generalize this situation to substitutions. Although the lattice multiple tiling has been more often considered in the literature for substitutive dynamical systems, we then chose to focus in Section 2 on the self-replicating multiple tiling by following the same scheme and formalism as in the beta-numeration case, in order to unify both approaches. We introduce for that purpose in Section 2.4 a numeration system, the Dumont-Thomas numeration system [DT89, DT93, Rau90], based on the substitution, which allows us to expand real numbers. The lattice multiple tiling has an interesting dynamical and spectral interpretation that we develop in Section 3. We focus on the connections with mathematical quasicrystals and model sets in Section 3.4. Section 4 is devoted to the links between the self-replicating multiple tiling and discrete geometry, more precisely, standard discrete planes. The point of view on the substitution self-replicating multiple tiling developed in Section 2 (that does not appear to our knowledge stated as such in the literature), allows us to introduce in Section 4.2 the substitutive counter-part to the well-known (F) property [FS92], which is a useful sufficient condition for tiling. We then discuss the Pisot conjecture and formulate several equivalent statements as well as sufficient conditions for tiling in Section 5.

1. Beta-numeration

Let us define now the central tile associated with the beta-numeration; we recall all the required background on the beta-shift in Section 1.1; we then introduce the definition of the central tile and of its translation vectors set in Section 1.2 and 1.3; the multiple self-replicating tiling is defined in Section 1.4; we then work out the Tribonacci example in Section 1.5., and conclude this section by evoking some particular finiteness properties that imply that the multiple tiling is indeed a tiling. We mainly follow here [Thu89] and [Aki98, Aki99, Aki02].

1.1. Beta-shift

Let $\beta > 1$ be a real number. The (Renyi) *β-expansion* [Rén57, Par60] of a real number $x \in [0,1]$ is defined as the sequence $(x_i)_{i\geq 1}$ with values in $\mathcal{A}_\beta := \{0, 1, \ldots, \lceil\beta\rceil - 1\}$ produced by the β-transformation $T_\beta :\ x \mapsto \beta x \pmod 1$ as follows

$$\forall i \geq 1,\ u_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor, \text{ and thus } x = \sum_{i\geq 1} u_i \beta^{-i}.$$

We denote the β-expansion of 1 by $d_\beta(1) = (t_i)_{i\geq 1}$. Numbers β such that $d_\beta(1)$ is ultimately periodic are called *Parry numbers* and those such that $d_\beta(1)$ is finite are called *simple Parry numbers* (in this latter case, we omit the ending zero's when writing $d_\beta(1)$).

We assume throughout this paper that β is an algebraic number. The algebraic integer β is said to be a *Pisot number* if all its algebraic conjugates have modulus less than 1: under this assumption, then β is either a Parry number or a simple Parry number [BM86]; more generally, every element in $\mathbb{Q}(\beta) \cap [0,1]$ has eventually periodic β-expansion according to [BM86, Sch80]. But conversely, if β is a Parry number we can only say that β is a *Perron number*, that is, an algebraic integer greater than 1 all conjugates of which have absolute value less than that number [Lin84, DCK76]. Indeed, as quoted in [Bla89], there exist Parry numbers which are neither Pisot nor even Salem; consider e.g., $\beta^4 = 3\beta^3 + 2\beta^2 + 3$ with $d_\beta(1) = 3203$; a *Salem number* is a Perron number, all conjugates of which have absolute value less than or equal to 1, and at least one has modulus 1. It is conjectured in [Sch80] that every Salem number is a Parry number. This conjecture is sustained by the fact that if each rational in $[0,1)$ has an ultimately periodic β-expansion, then β is either a Pisot or a Salem number. In particular, it is proved in [Boy89] that if β is a Salem number of degree 4, then β is a Parry number; see [Boy96] for the case of Salem numbers of degree 6. Note that the algebraic conjugates of a Parry number $\beta > 1$ are smaller than $\frac{1+\sqrt{5}}{2}$ in modulus, this upper bound being sharp [FLP94, Sol94].

Combinatorial characterization of β-expansions. We suppose that β is a Parry number. Let $d^*_\beta(1) = d_\beta(1)$, if $d_\beta(1)$ is infinite, and $d^*_\beta(1) = (t_1 \ldots t_{n-1}(t_n - 1))^\infty$, if

$d_\beta(1) = t_1 \dots t_{n-1} t_n$ is finite ($t_n \neq 0$). The set of β-expansions of real numbers in $[0, 1)$ is exactly the set of sequences $(u_i)_{i\geq 1}$ in $\mathcal{A}_\beta^{\mathbb{N}^+}$ (where $\mathbb{N}^+$ stands for the set of positive integers) such that

$$\forall k \geq 1,\ (u_i)_{i\geq k} <_{\text{lex}} d^*_\beta(1). \tag{2}$$

For more details on the β-numeration, see for instance [Bla89, Fro00, Fro02].

The β-shift. It is natural to introduce the following symbolic dynamical system known as the (right one-sided) *β-shift* (X^r_β, S) which is defined as the closure in $\mathcal{A}_\beta^{\mathbb{N}^+}$ of the set of β-expansions of real numbers in $[0, 1)$ on which the shift map S acts; let us recall that S maps the sequence $(y_i)_{i\in\mathbb{N}^+}$ onto $(y_{i+1})_{i\in\mathbb{N}^+}$. Hence X^r_β is equal to the set of sequences $(u_i)_{i\geq 1} \in \mathcal{A}_\beta^{\mathbb{N}^+}$ which satisfy

$$\forall k \geq 1,\ (u_i)_{i\geq k} \leq_{\text{lex}} d^*_\beta(1). \tag{3}$$

We can easily extend this admissibility condition to two-sided sequences and introduce the *two-sided symbolic β-shift* (X_β, S) (the shift map S maps now the sequence $(y_i)_{i\in\mathbb{Z}}$ onto $(y_{i+1})_{i\in\mathbb{Z}}$). The set X_β is then defined as the set of two-sided sequences $(y_i)_{i\in\mathbb{Z}}$ in $\mathcal{A}_\beta^{\mathbb{Z}}$ such that each left truncated sequence is less than or equal to $d^*_\beta(1)$, that is, $\forall k \in \mathbb{Z},\ (y_i)_{i\geq k} \leq_{\text{lex}} d^*_\beta(1)$.

We will use the following notation for the elements of X_β: if $y = (y_i)_{i\in\mathbb{Z}} \in X_\beta$, we set $u = (u_i)_{i\geq 1} = (y_i)_{i\geq 1}$ and $w = (w_i)_{i\geq 0} = (y_{-i})_{i\geq 0}$. One thus gets a two-sided sequence of the form

$$\dots w_3 w_2 w_1 w_0, u_1 u_2 u_3 \dots$$

and write it as $y = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) = (w, u)$. In other words, we will use the letters (w_i) for denoting the "past" and (u_i) for the "future" of the element $y = (w, u)$ of the two-sided shift X_β.

One similarly defines X^l_β as the set of one-sided sequences $w = (w_i)_{i\geq 0}$ such that there exists $u = (u_i)_{i\geq 1}$ with $(w, u) \in X_\beta$. We call it the *left one-sided β-shift*.

Sofic shift and language F_β. The β-shifts associated with Parry numbers have interesting combinatorial properties: indeed, (X_β, S) is *sofic* (that is, the set of finite factors of the sequences in X_β can be recognized by a finite automaton) if and only if β is a Parry number (simple or not) [BM86].

The minimal automaton $\mathcal{M}_\beta$ recognizing the *language* F_β, defined as the set of finite factors of the sequences in X_β, can easily be constructed (see Figure 3). The number of states n of this automaton is equal to the length of the period n of $d^*_\beta(1)$ if β is a simple Parry number with $d_\beta(1) = t_1 \dots t_{n-1} t_n$, $t_n \neq 0$, and to the sum n of its preperiod m plus its period p, if β is a non-simple Parry number with $d_\beta(1) = t_1 \dots t_m (t_{m+1} \dots t_{m+p})^\infty$ ($t_m \neq t_{m+p}$, $t_{m+1} \cdots t_{m+p} \neq 0^p$).

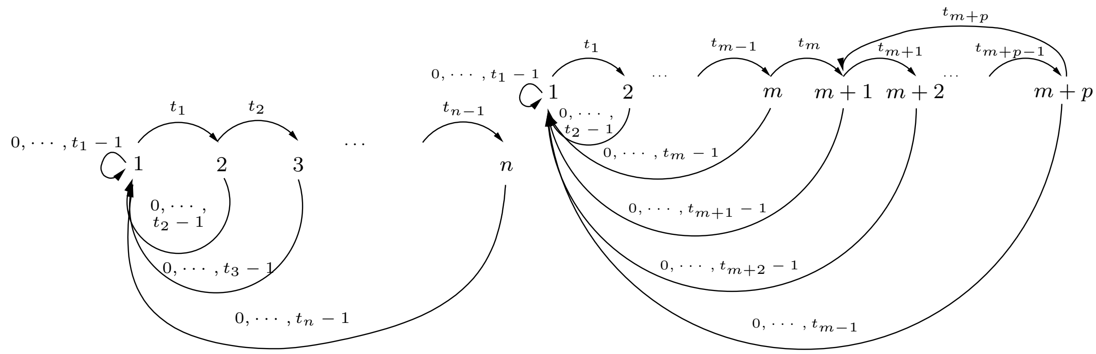

Figure 3: The automata $\mathcal{M}_\beta$ for β simple Parry number ($d_\beta(1) = t_1 \dots t_{n-1} t_n$) and for β non-simple Parry number ($d_\beta(1) = t_1 \dots t_m (t_{m+1} \dots t_{m+p})^\infty$).

1.2. A tiling of the line

We can define the β-expansion of a real number x greater than 1 as follows: let $k \in \mathbb{N}$ such $\beta^k \leq x < \beta^{k+1}$; one has $0 \leq \frac{x}{\beta^{k+1}} < 1$; we denote the β-expansion of $\frac{x}{\beta^{k+1}}$ by $u = (u_i)_{i \geq 1}$; the β-expansion of x is then defined as the sequence $(\dots 000 u_1 u_2 \dots u_{k+1}, u_{k+2} u_{k+3} \dots)$. We thus can associate with any positive real number a two-sided sequence in X_β which corresponds to its β-expansion (the converse being obviously untrue).

The sets Fin(β) and $\mathbb{Z}_\beta^+$. The *β-fractional part* of the positive real number x with β-expansion $(w, u) \in X_\beta$ is defined as the sequence u; it is said to be *finite* if the sequence u takes ultimately only zero values.

We denote the set of positive real numbers having a finite β-fractional part by Fin(β), and the set of positive real numbers which have a zero fractional part in their β-expansion by $\mathbb{Z}_\beta^+ \subset \mathrm{Fin}(\beta)$, that is,

$$\begin{aligned}
\mathrm{Fin}(\beta) &= \{w_M \beta^M + \cdots + w_0 + u_1 \beta^{-1} + \cdots + u_L \beta^{-L}; \\
&\qquad\qquad M \in \mathbb{N},\ (w_M \cdots w_0 u_1 \dots u_L) \in F_\beta\}, \\
\mathbb{Z}_\beta^+ &= \{w_M \beta^M + \cdots + w_0;\ M \in \mathbb{N},\ (w_M \cdots w_0) \in F_\beta\} \subset \mathrm{Fin}(\beta).
\end{aligned}$$

A tiling of the line. There is a tiling of the line that can be naturally associated with the β-numeration: let us place on the nonnegative half line the points of $\mathbb{Z}_\beta^+$; under the assumption that β is a Parry number, one gets a tiling by intervals that take a finite number of lengths. Indeed we define the successor map Succ : $\mathbb{Z}_\beta^+ \to \mathbb{Z}_\beta^+$ as the map which sends to an element x of $\mathbb{Z}_\beta^+$ the smallest element of $\mathbb{Z}_\beta^+$ strictly larger than x. When β is a Parry number, the set of values taken by $\mathrm{Succ}(x) - x$ on $\mathbb{Z}_\beta^+$ is finite and equal to $1, \beta - t_1, \beta^2 - t_1\beta - t_2, \dots, \beta^{n-1} - t_1 \beta^{n-2} - \cdots - t_{n-1}$, if $d_\beta(1) = t_1 \dots t_{n-1} t_n$ is finite ($t_n \neq 0$), and $1, \beta - t_1, \beta^2 - t_1\beta - t_2, \cdots, \beta^{m+p-1} - t_1 \beta^{m+p-2} - \cdots - t_{m+p-1}$ if $d_\beta(1) = t_1 \dots t_m (t_{m+1} \dots t_{m+p})^\infty$, $t_m \neq t_{m+p}$, $t_{m+1} \dots t_{m+p} \neq 0$, according to [Thu89], that is, to $\{T^i(\beta);\ 0 \leq i \leq n-1\}$, where $n = m + p$. We indeed divide $\mathbb{Z}_\beta^+$ according to the state *reached* in the automaton $\mathcal{M}_\beta$ when feeding the automaton by the digits of the elements of $\mathbb{Z}_\beta^+$ read from left to right, that is, the most significant digit first. This tiling of the line has many interesting features when β is furthermore assumed to be Pisot. Let us first recall a few definitions issued from the mathematical theory of quasicrystals.

Definition 1 *A set $X \subset \mathbb{R}^n$ is said to be* uniformly discrete *if there exists a positive real number r such that for any $x \in X$, the open ball located at x of radius r contains* at most *one point of X; a set $X \subset \mathbb{R}^n$ is said* relatively dense *it there exists a positive real number R such that for any x in $\mathbb{R}^n$, the open ball located at x of radius R contains* at least *one point of X.*

A subset of $\mathbb{R}^n$ is a Delaunay set *if it is uniformly discrete and relatively dense. A Delaunay set is a* Meyer set *if $X - X$ is also a Delaunay set.*

We deduce from the above results that if β is a Parry number, then $\pm\mathbb{Z}_\beta^+$ is a Delaunay set. More can be said when β is a Pisot number.

Proposition 1 ([BFGK98, VGG04]) *When β is a Pisot number, then $\pm\mathbb{Z}_\beta^+$ is a Meyer set.*

Proof. Since $\pm\mathbb{Z}_\beta^+$ is relatively dense, we first deduce that $(\pm\mathbb{Z}_\beta^+)-(\pm\mathbb{Z}_\beta^+)$ is also relatively dense. Now if S is a finite subset of $\mathbb{Z}$, then $\{P(\beta);\ P \in S[X]\}$ is easily seen to be a discrete set ([Sol97], Lemma 6.6): indeed $P(\beta)$ is an algebraic integer for any polynomial P with coefficients in $\mathbb{Z}$; furthermore, since β is assumed to be Pisot, there exists C such that $|P_1(\beta^{(i)}) - P_2(\beta^{(i)})| \leq C$ for any algebraic conjugate $\beta^{(i)}$ (distinct from β), and for $P_1, P_2 \in S[X]$; since $\prod_i (P_1 - P_2)(\beta^{(i)})(P_1 - P_2)(\beta) \in \mathbb{Z}$ and is non-zero for $P_1, P_2 \in S[X]$ with $P_1(\beta) \neq P_2(\beta)$, we deduce a positive uniform lower bound for $P_1(\beta) - P_2(\beta)$ with $P_1(\beta) \neq P_2(\beta)$, which is sufficient to conclude that $\{P(\beta);\ P \in S[X]\}$ is a discrete set.■

A Meyer set [Mey92, Mey95] is a mathematical model for quasicrystals [Moo97, BM00]; indeed a Meyer set is also equivalently defined as a Delaunay set for which there exists a finite set F such that $X - X \subset X + F$ [Mey92, Moo97]; this endows a Meyer set with a structure of "quasi-lattice": Meyer sets play indeed the role of the lattices in the theory of crystalline structure. For some families of β (mainly Pisot quadratic units), an internal law can even be produced formalizing this quasi-stability under subtraction and multiplication [BFGK98]. Beta-numeration reveals itself as a very efficient and promising tool for the modeling of families of quasicrystals thanks to beta-grids [BFGK98, BFGK00, VGG04].

Let us note that that Proposition 1 is proved in [VGG04] by exhibiting a cut-and-project scheme. A *cut and project scheme* consists of a direct product $\mathbb{R}^k \times H$, $k \geq 1$, where H is a locally compact abelian group, and a lattice D in $\mathbb{R}^k \times H$, such that with respect to the natural projections $p_0:\ \mathbb{R}^k \times H \to H$ and $p_1:\ \mathbb{R}^k \times H \to \mathbb{R}^k$:

1. $p_0(D)$ is dense in H;

2. p_1 restricted to D is one-to-one onto its image $p_1(D)$.

This cut and project scheme is denoted by $(\mathbb{R}^k \times H, D)$. A subset Γ of $\mathbb{R}^k$ is a *model set* if there exists a cut and project scheme $(\mathbb{R}^k \times H, D)$ and a relatively compact set (i.e., a set the closure of which is compact) Ω of H with non-empty interior such that

$$\Gamma = \{p_1(P);\ P \in D,\ p_0(P) \in \Omega\}.$$

The set Γ is called the *acceptance window* of the cut and project sheme. Meyer sets are proved to be the subsets of model set of $\mathbb{R}^k$, for some $k \geq 1$, that are relatively dense [Mey92, Mey95, Moo97]. For more details, see for instance [BM00, LW03, Sen95]. We detail in Section 3.4 connections between such a generation process for quasicrystals and lattice tilings.

An important issue is to characterize those β for which $\pm\mathbb{Z}_\beta^+$ is uniformly discrete or even a Meyer set. Observe that $\mathbb{Z}_\beta^+$ is at least always a discrete set. It can easily be seen that $\pm\mathbb{Z}_\beta^+$ is uniformly discrete if and only if the β-shift X_β is *specified*, that is, if the strings of zeros in $d_\beta(1)$ have bounded lengths; let us observe that the set of specified real numbers $\beta > 1$ with a noneventually periodic $d_\beta(1)$ has Hausdorff dimension 1 according to [Sch97]; for more details, see for instance [Bla89] and the discussion in [VGG04]. Let us note that if $\pm\mathbb{Z}_\beta^+$ is a Meyer set, then β is a Pisot or a Salem number [Mey95].

1.3. Geometric representation

The right one-sided shift X_β^r admits the interval $[0,1]$ as a natural geometric representation; namely, one associates with a sequence $(u_i)_{i\geq 1} \in X_\beta^r$ its real value $\sum_{i\geq 1} u_i\beta^{-i}$. We even have a measure-theoretical isomorphism between X_β^r endowed with the shift, and $[0,1]$ endowed with the map T_β. We want now to give a similar geometric interpretation of the set X_β^l as the central tile (defined as an explicit compact set in the product of Euclidean spaces) of a self-replicating multiple tiling. We first need to introduce some algebraic formalism in order to embed $\mathbb{Z}_\beta^+$ in a hyperplane spanned by the algebraic conjugates of β; the closure of the "projected" points will be defined as the central tile. Let us note that we shall give a geometric interpretation of this projection process in Section 3, and a geometric representation of the whole two-sided shift X_β in Section 5.1.

Canonical embedding. We denote the real conjugates of β by $\beta^{(2)}, \ldots, \beta^{(r)}$, and its complex conjugates by $\beta^{(r+1)}, \overline{\beta^{(r+1)}}, \ldots, \beta^{(r+s)}, \overline{\beta^{(r+s)}}$. Let d be the degree of β. One has $d = r + 2s$. We set $\beta^{(1)} = \beta$. Let $\mathbb{K}^{(k)}$ be equal to $\mathbb{R}$ if $1 \leq k \leq r$, and to $\mathbb{C}$, if $k > r$. We furthermore denote by $\mathbb{K}_\beta$ the *representation space*

$$\mathbb{K}_\beta := \mathbb{R}^{r-1} \times \mathbb{C}^s \simeq \mathbb{R}^{d-1}.$$

For $x \in \mathbb{Q}(\beta)$ and $1 \leq i \leq r$, let $x^{(i)}$ be the conjugate of x in $\mathbb{K}^{(i)}$. Let us consider now the following algebraic embeddings:

- The *canonical embedding* on $\mathbb{Q}(\beta)$ maps a polynomial to all its conjugates
$\Phi_\beta : \mathbb{Q}(\beta) \to \mathbb{K}_\beta,\ P(\beta) \mapsto (P(\beta^{(2)}), \dots, P(\beta^{(r)}), P(\beta^{(r+1)}), \dots, P(\beta^{r+s}))$.

- The series $\lim_{n\to+\infty} \Phi_\beta(\sum_{i=0}^{n} w_i\beta^i) = \sum_{i\geq 0} w_i \Phi_\beta(\beta^i)$ are convergent in $\mathbb{K}_\beta$ for every $(w_i)_{i\geq 0} \in X_\beta^l$. The *representation map of* X_β^l is then defined as
$$\varphi_\beta :\ X_\beta^l \to \mathbb{K}_\beta,\ (w_i)_{i\geq 0} \mapsto \lim_{n\to+\infty} \Phi_\beta(\textstyle\sum_{i=0}^{n} w_i\beta^i).$$

Note that the map φ_β is continuous, hence the image of a closed set in X_β^l, which thus is compact, is again a compact set. In particular, $\overline{\Phi_\beta(\mathbb{Z}_\beta^+)} = \varphi_\beta(X_\beta^l)$.

Definition 2 *We define the* central tile $\mathcal{T}_\beta$ *as*

$$\mathcal{T}_\beta = \overline{\Phi_\beta(\mathbb{Z}_\beta^+)} = \varphi_\beta(X_\beta^l).$$

There is a natural decomposition of $\mathbb{Z}_\beta^+$ according to the values taken by the function $x \mapsto \mathrm{Succ}(x) - x$. By definition, a sequence $w = (w_j)_{j\in\mathbb{N}} \in X_\beta^l$ is the label of an *infinite left-sided path* in $\mathcal{M}_\beta$, that is, there exists an infinite sequence of states $(q_j)_{j\in\mathbb{N}}$ of the automaton $\mathcal{M}_\beta$ such that for all j, w_j is the label of an edge from state q_{j+1} to q_j; it is said to *arrive at state* i if $q_0 = i$. Let n be the number of states in the minimal automaton $\mathcal{M}_\beta$. For any $1 \leq i \leq n$, then $\mathrm{Succ}(x) - x = T_\beta^{i-1}(1)$ if and only if the last state read is the state i, as labeled on the graphs depicted in Figure 3. Hence the central tile can be naturally divided into n pieces, called *basic tiles*, as follows for $1 \leq i \leq n$:

$$\begin{aligned}\mathcal{T}_\beta(i) &= \overline{\Phi_\beta\left(\{x \in \mathbb{Z}_\beta^+;\ \mathrm{Succ}(x) - x = T^{i-1}(\beta)\}\right)} \\ &= \varphi_\beta(\{w \in X_\beta^l;\ w \text{ is the label of an infinite left-sided path in } \mathcal{M}_\beta \\ &\qquad \text{arriving at state } i\}).\end{aligned}$$

1.4. The self-replicating multiple tiling

We assume in the remaining of this section that β is a Pisot number. In order to be able to cover $\mathbb{K}_\beta$ by translates of the basic tiles according to a Delaunay translation set, we would like to consider a set the image of which by Φ_β is dense in $\mathbb{K}_\beta$, without being too large: a good candidate is the set $\mathbb{Z}[\beta]_{\geq 0}$ of nonnegative real numbers in $\mathbb{Z}[\beta]$. Indeed it is proved in [Aki99] (Proposition 1) that $\Phi_\beta(\mathbb{Z}[\beta]_{\geq 0})$ is dense in $\mathbb{K}_\beta$, the proof being based on Kronecker's approximation theorem. According to [Aki02], we introduce the (countable) set $\mathrm{Frac}(\beta) \subset X_\beta^r$ defined as the set of β-expansions of real numbers in $\mathbb{Z}[\beta] \cap [0, 1)$,

$$\mathrm{Frac}(\beta) = \{d_\beta(x),\ x \in \mathbb{Z}[\beta] \cap [0, 1)\} \subset X_\beta^r.$$

Let $u = (u_i)_{i\geq 1} \in \mathrm{Frac}(\beta)$. By definition of $\mathrm{Frac}(\beta)$, we have $\sum_{i\geq 1} u_i\beta^{-i} \in \mathbb{Q}(\beta)$; hence we can apply Φ_β to $\sum_{i\geq 1} u_i\beta^{-i}$. We define the tile $\mathcal{T}_u$ as

$$\mathcal{T}_u :\ = \overline{\Phi_\beta(\{W_M\beta^M + \cdots + W_0 + u_1\beta^{-1} + \cdots + u_l\beta^{-L} + \cdots;\ (\cdots 000W_M \cdots W_0, u) \in X_\beta\})}.$$

An immediate consequence is

$$\mathcal{T}_u \;\; = \Phi_\beta(\textstyle\sum_{i\geq 1} u_i\beta^{-i}) + \varphi_\beta(\{w \in X_\beta^l; (w,u) \in X_\beta\}).$$

Hence the tiles $\mathcal{T}_u$ are finite unions of translates of the basic tiles $\mathcal{T}_\beta(i)$ for $1 \leq i \leq n$ by considering the minimal automaton $\mathcal{M}_\beta$; furthermore, it is proved in [Aki02] that there are precisely n such tiles up to translation.

Theorem 1 ([Aki02, Aki99]) *We assume that β is a Pisot unit. The set*

$$\Gamma_\beta := \Phi_\beta\left(\left\{\textstyle\sum_{i\geq 1} u_i\beta^{-i};\; u \in \mathrm{Frac}(\beta)\right\}\right) = \Phi_\beta\left(\mathbb{Z}[\beta] \cap [0,1)\right)$$

is a Delaunay set. The (up to translation finite) set of tiles $\mathcal{T}_u$, for $u \in \mathrm{Frac}(\beta)$, covers $\mathbb{K}_\beta$, that is,

$$\mathbb{K}_\beta = \bigcup_{u\in\mathrm{Frac}(\beta)} \mathcal{T}_u = \bigcup_{1\leq i\leq n} \;\; \bigcup_{\substack{u=(u_j)_{j\in\mathbb{N}}\in\mathrm{Frac}(\beta),\\ \gamma=\Phi_\beta(\sum_{j\geq 1} u_j\beta^{-j}),\\ u \text{ is the label of an infinite}\\ \text{left-sided path in } \mathcal{M}_\beta \text{ arriving at state } i}} \mathcal{T}_\beta(i) + \gamma. \qquad (4)$$

For each u, the tile $\mathcal{T}_u$ has a non-empty interior; hence it has non-zero measure.

Proof. We follow here mainly [Aki99]. Let us prove that the set of translation vectors Γ_β is uniformly discrete. For that purpose, it is sufficient to prove that for a given norm $||\ ||$ in $\mathbb{K}_\beta$, and for any constant $C > 0$, there are only finitely many differences of elements γ, γ' in Γ_β such that $||\Phi_\beta(\gamma - \gamma')(x)|| < C$. This latter statement is a direct consequence of the fact that if $x \in \mathbb{Z}[\beta]$, then x is an algebraic integer, and that there exist only finitely many algebraic integers x in $\mathbb{Q}(\beta)$ such that $|x| < 1$ and $||\Phi_\beta(x)|| < C'$, since β is assumed to be a Pisot unit.

We now use the fact that $\Phi_\beta(\mathbb{Z}[\beta]_{\geq 0})$ is dense in $\mathbb{K}_\beta$ ([Aki99], Proposition 1). We first deduce that Γ_β is relatively dense. We then prove that one has the covering (4). Indeed, let $x \in \mathbb{K}_\beta$. There exists a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials in $\mathbb{Z}[X]$ with $P_n(\beta) \geq 0$, for all n, such that $(\Phi_\beta(P_n(\beta)))_n$ tends towards x. For all n, $\Phi_\beta(P_n(\beta)) \in \mathcal{T}_{u^{(n)}}$, where $u^{(n)}$ is the β-fractional part of $P_n(\beta)$. By uniform discreteness of Γ_β, there exist infinitely many n such that $u^{(n)}$ takes the same value, say, u. Since the tiles are closed, $x \in \mathcal{T}_u$. We now deduce from Baire's theorem that each tile has a non-empty interior. ∎

IFS structure. Let us prove now that our basic tiles are *graph-directed attractors* for a graph-directed self-affine Iterated Function System, according to the formalism of [MW88, LW96].

We denote the set of states of the minimal automaton $\mathcal{M}_\beta$ by $\mathcal{S}_\beta := \{1, \ldots, n\}$. The notation $a \mapsto_i b$ stands for the fact that there exists an arrow labeled by i (in $\mathcal{A}_\beta$) from a to b (with $a, b \in \mathcal{S}_\beta$) in the minimal automaton $\mathcal{M}_\beta$. We denote by $h_\beta :\ \mathbb{K}_\beta \to \mathbb{K}_\beta$ the β-multiplication map that multiplies the coordinate of index i by $\beta^{(i)}$, for $2 \leq i \leq d$.

Theorem 2 *Let β be a Parry number. The basic tiles of the central tile $\mathcal{T}_\beta$ are solutions of the following graph-directed self-affine Iterated Function System:*

$$\forall a \in \mathcal{S}_\beta,\ \mathcal{T}_\beta(a) = \bigcup_{b\in\mathcal{S}_\beta,\ i,\ b\mapsto_i a} h_\beta(\mathcal{T}_\beta(b)) + \Phi_\beta(i). \tag{5}$$

If β is assumed to be a Pisot unit, then the basic tiles have disjoint interiors and each basic tile is the closure of its interior. Furthermore, there exists an integer $k \geq 1$ such that the covering (4) is almost everywhere k-to-one. This multiple tiling is repetitive: any finite collection of tiles up to translation reoccurs in the tiling at a bounded distance from any point of the tiling.

Let us observe that the subdivision matrix of the IFS is the transpose of the adjacency matrix of $\mathcal{M}_\beta$, the entry (a, b) of which is equal to the number of edges i such that $a \mapsto_i b$. Let us note that this matrix is *primitive*, that is, it admits a power with only positive entries; indeed, $\mathcal{M}_\beta$ is both strongly connected and aperiodic, that is, the lengths of its cycles are relatively prime since there exists a cycle of length 1 at state 1, as labeled on Fig. 3.

Proof. The proof of (5) as well as the proof of the fact that the basic tiles are the closure of their interior is an adaption of [SW02] concerned with Pisot substitution dynamical systems. The disjointness of the interiors of the basic tiles follows [Hos92, AI01].

Let $a \in \mathcal{S}_\beta$ be given. Let $w = (w_k)_{k\geq 0} \in X^l_\beta$ such that w is the label of an infinite left-sided path in the automaton $\mathcal{M}_\beta$ arriving at state a. One has:

$$\begin{aligned} \Phi_\beta(w) &= \Phi_\beta\left(\textstyle\sum_{k\geq 1} w_k\beta^k\right) + \Phi_\beta(w_0) = h_\beta \circ \Phi_\beta\left(\textstyle\sum_{k\geq 1} w_k\beta^{k-1}\right) + \Phi_\beta(w_0) \\ &= h_\beta \circ \varphi_\beta((w_k)_{k\geq 1}) + \Phi_\beta(w_0). \end{aligned}$$

One deduces (5) by noticing that $(w_k)_{k\geq 1}$ is the label of an infinite left-sided path in $\mathcal{M}_\beta$ arriving at state b with $b \mapsto_{w_0} a$ in $\mathcal{M}_\beta$.

We assume that β is a Pisot unit. According to Theorem 1, each basic tile has a non-empty interior. The same reasoning as above shows that the interiors of the pieces satisfy the same IFS equation. We deduce from the uniqueness of the solution of the IFS [MW88] that each basic tile is the closure of its interior.

Take two distinct basic tiles, say, $\mathcal{T}_\beta(c)$ and $\mathcal{T}_\beta(c)$, with $c \neq d$. We denote the state 1 by a as labeled on Fig. 3. From the shape of $\mathcal{M}_\beta$, one deduces that $c \mapsto_0 a$, and $d \mapsto_0 a$. Hence both basic tiles $\mathcal{T}_\beta(c)$ and $\mathcal{T}_\beta(d)$ occur in (5) applied to the letter a, with the same translation term which is equal to $0 = \Phi_\beta(0)$. Let $\mu_{\mathbb{K}_\beta}$ stand for the Lebesgue measure on $\mathbb{K}_\beta$: for every Borel set B of $\mathbb{K}_\beta$, $\mu_{\mathbb{K}_\beta}(h_\beta(B)) = \frac{1}{\beta}\mu_{\mathbb{K}_\beta}(B)$, following for instance [Sie03]. One has, according to (5)

$$\begin{aligned} \mu_{\mathbb{K}_\beta}(\mathcal{T}_\beta(a)) &\leq \textstyle\sum_{b,b\mapsto_i a} \mu_{\mathbb{K}_\beta}(h_\beta(\mathcal{T}_\beta(b))) \\ &\leq 1/\beta \textstyle\sum_{b,b\mapsto_i a} \mu_{\mathbb{K}_\beta}(\mathcal{T}_\beta(b)). \end{aligned} \tag{6}$$

Let $\mathbf{m} = (\mu_{\mathbb{K}_\beta}(\mathcal{T}_\beta(a)))_{a\in\mathcal{S}_\beta}$ be the vector with nonnegative entries in $\mathbb{R}^n$ of measures in $\mathbb{K}_\beta$ of the basic tiles; we know from Theorem 1 that $\mathbf{m}$ is a non-zero vector. According to Perron-Frobenius theorem, the previous inequality implies that $\mathbf{m}$ is a left eigenvector of the adjacency matrix of $\mathcal{M}_\beta$ which is primitive. We thus have equality in (6) which implies that the union in (5) applied to the letter a is a disjoint union up to sets of zero measure. We thus have proved that the n basic tiles have disjoint interiors.

Obviously it follows from Lemma 1 below that there exists an integer k such that this covering is almost everywhere k-to-one.

Lemma 1 *Let $(\Omega_i)_{i\in I}$ be a collection of open sets in $\mathbb{R}^k$ such that i) $\cup_{i\in I}\overline{\Omega_i} = \mathbb{R}^k$, ii) for any compact set K, $I_k := \{i \in I;\ \overline{\Omega_i} \cap K \neq \emptyset\}$ is finite. For $x \in \mathbb{R}^k$, let $f(x) := Card\{i \in I;\ x \in \overline{\Omega_i}\}$. Let $\Omega = \mathbb{R}^k \setminus \cup_{i\in I}\delta(\omega_i)$, where $\delta(\Omega_i)$ stands for the boundary of Ω_i. Then f is locally constant on Ω.*

Let us prove the repetitivity of the covering (4). For any positive real number r, we define the r-patch centered at the point $\gamma_0 \in \Gamma_\beta$ as

$$P_{\gamma_0}(r) := \{(\gamma, i) \in \Gamma_\beta \times \{1, \cdots, n\};\ \gamma \in B(\gamma_0, r),\ \mathcal{T}_\beta(i) + \gamma \text{ occurs in (4)}\},$$

where $B(\gamma_0, r)$ stands for the closed ball in $\mathbb{K}_\beta$ of a radius r centered at γ_0. The r-local configuration centered at the point $\gamma_0 \in \Gamma_\beta$ is then defined as

$$LC_{\gamma_0}(r) := \{(\gamma - \gamma_0, i);\ (\gamma, i) \in P_{\gamma_0}(r)\}.$$

Given a local configuration $LC_{\gamma_0}(r)$, we want to prove the existence of a positive number R such that for any $\gamma \in \Gamma_\beta$, there exists $\gamma_1 \in B(\gamma, R)\cap\Gamma_\beta$ such that $LC_{\gamma_1}(r) = LC_{\gamma_0}(r)$. We first notice that if $(\gamma, i) \in \Gamma_\beta\times\{1, \cdots, n\}$, then $\mathcal{T}_\beta(i)+\gamma$ occurs in (4) if and only if there exists $(x, i) \in (\mathbb{Z}[\beta] \cap [0, 1))$ such that $\gamma = \Phi_\beta(x)$ and $x \in [0, T^{i-1}(\beta))$. Hence for a given $\gamma_1 = \Phi_\beta(x_1) \in \Gamma_\beta$ and $(x, i) \in (\mathbb{Z}[\beta]\cap[0, 1))\times\{0, 1, \cdots n\}$, we have $\Phi_\beta(x+x_1, i) \in LC_{\gamma_1}(r)$ if and only if $\Phi_\beta(x) \in B(0, r)$ and $x+x_1 \in [0, T_\beta^{i-1}(1))$. In other words, $LC_{\gamma_1}(r) = LC_{\gamma_0}(r)$ if and only if x_1 belongs to the interval $I_{\gamma_0} := \cap_{(x,i)\in LC_{\gamma_0}(r)}\ (-x + [0, T_\beta^{i-1}(1)))$. According to [Sla50], there exists a finite set $\mathcal{K} \subset \mathbb{N}^+ \times \mathbb{Z}$ of return times to I_{γ_0}, such that for all $x \in [0, 1)$, there exists $(k, \ell) \in \mathcal{K}$ such that $x + \frac{k}{\beta} + \ell \in I_{\gamma_0}$. Let $R = \max_{(k,\ell)\in\mathcal{K}} ||\Phi_\beta(k/\beta + \ell)||$. Let $\gamma = \Phi_\beta(x) \in \Gamma_\beta$ where $x \in \mathbb{Z}[\beta] \cap [0, 1)$; there exists $(k, \ell) \in \mathcal{K}$ such that $x_1 := x + \frac{k}{\beta} + \ell \in I_{\gamma_0}$. Let $\gamma_1 := \Phi_\beta(x_1)$. One has $\gamma_1 \in B(\gamma, R)$ and $LC_{\gamma_1}(r) = LC_{\gamma_0}(r)$.

It only remains to prove Lemma 1. ∎

Proof of Lemma 1. According to ii), one deduces that Ω is an open set in $\mathbb{R}^k$. By i), for all x one has $f(x) \geq 1$, and by ii), f takes bounded values.

For every fixed $\ell \in \mathbb{N}$, the set $J_\ell := \{x \in \mathbb{R}^k; f(x) \geq \ell\}$ is a closed set of $\mathbb{R}^k$. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of elements in J_ℓ; we denote its limit by x. There exist $(i_1^{(n)})_{n\in\mathbb{N}}, \ldots, (i_\ell^{(n)})_{n\in\mathbb{N}}$ such that for all n, $x_n \in \overline{\Omega}_{i_j^{(n)}}$, for $1 \leq j \leq \ell$. By using ii), there exist $i_1, \dots, i_\ell$ such that for infinitely many n, $x_n \in \overline{\Omega_{i_j}}$, for $1 \leq j \leq \ell$, hence $x \in \cap_{1 \leq j \leq \ell} \overline{\Omega_{i_j}}$ and $f(x) \geq \ell$.

Let us introduce now for $x \in \mathbb{R}^k$, $g(x) := \mathrm{Card}\{i \in I;\ x \in \Omega_i\}$. Let us note that f and g do coincide over Ω. We similarly prove that any fixed $\ell \in \mathbb{N}$, $\{x \in \mathbb{R}^k; g(x) \geq \ell\}$ is an open set of $\mathbb{R}^k$.

Now let $x \in \Omega$; let $r > 0$ such that the open ball $B(x,r)$ of center x and radius r is included in Ω; such a ball exists since Ω is an open set. For all ℓ, $B_\ell := \{y \in B(x,r);\ f(y) \geq \ell\}$ is both an open and a closed set of $B(x,r)$, from what preceeds. Hence it is either equal to the empty set or to $B(x,r)$, by connectedness of $B(x,r)$. We have

$$\dots B_\ell \subset B_{\ell-1} \subset \ \cdots \subset B_1 \subset B_0 = B(x,r).$$

Now from ii), one cannot get for all ℓ, $B_\ell = B(x,r)$. Let $\ell_0 = \max\{\ell;\ B_\ell = B(x,r)\}$. For all $y \in B(x,r)$, one has $f(y) \geq \ell_0$, but $B_{\ell_0+1} = \emptyset$, hence $f(y) < \ell_0 + 1$; this thus implies that $f(y) = \ell_0$, for all $y \in B(x,r)$. ∎

1.5. An example: The Tribonacci number.

Let β be the *Tribonacci number*, that is, the Pisot root of the polynomial $X^3 - X^2 - X - 1$. One has $d_\beta(1) = 111$ (β is a simple Parry number) and $d^*_\beta(1) = (110)^\infty$. Hence X_β is the set of sequences in $\{0,1\}^{\mathbb{Z}}$ in which there are no three consecutive 1's. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is reduced to the $\mathbb{Q}$-isomorphism τ_α which maps β on α, where α is one of the complex roots of $X^3 - X^2 - X - 1$. The set $\mathcal{T}_\beta$ which satisfies

$$\mathcal{T}_\beta = \{\sum_{i \geq 0} w_i \alpha^i;\ \forall i,\ w_i \in \{0,1\},\ w_i w_{i+1} w_{i+2} = 0\}$$

is a compact subset of $\mathbb{C}$ called the *Rauzy fractal* [Rau82]. It is shown in Fig. 4 with its division into the three basic tiles $\mathcal{T}(i)$, $i = 1, 2, 3$, indicated by different shades. They correspond respectively to the sequences $(w_i)_{i \geq 0}$ such that either $w_0 = 0$, or $w_0 w_1 = 10$, or $w_0 w_1 = 11$; this is easily seen thanks to the automaton $\mathcal{M}_\beta$ shown in Fig. 4. One has

$$\begin{cases} \mathcal{T}_\beta(1) = \alpha(\mathcal{T}_\beta(1) \cup \mathcal{T}_\beta(2) \cup \mathcal{T}_\beta(3)) \\ \mathcal{T}_\beta(2) = \alpha(\mathcal{T}_\beta(1)) + 1 \\ \mathcal{T}_\beta(3) = \alpha(\mathcal{T}_\beta(2)) + 1. \end{cases}$$

One interesting property in this numeration is that the β-fractional parts of the elements of $\mathbb{Z}[\beta]_{\geq 0}$ are all finite [FS92], as detailed in Section 1.6: this implies that the self-replicating multiple tiling is a tiling.

If U is a finite word which is a β-fractional part, and which begins with the letter 0, then $\mathcal{T}_U = \mathcal{T} = \mathcal{T}_\beta(1) \cup \mathcal{T}_\beta(2) \cup \mathcal{T}_\beta(3)$; if U begins with the factor 10, then $\mathcal{T}_U = \mathcal{T}_\beta(1) \cup \mathcal{T}_\beta(2)$; if U begins with the factor 11, then $\mathcal{T}_U = \mathcal{T}_\beta(1)$. The corresponding self-replicating tiling is shown in Fig. 5: the different shades indicate its division into tiles $\mathcal{T}_u$ (right figure), whereas the division into basic tiles $\mathcal{T}_\beta(i)$ is depicted in the left figure.

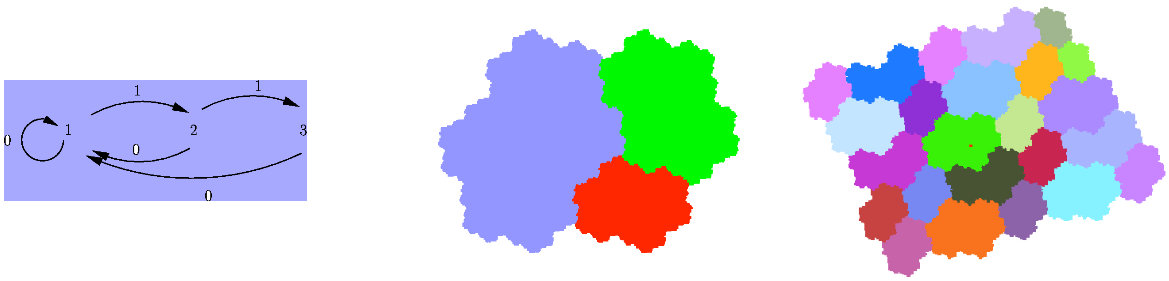

Figure 4: Tribonacci number: the minimal automaton $\mathcal{M}_\beta$, the central tile divided into its basic tiles; the self-replicating multiple tiling.

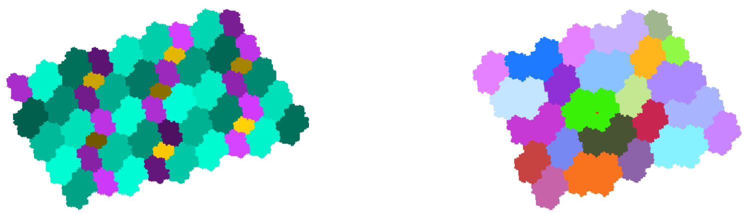

Figure 5: The Tribonacci self-replicating multiple tiling: its division into basic tiles $\mathcal{T}_\beta(i)$ (left) and into tiles $\mathcal{T}_U$ (right).

1.6. Finiteness conditions

Let us recall that the tiles in the covering (4) are labeled by the fractional parts of elements in $\mathbb{Z}[\beta]_{\geq 0}$. When the elements of $\mathbb{Z}[\beta]_{\geq 0}$ all have a finite fractional part, as in the Tribonacci case, then much more can be said. This finiteness condition is called the (F) property, *finiteness property*, and has been introduced by C. Frougny and B. Solomyak [FS92]: an algebraic integer $\beta > 1$ is said to satisfy the (F) property if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta]_{\geq 0}. \qquad \text{(F)}$$

Property (F) implies that β is both a Pisot number and a simple Parry number; hence not all Pisot numbers have property (F). A sufficient condition for β to satisfy the (F) property is the following: if $\beta > 1$ is the *dominant root* (that is, if it has the maximal modulus along all the roots) of the polynomial $X^d - t_1X^{d-1} - \cdots - t_d$, with $t_i \in \mathbb{N}$, $t_1 \geq t_2 \geq \cdots \geq t_d \geq 1$, then β satisfies (F) [FS92]; the same conclusion holds if more generally $t_1 > t_2 + \cdots + t_d$ [Hol96]. A complete characterization of some families of Pisot numbers with property (F) exists: the quadratic case is studied in [FS92], the case of cubic units has been treated in [Aki00].

The so-called (W) condition or *weak finiteness condition* has first been introduced by M. Hollander [Hol96]. He has proved that the (W) property implies the pure discreteness of the spectrum of the beta-shift. The (W) condition can be stated for Pisot number β as follows:

$$\forall z \in \mathbb{Z}[\beta^{-1}] \cap [0,1),\ \forall \varepsilon > 0,\ \exists x, y \in \mathrm{Fin}(\beta) \text{ such that } z = x - y \text{ and } y < \varepsilon. \quad \text{(W)}$$

Note that if β has property (W), then it must be a Pisot or a Salem number [ARS04]. All the quadratic units [FS92] and all the cubic units [ARS04] are known to satisfy (W). The (W) property has been proved in [Aki02] to be equivalent with the fact that the multiple tiling (4) is in fact a tiling. Hence the Pisot conjecture is equivalenty reformulated as:

Conjecture ([Aki02], [Sid03b]) *The (W) condition holds for every Pisot number β.*

An algorithm which can tell whether a given Pisot β has property (F) or (W) is described in [ARS04]. It is also proved in [ARS04] that (W) holds for all the cubic units and, in higher degree, for each dominant root of a polynomial $X^d - t_1X^{d-1} - \cdots - t_d$ with $t_i \in \mathbb{N}$, $t_1 > |t_2| + \cdots + |t_d|$, and $(t_1, t_2) \neq (2, -1)$. See also [BK05].

The (F) and (W) properties can be reformulated in topological terms: (F) is equivalent with the fact that the origin is an inner *exclusive point* of the central tile [Aki99, Aki02]; an inner point in a tile $\mathcal{T}_u$ is said *exclusive* if it is contained in no other tile $\mathcal{T}_v$ with $u \neq v$; (W) is equivalent to the fact that there exists an exclusive inner point in the central tile [Aki02].

More generally, the study of the topological properties of the central tile is an important issue of the field. The connectedness of the central tile in the Pisot unit case is studied in [AN04b]: it is proved that when β is a Pisot number of degree 3, then each central tile is arcwise connected, but examples of Pisot numbers of degree 4 with a disconnected central tile have been produced. In particular it is proved in [AN04a] that each tile corresponding to a Pisot unit is arcwise connected if $d_\beta(1)$ is finite and terminates with 1. A complete description of the β-expansion of 1 is also given for cubic and quartic Pisot units; for the general case of cubic Pisot numbers, see [Bas02].

These results are inspired by techniques used for some particular generalized radix representations, the so-called Canonical Number System case. Let us quote the recent attempt through the notion of Radix Number Systems [ABB$^+$05] to embrace both approaches, that is, beta-numeration and Canonical Number Systems: in both cases tiling properties can be ensured by similar finiteness properties, that can be expressed in terms of the orbit of a certain dynamical system.

2. Substitution numeration systems and Rauzy fractals

The first example of a central tile associated with a substitution is the Tribonacci tile (also known as *Rauzy fractal*), which is due to Rauzy [Rau82]. (Let us note that the central tile is usually called Rauzy fractal when associated with a substitution, but for the sake of consistency, we call it here again central tile, and introduce the term Rauzy fractal in Section 3.1 for its geometric representation.) The associated tiling is a lattice tiling having some deep dynamical interpretation. We discuss it in Section 3. One natural question is to figure out what structure can play the role of $\mathbb{Z}[\beta]_{\geq 0}$. We thus introduce for that purpose the so-called Dumont-Thomas numeration system based on the substitution, that allows one to expand real numbers: the only difference is that the digits will not only belong to $\mathbb{Z}$ but to some finite subset of $\mathbb{Z}[\beta]$.

We recall in Section 2.1 basic definitions on substitutive dynamical systems; we work out the notion of desubstitution in Section 2.2; a family of substitutions, the so-called β-substitutions, which allows us to recover the beta-numeration developed in Section 1, is described in Section 2.3, in order to introduce in Section 2.4 the Dumont-Thomas numeration; we then consider the self-replicating multiple tiling in Section 2.5. Let us note that we need an extra combinatorial assumption, the so-called strong coincidence condition, so that the basic tiles have distinct interiors. This property always holds for all β-substitutions.

The main assumption made in this section is the following: the substitution σ is supposed to be primitive and to have a Perron-Frobenius dominant eigenvalue β which is a Pisot unit; the characteristic polynomial of the substitution may thus be reducible.

2.1. Substitutions

A *substitution* σ is an endomorphism of the free monoid $\mathcal{A}^*$ such that the image of any letter of $\mathcal{A}$ never equals the empty word ε, and for at least one letter a, we have $|\sigma^n(a)| \to +\infty$. A substitution naturally extends to the set of bi-infinite words $\mathcal{A}^{\mathbb{Z}}$:

$$\sigma(\ldots w_{-2}w_{-1}.w_0w_1\ldots) = \ldots \sigma(w_{-2})\sigma(w_{-1}).\sigma(w_0)\sigma(w_1)\ldots$$

The two assumptions above guarantee the existence of bi-infinite words generated by iterating the substitution. To be more precise, a *periodic point* of σ is a bi-infinite word $u = (u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ that satifies $\sigma^n(u) = u$ for some $n > 0$; if $\sigma(u) = u$, then u is a *fixed point* of σ. Every substitution has at least one periodic point [Que87]. The substitution is said *shift-periodic* when there exists a bi-infinite word that is periodic for both the shift map S and the substitution σ.

A substitution σ is called *primitive* if there exists an integer n (independent of the letters) such that $\sigma^n(a)$ contains at least one occurrence of the letter b for every pair $(a,b) \in \mathcal{A}^2$. In that case, if u is a periodic point for σ, then the closure in $\mathcal{A}^{\mathbb{Z}}$ of the shift orbit of u does not depend on u. We thus denote it by X_σ. The *symbolic dynamical system generated by* σ is defined as (X_σ, S). The system (X_σ, S) is *minimal* (every non-empty closed shift-invariant subset equals the whole set) and *uniquely ergodic* (there exists a unique shift-invariant probability measure μ_{X_σ} on X_σ [Que87]); it is made of all the bi-infinite words, the set of factors of which coincides with the set of factors F_σ of u (which does not depend on the choice of u by primitivity).

Incidence matrix. Let $\mathbf{l} : \mathcal{A}^* \to \mathbb{N}^n$ be the natural homomorphism obtained by abelianization of the free monoid. In the sequel, we assume that $\mathcal{A} = \{1,\ldots,n\}$. If $|W|_a$ stands for the number of occurrences of the letter $a \in \mathcal{A}$ in a finite word W, then we have $\mathbf{l}(W) = (|W|_k)_{k=1,\ldots,n} \in \mathbb{N}^n$. A *abelianization linear map* is canonically associated with each substitution σ on $\mathcal{A}$. Its matrix $\mathbf{M}_\sigma = (m_{i,j})_{1\leq i,j\leq n}$ (called *incidence matrix of* σ) is defined by $m_{i,j} = |\sigma(j)|_i$, so that we have $\mathbf{l}(\sigma(W)) = \mathbf{M}_\sigma \mathbf{l}(W)$ for every $W \in \mathcal{A}^*$. If σ is primitive, the *Perron-Frobenius theorem* says that the incidence matrix $\mathbf{M}_\sigma$ has a simple real positive dominant eigenvalue β.

A substitution σ is called *unimodular* if det $\mathbf{M}_\sigma = \pm 1$. A substitution σ is said to be *Pisot* if its incidence matrix $\mathbf{M}_\sigma$ has a dominant eigenvalue β such that for every other eigenvalue λ, one gets: $0 < \lambda < 1 < \beta$. The characteristic polynomial of the incidence matrix of such a substitution is irreducible over $\mathbb{Q}$. We deduce [Fog02] that the dominant eigenvalue β is a Pisot number, Pisot substitutions are primitive, and that Pisot substitutions are not shift-periodic. For this last point, it is easy to recognize whether a substitution is not shift-periodic: indeed, if σ is a primitive substitution the matrix of which has a non-zero eigenvalue of modulus less that 1, then no fixed point of σ is shift-periodic, according to [HZ98]. Hence, if a substitution is a Pisot substitution then its characteristic polynomial is irreducible, whereas when the dominant eigenvalue of a primitive substitution is assumed to be a Pisot number, it may be reducible. In all that follows, by the reducible (resp. irreducible) case, we mean that the characteristic polynomial of the incidence matrix of the substitution is reducible (resp. irreducible). We do not need any irreducibility assumption in all this section. This assumption will be required during Section 3 and in Section 4.

2.2. Combinatorial numeration system: desubstitution

We need to be able to desubstitute, that is, to define a notion of inverse map for the action of the substitution σ on X_σ. For that purpose, we decompose any $w \in X_\sigma$ as a combinatorial power series. Hence a combinatorial expansion defined on X_σ plays the role of an exotic numeration system acting on the bi-infinite words w.

Desubstitution: a combinatorial division by σ. We follow here the approach and notation of [CS01a, CS01b]. Every bi-infinite word $w \in X_\sigma$ has a unique decomposition $w = S^\nu(\sigma(v))$, with $v \in X_\sigma$ and $0 \leq \nu < |\sigma(v_0)|$, where v_0 is the 0-th coordinate of v [Mos92]. This means that any word of the dynamical system can be uniquely written in the following form for some $\ldots v_{-n} \ldots v_{-1}.v_0 v_1 \ldots v_n \cdots \in X_\sigma$:

$$w = \ldots \mid \underbrace{\ldots}_{\sigma(v_{-1})} \mid \underbrace{w_{-\nu} \ldots w_{-1}.w_0 \ldots w_{\nu'}}_{\sigma(v_0)} \mid \underbrace{\ldots}_{\sigma(v_1)} \mid \underbrace{\ldots}_{\sigma(v_2)} \mid \ldots$$

Here, the doubly infinite word v can be considered as the "quotient" of w after the "division" by σ. The "rest" of this division consists in the the triple (p, w_0, s), where $p = w_{-\nu} \ldots w_{-1}$ (prefix) and $s = w_1 \ldots w_{\nu'}$ (suffix). The word w is completely determined by the quotient v and the rest (p, w_0, s).

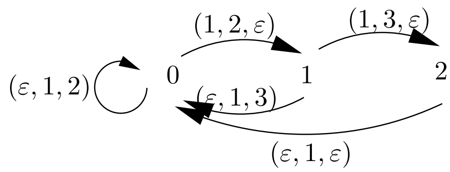

Figure 6: Prefix-suffix automaton for the Tribonacci substitution.

Let $\mathcal{P}$ be the finite set of all rests or *digits* associated with σ:

$$\mathcal{P} = \{(p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*;\ \exists\, b \in \mathcal{A},\ \sigma(b) = pas\}\,.$$

The *desubstitution map* $\theta : X_\sigma \to X_\sigma$ maps a bi-infinite word w to its quotient v. The decomposition of $\sigma(v_0)$ of the form pw_0s is denoted as $\gamma : X_\sigma \to \mathcal{P}$ (mapping w to (p, w_0, s)).

Prefix-suffix expansion. We denote by $X^l_{\mathcal{P}}$ the set of infinite left-sided sequences with values in $\mathcal{P}$ $(p_i, a_i, s_i)_{i\geq 0}$ that satisfy $\sigma(a_{i+1}) = p_i a_i s_i$, for all $i \geq 0$. The *prefix-suffix expansion* is the map $E_{\mathcal{P}} : X_\sigma \to X^l_{\mathcal{P}}$ which maps a word $w \in X_\sigma$ to the sequence $(\gamma\,(\,\theta^i w)\,)_{i\geq 0}$, that is, the orbits of w through the desubstitution map according to the partition defined by γ.

Let $w \in X_\sigma$ and $E_{\mathcal{P}}(w) = (p_i, a_i, s_i)_{i\geq 0}$ be its prefix-suffix expansion. If there are infinitely many prefixes and suffixes that are non-empty, then w and $E_{\mathcal{P}}(w)$ satisfy:

$$w = \lim_{n\to+\infty} \sigma^n(p_n)\ldots\sigma(p_1)p_0.a_0s_0\sigma(s_1)\ldots\sigma^n(s_n).$$

Hence, the prefix-suffix expansion can be considered as an expansion of the points of X_σ in a "combinatorial" power series. The triples (p_i, a_i, s_i) play the role of digits in this combinatorial expansion. Let us observe that any element in $X^l_{\mathcal{P}}$ is the expansion of a bi-infinite word in X_σ, since the map $E_{\mathcal{P}}$ is continuous and onto $X_{\mathcal{P}}$, according to [CS01a]. Furthermore a countable number of bi-infinite words is not characterized by their prefix-suffix expansions: $E_{\mathcal{P}}$ is one-to-one except on the orbit of periodic points of σ, where it is n-to-one with $n > 1$ (see the proofs in [CS01a, HZ01]). Observe that the prefix-suffix expansion of periodic points for σ has only empty prefixes.

Prefix-suffix automaton. Any prefix-suffix expansion is the label of an infinite path in the so-called *prefix-suffix automaton* $\mathcal{M}_\sigma$ of σ which is defined as follows. Its set of vertices is the alphabet $\mathcal{A}$ and its edges satisfy the following: there exists an edge labeled by $(p, a, s) \in \mathcal{P}$ from b toward a if $pas = \sigma(b)$; we set $b \mapsto_{(p,a,s)} a$. The automaton for the Tribonacci substitution $1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1$ is given in Figure 6.

Let us note that the adjacency matrix of the prefix-suffix automaton is the transpose of the incidence matrix of the substitution.

A subshift of finite type. The set $X^l_{\mathcal{P}}$ consists of labels of infinite left-sided paths $(p_i, a_i, s_i)_{i\geq 0}$ in the prefix-suffix automaton. Similarly as in Section 1, let $X^r_{\mathcal{P}}$ be the set of labels of infinite right-sided paths in the prefix-suffix automaton; we denote its elements as sequences $(q_i, b_i, r_i)_{i\geq 1} \in \mathcal{P}^{\mathbb{N}^+}$. Lastly we define $X_{\mathcal{P}}$ as the set of labels of two-sided paths in the prefix-suffix automaton; we denote its elements as sequences $((p_i, a_i, s_i)_{i\geq 0}, (q_i, b_i, r_i)_{i\geq 1}) \in \mathcal{P}^{\mathbb{Z}}$. These sets are the support of a subshift of finite type. We denote by $F_{\mathcal{P}}$ the set of factors of $X_{\mathcal{P}}$.

2.3. A specific case: β-substitution

Let $\beta > 1$ be a Parry number as defined in Section 1. As introduced for instance in [Thu89] and in [Fab95], one can associate in a natural way with (X_β, S) a substitution σ_β over the alphabet $\{1, \cdots, n\}$, called *β-substitution*, where n stands for the number of states of the automaton $\mathcal{M}_\beta$ (see Fig. 3): j is the k-th letter occurring in $\sigma_\beta(i)$ (that is, $\sigma_\beta(i) = pjs$, where $p, s \in \{1, \cdots, n\}^*$ and $|p| = k-1$) if and only if there is an arrow in $\mathcal{M}_\beta$ from the state i to the state j labeled by $k-1$. One easily checks that this definition is consistent.

An explicit formula for σ_β can be computed by considering the two different cases, β simple and β non-simple.

- Assume $d_\beta(1) = t_1 \dots t_{n-1} t_n$ is finite, with $t_n \neq 0$. Thus $d^*_\beta(1) = (t_1 \dots t_{n-1}(t_n - 1))^\infty$. One defines σ_β over the alphabet $\{1, 2, \dots, n\}$ as shown in (7).
- Assume $d_\beta(1)$ is infinite. Then it cannot be purely periodic (according to Remark 7.2.5 [Fro02]). Hence $d_\beta(1) = d^*_\beta(1) = t_1 \dots t_m (t_{m+1} \dots t_{m+p})^\infty$, with $m \geq 1$, $t_m \neq t_{m+p}$ and $t_{m+1} \dots t_{m+p} \neq 0^p$. One defines σ_β over the alphabet $\{1, 2, \dots, m+p\}$ as shown in (7).

$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \vdots \\ n-1 & \mapsto 1^{t_{n-1}}n \\ n & \mapsto 1^{t_n}. \end{cases} \qquad \sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \vdots \\ m+p-1 & \mapsto 1^{t_{m+p-1}}(m+p) \\ m+p & \mapsto 1^{t_{m+p}}(m+1). \end{cases} \tag{7}$$

Substitution associated with a simple Parry number

Substitution associated with a non-simple Parry number

If the number of letters n equals the degree of β, then σ_β is a Pisot substitution. Otherwise the characteristic polynomial of the incidence matrix of σ_β may be reducible. In the latter case we cannot apply directly the substitutive formalism to the substitution σ_β. The dominant eigenvalue of σ_β is anyway a Pisot number but other eigenvalues ≥ 1 may occur, as in the smallest Pisot case: let β be the Pisot root of $X^3 - X - 1$; one has

$d_\beta(1) = 10001$ (β is a simple Parry number) and $d^*_\beta(1) = (10000)^\infty$; we have $\sigma_\beta :\ 1 \mapsto 12$, $2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$; the characteristic polynomial of its incidence matrix is $(X^3 - X - 1)(X^2 - X + 1)$, hence σ_β is not a Pisot substitution. Furthermore, some extra roots may be outside the unit circle: consider for instance, as quoted in [Boy89, Boy96], the dominant root β of $P(X) = X^7 - 2X^5 - 2X^4 - X - 1$; then the complementary factor $Q(X)$ (such that $P(X)Q(X)$ is the characteristic polynomial of the incidence matrix of σ_β) is non-reciprocal, so that there exist roots outside the unit circle.

The prefix-suffix automaton of the substitution σ_β is strongly connected with the finite automaton $\mathcal{M}_\beta$ recognizing the set F_β of finite factors of the β-shift X_β (compare for instance Fig. 3 and 4). Let us first note that the proper prefixes of the images of letters contain only the letter 1; it is thus natural to code a proper prefix by its length: if $(p, a, s) \in \mathcal{P}$, then $p = 1^{|p|}$, where the notation $1^{|p|}$ stands for the fact that p consists of exactly $|p|$ occurrences of the letter 1. Hence it is easily seen that one recovers the automaton $\mathcal{M}_\beta$ by replacing in the prefix-suffix automaton the set of labeled edges

$$\mathcal{P} = \{(p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*;\ \exists\, b \in \mathcal{A},\ \sigma(b) = pas\}$$

by the following set of labeled edges

$$\{|p|;\ \exists b \in \mathcal{A} \,:\, \sigma(b) = pas \text{ with } (p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*\}\,.$$

Hence, the following relation holds between the set F_β of finite factors of the β-shift X_β and the set $F_{\mathcal{P}}$ of factors of $X_{\mathcal{P}}$, that is, the set of finite words recognized by the prefix-suffix automaton:

$$w_M \dots w_0 \in F_\beta \Longleftrightarrow \exists\, a_0 \dots a_M \in \mathcal{A},\ s_0 \dots s_M \in \mathcal{A}^\star\ (1^{w_M}, a_M, s_M) \dots (1^{w_0}, a_0, s_0) \in F_{\mathcal{P}}. \tag{8}$$

From this relation, one can interpret any point $w_M\beta^M + \cdots + w_0$ in $\mathbb{Z}^+_\beta$ as the real value of the combinatorial expansion $(1^{w_M}, a_M, s_M) \dots (1^{w_0}, a_0, s_0)$ associated with the β-substitution. We formalize this interpretation in the next section.

2.4. Dumont-Thomas numeration

The Dumont-Thomas numeration system [DT89, DT93, Rau90] generalizes the approach given above to any primitive substitution, the dominant eigenvalue of which is a Pisot number.

Let us first define the Dumont-Thomas numeration on $\mathbb{N}$. Let v be a one-sided fixed point of σ; we denote its first letter by v_0. This numeration depends on this particular choice of a fixed point, and more precisely on the letter v_0. One checks ([DT89] Theorem 1.5) that every finite prefix of v can be uniquely expanded as

$$\sigma^n(p_n)\sigma^{n-1}(p_{n-1}) \cdots p_0,$$

where $p_n \neq \varepsilon$, $\sigma(v_0) = p_n a_n s_n$, and $(p_n, a_n, s_n) \cdots (p_0, a_0, s_0) \in F_{\mathcal{P}}$ is the sequence of labels of a path in the prefix-suffix automaton $\mathcal{M}_\sigma$ starting from the state v_0; one has for all i, $\sigma(p_i) = p_{i-1} a_{i-1} s_{i-1}$, that is,

$$v_0 \mapsto_{(a_n, p_n, s_n)} a_n \mapsto_{(p_{n-1} a_{n-1} s_{n-1})} a_{n-1} \cdots \mapsto_{(p_0, a_0, s_0)} a_0.$$

Conversely, any path in $\mathcal{M}_\sigma$ starting from v_0 generates a finite prefix of v. This numeration works a priori on finite words but we can expand the nonnegative natural integer N as $N = |\sigma^n(p_n)| + \cdots + |p_0|$, where N stands for the length of the prefix $\sigma^n(p_n)\sigma^{n-1}(p_{n-1}) \cdots p_0$ of v. The expansions of prefixes of fixed points of σ play here the role of $\mathbb{Z}_\beta^+$ in the beta-numeration case.

Let us now expand real numbers. We denote by β the dominant eigenvalue of the incidence matrix $\mathbf{M}_\sigma$ of the primitive substitution σ. We assume that β is a Pisot number. We want to expand real numbers in base β, with digits which may not belong to $\mathbb{Z}$ anymore, but do belong to a finite subset of $\mathbb{Q}(\beta)$. Let $F_{\mathcal{P}}$ stand for the set of finite words recognized by the prefix-suffix automaton $\mathcal{M}_\sigma$. We want to define a map $\delta_\sigma : \mathcal{A}^* \to \mathbb{Q}(\beta)$ such that one can associate with a combinatorial expansion $(p_n, a_n, s_n) \ldots (p_0, a_0, a_0) \in F_{\mathcal{P}}$ the real value $\delta_\sigma(p_n)\beta^n + \cdots + \delta_\sigma(p_0) \in \mathbb{Q}[\beta]$. A natural and suitable choice is given by

$$\delta_\sigma : \quad \mathcal{A}^* \to \mathbb{Q}(\beta), \ \delta_\sigma(p) \mapsto < \mathbf{l}(p), \mathbf{v}_\beta >, \tag{9}$$

where $\mathbf{v}_\beta$ is a (simple) dominant eigenvector for the transpose of the matrix $\mathbf{M}_\sigma$, i.e., $\mathbf{v}_\beta$ is a left eigenvector associated with β. To recover the β-expansion in case of a β-substitution, $\mathbf{v}_\beta$ has to be normalized so that its first coordinate is equal to 1: the coordinates of $\mathbf{v}_\beta$ are then equal to $T_\beta^{i-1}(1)$, for $1 \leq i \leq n$, where n denotes the number of states in the automaton $\mathcal{M}_\beta$. In the substitutive case, we just normalize $\mathbf{v}_\beta$ so that its coordinates belong to $\mathbb{Q}(\beta)$. The map δ_σ sends the letter a to the corresponding coordinate of the left eigenvector $\mathbf{v}_\beta$. We now get the following representation:

Theorem 3 ([DT89]) *Let σ be a primitive substitution on the alphabet $\mathcal{A}$, the dominant eigenvalue of which is a Pisot number. Let us fix $a \in \mathcal{A}$. Every real number $x \in [0, \delta_\sigma(a))$ can be uniquely expanded as*

$$x = \sum_{i \geq 1} \delta_\sigma(q_i)\beta^{-i},$$

where the sequence of digits $(q_i)_{i \geq 1}$ is the projection on the first component of an element $(q_i, b_i, r_i)_{i \geq 1}$ in $X_{\mathcal{P}}^r$ with $\sigma(a) = q_1 b_1 r_1$, and with the extra condition that there exist infinitely non-empty suffixes in the sequence $(r_i)_{i \geq 1}$. We call this expansion the (σ, a)-expansion of x and denote it by $d_{(\sigma,a)}(x)$.

This theorem provides an analogue of the Parry condition (2), the proof being also based on the greedy algorithm. The underlying dynamics depends of each interval $[0, \delta_\sigma(a))$, and is defined as follows:

$$\begin{array}{rcl} T_\sigma : \bigcup_{a\in\mathcal{A}}[0,\delta_\sigma(a))\times\{a\} & \to & \bigcup_{a\in\mathcal{A}}[0,\delta_\sigma(a))\times\{a\} \\ (x,b) & \mapsto & (\beta x-\delta_\sigma(p),c) \text{ with } \left\{\begin{array}{l}\sigma(b)=pcs\\ \beta x-\delta_\sigma(p)\in[0,\delta_\sigma(c)).\end{array}\right. \end{array}$$

Theorem 3 states that this map is well defined, meaning that for every (x,b) there exists a unique (y,c) satisfying the above conditions.

Let us note that one may obtain a different type of numeration for each letter. Nevertheless, one easily checks that for a β-substitution all the associated numerations are consistent with the β-numeration: in particular $\mathbf{v}_\beta$ is normalized, so that $\delta_\sigma(1)=1$, and the numeration associated with the letter 1 is exactly the β-numeration.

The Dumont-Thomas numeration shares many properties with the β-numeration. In particular, when β is a Pisot number, then for every $a\in\mathcal{A}$, every element of $\mathbb{Q}(\beta)\cap[0,\delta_\sigma(a))$ admits an eventually periodic expansion. The proof can be conducted exactly in the same way as in [Sch80]. See also [RS05] for a similar result in the framework of Pisot abstract numeration systems.

2.5. The self-replicating substitution multiple tiling

We now have gathered all the required tools to be able to define the central tile as the image under a suitable representation map of the one-dimensional prefix-suffix expansions.

Let σ be a primitive unimodular substitution, the dominant eigenvalue β of which is a Pisot unit. The cardinality of the alphabet on which σ is defined is denoted by n whereas d stands for the algebraic degree of β. We use the same notation as in Section 1.5 concerning the *canonical embedding* and the *representation space* denoted respectively by Φ_σ and $\mathbb{K}_\sigma$ (one has $\mathbb{K}_\sigma\simeq\mathbb{R}^{d-1}$). We define the *representation map* as

$$\varphi_\sigma:\ X^l_{\mathcal{P}}\to\mathbb{K}_\beta,\ (p_i,a_i,s_i)_{i\geq 0}\mapsto\lim_{n\to+\infty}\Phi_\sigma\left(\sum_{i\geq 0}^{n}\delta_\sigma(p_i)\beta^i\right).$$

We define similarly as in the beta-numeration case:

$$\mathbb{Z}^+_\sigma=\{\delta_\sigma(p_M)\beta^M+\cdots+\delta_\sigma(p_0);\ M\in\mathbb{N},\ (p_M,a_M,s_M)\dots(p_0,a_0,s_0)\in F_{\mathcal{P}}\}$$

Definition 3 *We define the* central tile $\mathcal{T}_\sigma$ *as*

$$\mathcal{T}_\sigma=\overline{\Phi_\sigma(\mathbb{Z}^+_\sigma)}=\varphi_\sigma(X^l_{\mathcal{P}}).$$

Recall that n stands for the number of letters in the alphabet $\mathcal{A}$ on which σ is defined. The central tile is here again divided into n pieces, called *basic tiles*, as follows: for $a \in \mathcal{A}$,

$$\begin{aligned}\mathcal{T}_\sigma(a) \ &= \varphi_\sigma(\{(p_i, a_i, s_i)_{i\geq 0} \in X_{\mathcal{P}}^l;\ (p_i, a_i, s_i)_{i\geq 0} \text{ is the label of an infinite} \\ &\qquad \text{left-sided path in } \mathcal{M}_\sigma \text{ arriving at state } a_0 = a\}).\end{aligned}$$

To fit with the formalism and the proofs developed for the β-numeration, we intend, for each $a \in \mathcal{A}$, to introduce a set $\mathrm{Frac}(\sigma, a)$ defined as the set of fractional (σ, a)-expansions of a suitable set (analogous to $\mathrm{Frac}(\beta)$) the image of which under Φ_σ has to be relatively dense and uniformly discrete. We thus introduce the following countable sets:

$$\forall a \in \mathcal{A},\ \mathrm{Frac}(\sigma, a) = d_{(\sigma,a)}\left(\mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] \cap [0, \delta_\sigma(a))\right).$$

$$\mathrm{Frac}(\sigma) = \bigcup_{a\in\mathcal{A}} \mathrm{Frac}(\sigma, a).$$

In Section 4 we shall see a motivation for the introduction of this notion.

Notice that for a β-substitution, we have $\mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] = \mathbb{Z}[\beta]$, and $\mathrm{Frac}(\sigma) = \mathrm{Frac}(\beta)$ as introduced in Section 1. Indeed, one checks that the coordinates $\delta_\sigma(i)$ of the left eigenvector $\mathbf{v}_\beta$ satisfy $\delta_\sigma(i) = T^{i-1}(\beta) \in \mathbb{Z}[\beta]$, for $1 \leq i \leq n$.

Let $u = (q_i, b_i, r_i)_{i\geq 1} \in \mathrm{Frac}(\sigma)$. Then $\sum_{i\geq 1} \delta_\sigma(q_i)(\beta^{-i}) \in \mathbb{Q}(\beta)$. We define the tile $\mathcal{T}_u$ as

$$\begin{aligned}\mathcal{T}_u \ &= \Phi_\sigma(\textstyle\sum_{i\geq 1} \delta_\sigma(q_i)\beta^{-i}) + \varphi_\sigma(\{(p_i, a_i, s_i) \in X_{\mathcal{P}}^l; \\ &\qquad ((p_i, a_i, r_i)_{i\geq 0}, (q_i, b_i, r_i)_{i\geq 1}) \in X_{\mathcal{P}}\}.\end{aligned}$$

Coincidence. In order to get basic tiles with disjoint interiors we need here an extra condition, called the strong coincidence condition, that is satisfied by β-substitutions in particular. The condition of coincidence was introduced in [Dek78] for substitutions of constant length. It was generalized to non-constant length substitutions by Host in unpublished manuscripts. A formal and precise definition appears in [AI01]: a substitution is said to satisfy the *strong coincidence condition* if for any pair of letters (i, j), there exist two integers k, n such that $\sigma^n(i)$ and $\sigma^n(i)$ have the same k-th letter, and the prefixes of length $k-1$ of $\sigma^n(i)$ and $\sigma^n(j)$ have the same image under the abelianization map. It is conjectured that every Pisot substitution satisfies the strong coincidence condition; the conjecture holds for two-letter substitutions [BD02].

The following theorem can be proved similarly as Theorem 1 and 2, thanks to the appropriate choice of $\mathrm{Frac}(\sigma)$.

Theorem 4 ([SW02, IR06]) *We assume that σ is a primitive substitution, the dominant eigenvalue β of which is a Pisot unit. Then the set*

$$\begin{aligned}\Gamma_\sigma \ &:= \Phi_\sigma\left(\textstyle\sum_{i\geq 1} \delta_\sigma(p_i)\beta^{-1};\ (q_i, b_i, r_i)_{i\geq 1} \in \mathrm{Frac}(\sigma)\right) \\ &= \Phi_\sigma\ (\cup_{a\in\mathcal{A}}\mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] \cap [0, \delta_\sigma(a))).\end{aligned}$$

is a Delaunay set. The (up to translation finite) set of tiles $\mathcal{T}_u$, for $u \in \mathrm{Frac}(\sigma)$, covers $\mathbb{K}_\sigma$:

$$\mathbb{K}_\sigma = \bigcup_{u \in \mathrm{Frac}(\sigma)} \mathcal{T}_u = \bigcup_{a \in \mathcal{A}} \bigcup_{\substack{u = (q_j, b_j, r_j)_{j \geq 1} \in \mathrm{Frac}(\sigma, a), \\ \gamma = \Phi_\sigma(\sum_{j \geq 1} \delta_\sigma(q_j)\beta^{-j}),}} \mathcal{T}_\sigma(a) + \gamma. \tag{10}$$

For each u, the tile $\mathcal{T}_u$ has a non-empty interior, hence it has non-zero measure. The basic tiles of the central tile $\mathcal{T}_\sigma$ are solutions of the following graph-directed self-affine Iterated Function System:

$$\forall a \in \mathcal{A},\ \mathcal{T}_\sigma(a) = \bigcup_{\substack{b \in \mathcal{A}, \\ b \mapsto_{(p,a,s)} a}} h_\beta(\mathcal{T}_\sigma(b)) + \Phi_\sigma(\delta_\sigma(p)).$$

Each basic tile is the closure of its interior. We assume furthermore that σ satisfies the strong coincidence condition. Then the basic tiles have disjoint interiors. Furthermore, there exists an integer $k \geq 1$ such that the covering (10) is almost everywhere k-to-one. This multiple tiling is repetitive.

It can be proved that it is dense for $\Phi_\sigma\ (\mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] \cap [0, \delta_\sigma(a)))$ in $\mathbb{K}_\sigma$ similarly as for $\Psi_\sigma(\mathbb{Z}[\beta]_{\geq 0})$ ([Aki99], Proposition 1). In Section 3.1 we give an elementary proof of this result (Lemma 2) by introducing a suitable basis of the representation space $\mathbb{K}_\sigma \simeq \mathbb{R}^{d-1}$.

3. The lattice multiple tiling: a dynamical point of view

In this section, we give a geometric and dynamical interpretation of the central tile of a substitution; for that purpose, we introduce a lattice multiple tiling that provides a geometric representation of the substitutive dynamical system; the shift is thus proved to be measure-theoretically isomorphic to an exchange of domains acting on the basic tiles.

3.1. Geometric construction of the Rauzy fractal

In all that follows σ is a *primitive substitution the dominant eigenvalue of which is a unit Pisot number* on the alphabet $\mathcal{A} = \{1, \cdots, n\}$. Let u be a two-sided periodic point of σ. This bi-infinite word u is embedded as a broken line in $\mathbb{R}^n$ by replacing each letter in the periodic point by the corresponding vector in the canonical basis $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ in $\mathbb{R}^n$. More precisely, the *broken line* is defined as follows (see Fig. 7):

$$\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N}\}. \tag{11}$$

Notice that the following notations where introduced in Section 2.1.

Algebraic normalized eigenbasis. We need now to introduce a suitable decomposition of the representation space $\mathbb{K}_\sigma$ with respect to eigenspaces associated with the substitution σ and its (simple) dominant eigenvalue β. We denote by d the algebraic degree of β; we recall that n stands for the cardinality of the alphabet on which σ is defined; one has $d \leq n$, the characteristic polynomial of $\mathbf{M}_\sigma$ may be reducible. Let $\mathbf{v}_\beta \in \mathbb{Q}(\beta)^n$ be an expanding left eigenvector of the incidence matrix $\mathbf{M}_\sigma$. Let $\mathbf{u}_\beta \in \mathbb{Q}(\beta)^d$ be the unique right eigenvector of $\mathbf{M}_\sigma$ associated with β, normalized so that $\langle \mathbf{u}_\beta, \mathbf{v}_\beta \rangle = 1$, where $\langle\ \rangle$ stands for the usual Hermitian scalar product. An eigenvector $\mathbf{u}_{\beta^{(k)}}$ for each eigenvalue $\beta^{(k)}$ $(1 \leq k \leq d)$ is obtained by replacing β by $\beta^{(k)}$ in $\mathbf{u}_\beta$ (we set $\beta^{(1)} = \beta$). Similarly, a left eigenvector $\mathbf{v}_{\beta^{(k)}}$ for each eigenvalue $\beta^{(k)}$ $(1 \leq k \leq d)$ is obtained by replacing β by $\beta^{(k)}$ in $\mathbf{v}_\beta$.

Let us recall that $\beta^{(2)}, \ldots, \beta^{(r)}$ are the real conjugates of β, and that $\beta^{(r+1)}, \overline{\beta^{(r+1)}}, \ldots, \beta^{(r+s)}, \overline{\beta^{(r+s)}}$ are its complex conjugates. Let $\mathbb{H}_c$ stand for the subspace of $\mathbb{R}^n$ generated by the vectors $\mathbf{u}_{\beta^{(2)}}, \ldots, \mathbf{u}_{\beta^{(d)}}$, that is, $\mathbb{H}_c = \{\sum_{i=2}^{r} x_i \mathbf{u}_{\beta^{(i)}} + \sum_{i=r+1}^{r+s} x_i \mathbf{u}_{\beta^{(i)}} + \overline{x_i}\,\overline{\mathbf{u}_{\beta^{(i)}}};\ (x_1, \cdots, x_r) \in \mathbb{R}^r,\ (x_{r+1} \cdots x_{r+s}) \in \mathbb{C}^s\}$; we call it the β-*contracting plane*. We denote by $\mathbb{H}_e$ the real expanding line generated by $\mathbf{u}_\beta$, similarly called the β-*expanding line*. Let $\mathbb{H}_r$ be a complement subspace in $\mathbb{R}^n$ of $\mathbb{H}_c \oplus \mathbb{H}_e$. Let $\pi : \mathbb{R}^n \to \mathbb{H}_c$ be the projection onto $\mathbb{H}_c$ along $\mathbb{H}_e \oplus \mathbb{H}_r$, according to the natural decomposition $\mathbb{R}^n = \mathbb{H}_c \oplus \mathbb{H}_e \oplus \mathbb{H}_r$, and π' the projection onto the expanding line $\mathbb{H}_e$ along $\mathbb{H}_c \oplus \mathbb{H}_r$. Then π and π' can easily be expressed with respect to the algebraic normalized eigenbasis $\mathbf{u}_{\beta^{(1)}}, \cdots, \mathbf{u}_{\beta^{(d)}}$ of $\mathbb{H}_c \oplus \mathbb{H}_e$: for any $\mathbf{x} \in \mathbb{R}^n$, one has

$$\pi(\mathbf{x}) = \sum_{2 \leq k \leq d} \langle \mathbf{x}, \mathbf{v}_{\beta^{(k)}} \rangle \mathbf{u}_{\beta^{(k)}} \quad \text{and} \quad \pi'(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_\beta \rangle \mathbf{u}_\beta. \tag{12}$$

Indeed $\langle \mathbf{u}_{\beta^{(i)}}, \mathbf{v}_{\beta^{(j)}} \rangle = 0$ for every i, j with $i \neq j$ and $1 \leq i, j \leq d$. As a consequence, the projections $\pi(\mathbf{x})$ and $\pi'(\mathbf{x})$ of a rational vector $\mathbf{x} \in \mathbb{Q}^n$ are completely determined by the algebraic conjuguates of $\langle \mathbf{x}, \mathbf{v}_\beta \rangle$.

Rauzy fractal and projection of the broken line. An interesting property of the broken line (11) is that after projection by π one obtains a bounded set in $\mathbb{H}_c$ (Fig. 7). It appears that the closure of this set is exactly the central tile, after identification of $\mathbb{H}_c$ and $\mathbb{K}_\sigma$. More precisely, we denote by Ψ_σ the one-to-one identification map from $\mathbb{K}_\sigma$ to $\mathbb{H}_c$ that gives a geometric representation in $\mathbb{H}_c$ of points with coordinates in the right eigenvector basis of $\mathbb{H}_c$ which belong to $\mathbb{K}_\sigma$:

$$\begin{aligned}\Psi_\sigma : (x_2, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}) \in \mathbb{K}_\sigma \mapsto \quad & x_2 \mathbf{u}_{\beta^{(2)}} + \cdots + x_r \mathbf{u}_{\beta^{(r)}} + x_{r+1} \mathbf{u}_{\beta^{(r+1)}} \\ & + \overline{x_{r+1}}\,\overline{\mathbf{u}_{\beta^{(r+1)}}} + \ldots x_{r+s} \mathbf{u}_{\beta^{(r+s)}} + \overline{x_{r+s}}\,\overline{\mathbf{u}_{\beta^{(r+s)}}} \in \mathbb{H}_c.\end{aligned}$$

Let us observe that

$$\forall \mathbf{x} \in \mathbb{Z}^n,\ \pi(\mathbf{x}) = \Psi_\sigma \circ \Phi_\sigma(\langle \mathbf{x}, \mathbf{v}_\beta \rangle). \tag{13}$$

In particular

$$\forall a \in \mathcal{A},\ \Psi_\sigma \circ \Phi_\sigma \circ \delta_\sigma(a) = \pi(\mathbf{e_a}).$$

We now can easily prove the following density result mentioned after the proof of Theorem 4.

Lemma 2 *The set* $\Phi_\sigma(\mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)])$ *is dense in* $\mathbb{K}_\sigma$.

Proof. Let us prove that the rank r of the subgroup generated by $\Phi_\sigma \circ \delta_\sigma(1), \cdots, \Phi_\sigma \circ \delta_\sigma(n)$ in $\mathbb{K}_\sigma$ satisfies $r \geq d$. We first observe that the rank of the subgroup of $\mathbb{R} \times \mathbb{K}_\sigma$ generated by $(\delta_\sigma(1), \Phi_\sigma \circ \delta_\sigma(1)), \cdots, (\delta_\sigma(1), \Phi_\sigma \circ \delta_\sigma(1))$ is equal to r (we use here the canonical morphisms $\mathbb{Q}(\beta) \rightarrow \mathbb{K}^{(i)}$, $x \mapsto x^{(i)}$), which yields that the rank of the subgroup G generated by $(\pi + \pi')(\mathbf{e_1}), \cdots, (\pi + \pi')(\mathbf{e_n})$ also equals r. For all $1 \leq j \leq n$, there exists $\mathbf{f}_j$ in $\mathbb{H}_r$ such that $\mathbf{e}_j = \sum_{k=1}^{d} \langle \mathbf{e}_j, \mathbf{v}_{\beta^{(k)}} \rangle \mathbf{u}_{\beta^{(k)}} + \mathbf{f}_j$. Hence $\mathbb{Z}^n$ is included in the direct sum of G with the group of rank at most $n - d$ generated by the vectors $\mathbf{f}_j$, for $1 \leq j \leq n$, which yields $d \leq r$.

Theorem 5 *Let* σ *be a primitive substitution the dominant eigenvalue of which is a Pisot unit. Let* $u = (u_i)_{i \in \mathbb{Z}}$ *be a periodic point of* σ. *Then one has*

$$\mathcal{R}_\sigma := \Psi_\sigma(\mathcal{T}_\sigma) = \overline{\pi\left(\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N}\}\right)}, \tag{14}$$

and for $a \in \mathcal{A}$

$$\mathcal{R}_\sigma(a) := \Psi_\sigma(\mathcal{T}_\sigma(a)) = \overline{\pi\left(\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N},\ u_N = a\}\right)}.$$

The embedding of the central tile in the contracting hyperplane is called the Rauzy fractal and is denoted by $\mathcal{R}_\sigma$.

Proof. We first observe that the central tiles $\mathcal{T}_\sigma$ and $\mathcal{T}_{\sigma^\ell}$ coincide for any positive integer ℓ; we deduce this result from the uniqueness of the solution of (10) [MW88], when this equation is applied to σ^ℓ. We thus assume in the present proof w.l.o.g. that u is a fixed point of σ and that the incidence matrix of σ has positive entries.

Let N be fixed. Let us use the Dumont-Thomas numeration system to expand $u_0 \ldots u_{N-1}$ as $\sigma^n(p_n) \ldots p_0$, with $(p_n, a_n, s_n) \ldots (p_0, a_0, s_0) \in F_\mathcal{P}$, where $\sigma(u_0) = p_n a_n s_n$. One has according to (12)

$$\begin{aligned}
\pi(\mathbf{l}(u_0 \ldots u_{N-1})) &= \sum_{2 \leq j \leq d} \langle \mathbf{l}(\sigma^n(p_n) \ldots \sigma(p_1) p_0), \mathbf{v}_{\beta^{(j)}} \rangle \mathbf{u}_{\beta^{(j)}} \\
&= \sum_{2 \leq j \leq d} \left((\beta^{(j)})^n \langle \mathbf{l}(p_n), \mathbf{v}_{\beta^{(j)}} \rangle + \cdots + \langle \mathbf{l}(p_0), \mathbf{v}_{\beta^{(j)}} \rangle \right) \mathbf{u}_{\beta^{(j)}} \\
&= \Psi_\sigma \left[(\beta^{(j)})^n \langle \mathbf{l}(p_n), \mathbf{v}_{\beta^{(j)}} \rangle + \cdots + \langle \mathbf{l}(p_0), \mathbf{v}_{\beta^{(j)}} \rangle \right]_{2 \leq j \leq r+s} \\
&= \Psi_\sigma \circ \Phi_\sigma(\delta_\sigma(p_n)\beta^n + \cdots + \delta_\sigma(p_0)) \in \Psi_\sigma \circ \Phi_\sigma(\mathbb{Z}_\sigma^+),
\end{aligned}$$

which implies that $\mathcal{R}_\sigma \subset \Psi_\sigma(\mathcal{T}_\sigma)$.

Conversely, let $(p_n, a_n, s_n) \ldots (p_0, a_0, s_0) \in F_\mathcal{P}$; by positivity of the incidence matrix of σ, $\sigma(u_0)$ contains the letter a_{n+1} defined by $\sigma(a_{n+1}) = p_n a_n s_n$; there thus exists $(p_{n+1}, a_{n+1}, s_{n+1}) \in \mathcal{P}$ with $\sigma(u_0) = p_{n+1} a_{n+1} s_{n+1}$ such that

$$(p_{n+1}, a_{n+1}, s_{n+1}) \ldots (p_0, a_0, s_0) \in F_\mathcal{P}.$$

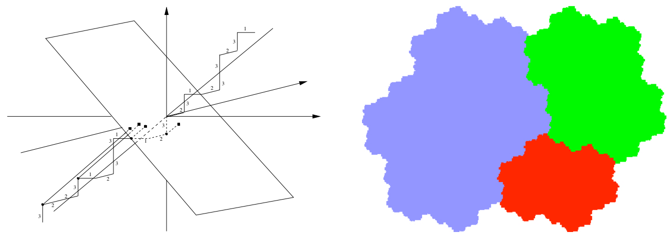

Figure 7: The projection method to get the Rauzy fractal for the Tribonacci substitution.

Hence $\sigma^{n+1}(p_{n+1})\dots p_0$ is a prefix of u, and we deduce $\Psi_\sigma(\mathcal{T}_\sigma) \subset \mathcal{R}_\sigma$ from

$$\overline{\Phi_\sigma(\mathbb{Z}_\sigma^+)} = \overline{\Phi_\sigma\{\sum_{i=0}^{M} \delta_\sigma(p_i)\beta^i;\, M \in \mathbb{N}, (p_M, a_M, s_M)\dots(p_0, a_0, s_0) \in F_{\mathcal{P}}, \sigma(u_0) = p_M a_M s_M\}}.$$

One similarly proves that $\mathcal{R}_\sigma(a) = \Psi_\sigma(\mathcal{T}_\sigma(a))$, for $a \in \mathcal{A}$, by noticing that if one expands $u_0 \dots u_{N-1}$ as $\sigma^n(p_n)\dots p_0$ in the Dumont-Thomas numeration system with $(p_n, a_n, s_n)\dots(p_0, a_0, s_0) \in F_{\mathcal{P}}$, then $u_N = a_0$. ∎

3.2. Domain exchange dynamical system

In Section 3.1 we give a geometric interpretation of the combinatorial expansions of finite prefixes as projections on the contracting space $\mathbb{H}_c$ of the vertices of the broken line (11). This allows us to introduce a dynamics on the central tile as a domain exchange. Indeed, from (14) one deduces that one can translate any point of a tile $\mathcal{R}_\sigma(a)$ by the projection of the a-th canonical vector $\mathbf{e}_a$ without exiting from the Rauzy fractal:

$$\begin{array}{rcl} \mathcal{R}_\sigma(a) + \pi(\mathbf{e}_a) & = & \overline{\pi\left(\{\mathbf{l}(u_0 \cdots u_{k-1}a);\ k \in \mathbb{N},\ u_k = a\}\right)} \\ & \subset & \overline{\pi\left(\{\mathbf{l}(u_0 \cdots u_{k-1}u_k);\ k \in \mathbb{N}\}\right)} = \mathcal{R}_\sigma. \end{array}$$

Since $\pi(\mathbf{e}_a) = \Psi_\sigma \circ \Phi_\sigma \circ \delta_\sigma(a)$, one gets in $\mathbb{K}_\sigma$

$$\forall a \in \mathcal{A},\ \mathcal{T}_\sigma(a) + \Phi_\sigma \circ \delta_\sigma(a) \subset \mathcal{T}_\sigma. \tag{15}$$

If furthermore, the substitution σ satisfies the strong coincidence condition, the basic tiles are almost everywhere disjoint according to Theorem 4, so that (15) defines a domain exchange on the central tile as

$$E_\sigma :\ \mathcal{T}_\sigma \to \mathcal{T}_\sigma,\ x \in \mathcal{T}_\sigma(a) \mapsto x + \Phi_\sigma \circ \delta_\sigma(a) \in \mathcal{T}_\sigma.$$

It is natural to code, up to the partition provided by the n basic tiles, the action of the domain exchange over the central tile $\mathcal{T}_\sigma$. Theorem 6 below says that the codings of the orbits of the points in the central tile under the action of this domain exchange are described by the substitutive dynamical system, that is, the coding map, from $\mathcal{T}_\sigma$ onto the

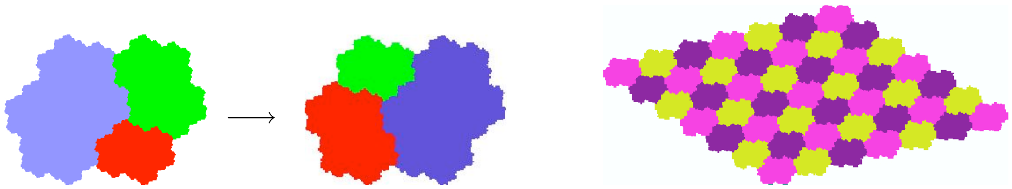

Figure 8: Domain exchange over the Rauzy fractal; lattice substitution multiple tiling

n-letter full shift $\{1,\ldots,n\}^{\mathbb{Z}}$ is almost everywhere one-to-one, and onto the substitutive dynamical system (X_σ, S). We thus have given an interpretation of the action of the shift map on X_σ as an exchange of domains acting on the basic tiles.

Theorem 6 ([AI01, CS01b]) *Let σ be a primitive substitution that satisfies the strong coincidence condition, and that has a dominant eigenvalue which is a Pisot unit. Then the domain exchange $E_\sigma : \mathcal{T}_\sigma \to \mathcal{T}_\sigma$, $x \in \mathcal{T}_\sigma(a) \mapsto x + \Phi_\sigma \circ \delta_\sigma(a) \in \mathcal{T}_\sigma$ is defined almost everywhere on the central tile. The substitutive dynamical system (X_σ, S) is measure-theoretically isomorphic to $(\mathcal{T}_\sigma, E_\sigma)$, that is, the representation map $\mu_\sigma : X_\sigma \to \mathcal{T}_\sigma$ of the substitutive dynamical system (X_σ, S) is continuous, onto and almost everywhere one-to-one, and satisfies $\mu_\sigma \circ S = E_\sigma \circ \mu_\sigma$.*

The Dumont-Thomas expansion appears as a realization of the isomorphism between the set of trajectories under E_σ and the central tile. Indeed, recall that

$$\forall (p_i, a_i, s_i)_{i\geq 0} \in X^l_{\mathcal{P}},\ \varphi_\sigma((p_i, a_i, s_i)_{i\geq 0}) = \lim_{n\to+\infty} \Phi_\sigma(\sum_{i\geq 0} \delta_\sigma(p_i)\beta^i) \in \mathcal{T}_\sigma$$

where $(p_i, a_i, s_i)_{i\geq 0}$ is the combinatorial prefix-suffix expansion of $w \in X_\sigma$, that is,

$$w = \lim_{n\to+\infty} \sigma^n(p_n)\ldots\sigma(p_1)p_0.a_0s_0\sigma(s_1)\ldots\sigma^n(s_n).$$

3.3. The lattice substitution multiple tiling

We want now to associate with the substitution σ a lattice covering of $\mathbb{K}_\sigma$. This will also allow us to associate with a beta-numeration a lattice tiling by considering the associated β-substitution.

The domain exchange E_σ is defined almost everywhere, but not everywhere, which prevents us from defining a continuous dynamics on the full central tile. A solution to this problem consists of factorizing the central tile $\mathcal{T}_\sigma$ by the smallest possible lattice so that the translation vectors $\Phi_\sigma \circ \delta_\sigma(a)$ for $a \in \mathcal{A}$ do coincide: we thus consider the subgroup

$$\sum_{i=1}^{n-1} \mathbb{Z}\Phi_\sigma(\delta_\sigma(i) - \delta_\sigma(n))$$

of $\mathbb{K}_\sigma$. In the case where this group is discrete, the quotient is a compact group and the domain exchange factorizes into a minimal translation on a compact group. This group is discrete if and only if $(\delta_\sigma(i) - \delta_\sigma(n))$, for $i = 1, \cdots, n-1$, are rationally independent with 1 in $\mathbb{Z}[\beta]$. A sufficient condition is thus that n equals the degree d of β, that is, that every eigenvalue of the substitution is a conjugate of the dominant eigenvalue β. We thus assume that we are in the irreducible case, that is, that σ is a unimodular Pisot substitution. Let us note that this implies severe restrictions for the beta-numeration: the cardinality of the alphabet of the β-substitution is equal to the degree of β.

Theorem 7 ([AI01, CS01b]) *Let σ be a unimodular Pisot substitution on the alphabet $\mathcal{A} = \{1, 2, \cdots, n\}$. Then the central tile generates a lattice multiple tiling of $\mathbb{K}_\sigma$:*

$$\mathcal{T}_\sigma + \sum_{i=1}^{n-1} \mathbb{Z}\Phi_\sigma(\delta_\sigma(i) - \delta_\sigma(d)) = \mathbb{K}_\sigma. \tag{16}$$

This multiple tiling is classical for substitutions but it is rarely associated with a beta-numeration, although it has a nice spectral interpretation in the irreducible case: the multiple tiling is a tiling if and only if the beta-shift has pure discrete spectrum ([BK04, Sie04]).

Proof. An intuitive approach of the proof is given by its interpretation in the full space $\mathbb{R}^n$. It is equivalent to prove $\cup_{\gamma \in \mathcal{L}} \mathcal{R}_\sigma + \gamma = \mathbb{H}_c$, where $\mathcal{L} := \mathbb{Z}\pi(\mathbf{e}_1) + \cdots + \mathbb{Z}\pi(\mathbf{e}_n)$. Indeed, the Pisot hypothesis and the fact that $d = n$ imply that $\mathcal{L}$ is a lattice (the proof is similar to that of Lemma 2).

The translates along the lattice $\mathcal{L}_0 = \sum_{i=1}^{n-1} \mathbb{Z}(\mathbf{e}_i - \mathbf{e}_d)$ of the vertices of the broken line $\mathbf{l}(u_0 u_1 \ldots u_{N-1})$, $N \in \mathbb{N}$, cover the following upper half-space:

$$\{\mathbf{l}(u_0 u_1 \ldots u_{N-1}) + \gamma;\ N \in \mathbb{N},\ \gamma \in \mathcal{L}_0\} = \{x \in \mathbb{Z}^n;\ \langle x, \mathbf{v}_\beta \rangle \geq 0\}. \tag{17}$$

As an application of the Kronecker theorem, the projection by π of the upper half-space is dense in the contracting plane $\mathbb{H}_c$. By definition, the lattice $\mathcal{L}$ is a Delaunay set. Consequently, given any point P of $\mathbb{H}_c$, there exists a sequence of points $(\pi(\mathbf{l}(u_0 u_1 \ldots u_{N_k - 1})) + \gamma_k)_k$ with γ_k in the lattice $\mathcal{L}$ which converges to P in $\mathbb{H}_c$. Since $\mathcal{R}_\sigma$ is a bounded set, there are infinitely many k for which the points γ_k of the lattice $\mathcal{L}_0$ take the same value, say, γ; we thus get $P \in \mathcal{R}_\sigma + \gamma$, which implies (16). ∎

In some specific situations in the reducible case $d < n$, some of the numbers $\delta_\sigma(i)$ are equal, so that a factorization can be performed even if the substitution is not a Pisot substitution. See for instance [EI05] for a detailed study of the β-substitution associated with the smallest Pisot number.

3.4. Model sets

The aim of this section is to reformulate some of the previous results in terms of a cut and project scheme, as defined in Section 1.2. We assume here that σ is a unimodular Pisot substitution. We recall that π' stands for the projection in $\mathbb{R}^d$ on the expanding line generated by $\mathbf{u}_\beta$ along the plane $\mathbb{H}_c$, whereas we denote by π the projection on the plane $\mathbb{H}_c$ along the expanding line. Theorem 8 states that the vertices of the broken line are exactly the points of $\mathbb{Z}^n$ selected by shifting the central tile $\mathcal{R}_\sigma$ (considered as an acceptance window), along the expanding eigendirection $\mathbf{u}_\beta$.

Theorem 8 *Let σ be a unimodular Pisot substitution over an n-letter alphabet such that the lattice multiple tiling is indeed a tiling. We are in particular in the irreducible case $d = n$. We assume furthermore that the projections by π of the vertices of the broken line belong to the interior of the Rauzy fractal. The subset $\pi'(\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N}\})$ of the expanding eigenline obtained by projecting the vertices of the broken line given by a periodic point of the substitution σ is a Meyer set associated with the cut and project scheme $(\mathbb{R} \times \mathbb{R}^{n-1}, \mathbb{Z}^n)$, whose acceptance window is the interior of the Rauzy fractal $\mathcal{R}_\sigma = \Psi_\sigma(\mathcal{T}_\sigma)$. In other words,*

$$\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N}\} = \{P = (x_1, \cdots, x_n)) \in \mathbb{Z}^n;\ \sum_{1 \le i \le n} x_i \ge 0; \pi(P) \in \mathrm{Int}(\mathcal{R}_\sigma)\}. \tag{18}$$

Proof. Let $H = \mathbb{R}^{n-1}$, $D = \mathbb{Z}^n$, $k = 1$. The set $H = \mathbb{R}^{n-1}$ is in one-to-one correspondence with the plane $\mathbb{H}_c$, whereas $\mathbb{R}$ is in one-to-one correspondence with the expanding eigenline. Up to these two isomorphisms, the natural projections p_0 and p_1 become respectively π and π' and are easily seen to satisfy the required conditions. It remains to prove (18).

By assumption, for every N, $\pi(\mathbf{l}(u_0 \cdots u_{N-1})) \in \mathrm{Int}(\mathcal{R}_\sigma)$. Conversely, let $P = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ with $\sum x_i \ge 0$ such that $\pi(P) \in \mathrm{Int}(\mathcal{R}_\sigma)$. Let $N = x_1 + \ldots x_n$. According to (17), there exists $\gamma \in \mathcal{L}_0$ such that $P = \mathbf{l}(u_0 \cdots u_{N-1}) + \gamma$. Since $\pi(P) = \pi(\mathbf{l}(u_0 \cdots u_{N-1})) + \pi(\gamma) \in \mathrm{Int}(\mathcal{R}_\sigma)$, one gets $\gamma = 0$ (we have assumed that we have a tiling) and $P = \mathbf{l}(u_0 \cdots u_{N-1})$. ∎

4. Discrete planes and $\mathbb{Z}^2$-actions

The aim of this section is to present an alternative construction and interpretation of the self-replicating multiple tiling associated with a primitive Pisot unimodular substitution as described in Section 2. This construction is based on the notion of a geometric generalized substitution due to [AI01], see also [IR06]. We respectively introduce the notion of a discrete plane in Section 4.1, of a substitution acting on it in Section 4.2, and end this section by giving a substitutive version of the finiteness condition (F) introduced in Section 1.6.

4.1 Discrete planes and discrete multiple tilings

We assume that σ is a unimodular Pisot substitution on the alphabet $\mathcal{A} = \{1, 2, \cdots, n\}$. The algebraic degree d of the dominant eigenvalue β of σ satisfies $d = n$ (irreducible case).

Let us first consider a discretisation of the contracting hyperplane $\mathbb{H}_c = \{\mathbf{x} \in \mathbb{Z}^n;\ \langle \mathbf{x}, \mathbf{v}_\beta \rangle = 0\}$ corresponding to the notion of an arithmetic plane introduced in [Rev91]; this notion consists in approximating the plane $\mathbb{H}_c$ by selecting points with integral coordinates above and within a bounded distance of the plane. More generally, given $\mathbf{v} \in \mathbb{R}^n$, $\mu, \omega \in \mathbb{R}$, the *lower* (resp. *upper*) *discrete hyperplane* $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is the set of points $\mathbf{x} \in \mathbb{Z}^d$ satisfying $0 \leq \langle \mathbf{x}, \mathbf{v} \rangle + \mu < \omega$ (resp. $0 < \langle \mathbf{x}, \mathbf{v} \rangle + \mu \leq \omega$). The parameter μ is called the *translation parameter* whereas ω is called the *thickness*. Moreover, if $\omega = \max\{v_i\} = |\mathbf{v}|_\infty$ (resp. $\omega = \sum v_i = |\mathbf{v}|_1$) then $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is said to be *naive* (resp. *standard*). For more details, see for instance the survey [BCK04].

We consider only standard discrete planes in this paper, hence we call them discrete planes, for the sake of simplicity. We consider more precisely the standard lower arithmetic discrete plane with parameter $\mu = 0$ associated with $\mathbf{v}_\beta$ that we denote for short by $\mathfrak{P}_\sigma$. We now associate with $\mathfrak{P}_\sigma \subset \mathbb{Z}^n$ the *stepped plane* $\mathcal{S}_\sigma \subset \mathbb{R}^n$ defined as the union of faces of integral cubes that connect the points points of $\mathfrak{P}_\sigma$, as depicted in Fig. 9, an *integral cube* being any translate of the fundamental unit cube $\mathcal{C} = \{\sum_{1 \leq i \leq n} \lambda_i \mathbf{e}_i;\ \lambda_i \in [0, 1],\ \text{ for all } i\}$ with integral vertices. The stepped surface $\mathcal{S}_\sigma$ is thus defined as the boundary of the set of integral cubes that intersect the lower open half-space $\{\mathbf{x} \in \mathbb{Z}^n;\ \langle \mathbf{x}, \mathbf{v}_\beta \rangle < 0\}$. The vertices of $\mathcal{S}_\sigma$ (that is, the points with integer coordinates of $\mathcal{S}_\sigma$) are exactly the points of the arithmetic discrete plane $\mathfrak{P}_\sigma$, according for instance to [BV00, AI01, ABI02, ABS04].

Theorem 9 ([BV00, AI01, ABI02, ABS04]) *Let σ be a unimodular Pisot substitution. The stepped surface $\mathcal{S}_\sigma$ is spanned by:*

$$\mathcal{S}_\sigma = \bigcup_{(\mathbf{x},i) \in \mathbb{Z}^n \times \mathcal{A},\ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} (\mathbf{x}, i), \tag{19}$$

where for $\mathbf{x} \in \mathbb{Z}^n$ and for $1 \leq i \leq n$:

$$(\mathbf{x}, i) := \{\mathbf{x} + \sum_{j \neq i} \lambda_j \mathbf{e}_j;\ 0 \leq \lambda_j \leq 1,\ \ \textit{for } 1 \leq j \leq n,\ j \neq i\}.$$

This union is a disjoint union up to the boundaries of the faces.

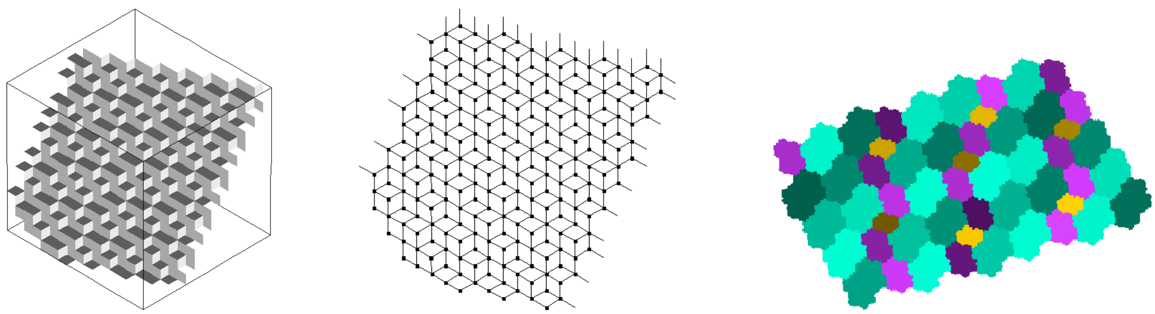

Figure 9: Stepped surface; pointed faces in the stepped surface; discrete multiple tiling.

Let us project now the discrete plane $\mathfrak{P}_\sigma$ on the contracting space $\mathbb{H}_c$ and replace each face $(\mathbf{x}, i)$ by the corresponding basic piece of the Rauzy fractal $\mathcal{R}_\sigma(i)$. The tiling (19) becomes

$$\mathbb{H}_c = \bigcup_{(\mathbf{x},i)\in\mathbb{Z}^n\times\mathcal{A},\ 0\leq\langle\mathbf{x},\mathbf{v}_\beta\rangle<\langle\mathbf{e}_i,\mathbf{v}_\beta\rangle} \pi(\mathbf{x}) + \mathcal{R}_\sigma(i). \tag{20}$$

It is shown in [IR05] that the covering (20) provides a multiple tiling of the contracting hyperplane, and that the translation vector set $\pi(\{\mathbf{x} \in \mathbb{Z}^n;\ \exists i \in \mathcal{A},\ (\mathbf{x}, i) \in \mathcal{S}_\sigma)\}$ is a Delaunay set. The aim of the present section is to give a simple proof of this result.

Theorem 10 *Let σ be a Pisot unimodular substitution. The projection by π of the discrete plane $\mathfrak{P}_\sigma$ associated with the contracting space $\mathbb{H}_c$ generates a covering of the contracting hyperplane, called the* discrete multiple tiling *of the substitution:*

$$\mathbb{H}_c \simeq \mathbb{R}^{n-1} = \bigcup_{(\mathbf{x},i)\in\mathbb{Z}^n\times\mathcal{A},\,0\leq\langle\mathbf{x},\mathbf{v}_\beta\rangle<\langle\mathbf{e}_i,\mathbf{v}_\beta\rangle} \pi(\mathbf{x}) + \mathcal{R}_\sigma(i).$$

The discrete multiple tiling is the embedding under the action of Ψ_σ of the self-replicating multiple tiling, that is,

$$\mathbb{K}_\sigma = \bigcup_{\mathbf{x}\in\mathbb{Z}^n\times\mathcal{A},\ 0\leq\langle\mathbf{x},\mathbf{v}_\beta\rangle<\langle\mathbf{e}_i,\mathbf{v}_\beta\rangle} \Phi_\sigma(<\mathbf{x}, \mathbf{v}_\beta>) + \mathcal{T}_\sigma(i). \tag{21}$$

Proof. The proof becomes easy in our context by noticing that (20) becomes under the action of Ψ_σ^{-1}

$$\mathbb{K}_\sigma = \bigcup_{(\mathbf{x},i)\in\mathbb{Z}^n\times\mathcal{A},\ 0\leq\langle\mathbf{x},\mathbf{v}_\beta\rangle<\langle\mathbf{e}_i,\mathbf{v}_\beta\rangle} \Phi_\sigma(<\mathbf{x}, \mathbf{v}_\beta>) + \mathcal{T}_\sigma(i).$$

We thus deduce (21) from (10) and from the following lemma.

Lemma 3 *One has for every $i \in \mathcal{A} = \{1, \cdots, n\}$*

$$\Psi_\sigma \circ \Phi_\sigma\left(\mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] \cap [0, \delta_\sigma(i))\right) = \pi(\{\mathbf{x} \in \mathbb{Z}^n;\ 0 \leq \langle\mathbf{x}, \mathbf{v}_\beta\rangle < \langle\mathbf{e}_i, \mathbf{v}_\beta\rangle\}).$$

Proof. Let $\mathbf{x} \in \mathbb{Z}^n$ such that $0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle = \delta_\sigma(i)$. By definition $z := \langle \mathbf{x}, \mathbf{v}_\beta \rangle \in \mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] \cap [0, \delta_\sigma(i))$. Since $\mathbf{x}$ has rational coordinates, its coordinates in the algebraic normalized eigenbasis are conjugate according to (12), so that $\mathbf{x} = z\mathbf{u}_\beta + \Psi_\sigma \circ \Phi_\sigma(z)$ and $\pi(\mathbf{x}) = \Psi_\sigma \circ \Phi_\sigma(z) \in \Psi_\sigma \circ \Phi_\sigma \left(\mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] \cap [0, \delta_\sigma(i)\right)$.

Conversely, let $z \in \mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] \cap [0, \delta_\sigma(i))$. There exists $\mathbf{x} \in \mathbb{Z}^n$ such that $z = \langle \mathbf{x}, \mathbf{v}_\beta \rangle$ according to (9). Since $\mathbf{x}$ has rational coordinates, $\pi(\mathbf{x}) = \Psi_\sigma \circ \Phi_\sigma(z)$, and the conclusion follows. ∎

One interest of this approach is that it establishes a correspondence between two multiple tilings that a priori have nothing in common, a multiple tiling provided on the one hand by a discrete approximation of the contracting plane of the substitution, and on the other hand, a multiple tiling obtained via the Dumont-Thomas numeration as a generalization of the β-numeration.

4.2. Substitutive construction of the discrete plane and extended (F) property

Substitution rule. We define $\mathcal{F}^*$ as the $\mathbb{R}$-vector space generated by $\{(\mathbf{x}, i);\ \mathbf{x} \in \mathbb{Z}^n,\ i \in \{1, 2, \cdots, n\}\}$. We define the following generation process which can be considered as a geometric realization of the substitution σ on the geometric set $\mathcal{F}^*$ consisting of finite sums of faces:

$$\forall (\mathbf{x}, a) \in \mathbb{Z}^n \times \mathcal{A},\ E_1^*(\sigma)(\mathbf{x}, a) = \bigcup_{\sigma(b) = pas} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{l}(p)), b).$$

Proposition 2 *[AI01] The stepped surface $\mathcal{S}_\sigma$ is stable under the action of $E_1(\sigma)^*$ and contains the faces $(\mathbf{0}, 1)$, ..., $(\mathbf{0}, n)$.*

Theorem 11 *Let σ be a unimodular Pisot substitution. Let* (F) *be the following extended* (F)*-property:*

$$\text{(F)} \qquad \forall a \in \mathcal{A} = \{1 \ldots n\}, \forall z \in \mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] \cap [0, \delta_\sigma(a)), \qquad d_{(\sigma,a)}(z) \text{ is finite.}$$

The extended (F)*-property is satisfied if and only if the images of the unit cube located at the origin $\mathcal{C} = \cup_{1 \leq i \leq n} (\mathbf{0}, i)$ under the iterated action of $E_1^*(\sigma)$ cover the full stepped plane, that is,*

$$\mathcal{S}_\sigma = \bigcup_{k \in \mathbb{N}} E_1^*(\sigma)^k((\mathbf{0}, 1) \cup \cdots \cup (\mathbf{0}, n)).$$

If the extended (F)*-property is satisfied and if the substitution satisfies the strong coincidence property, then the self-replicating substitution multiple tiling, which coincides with the discrete multiple tiling, is indeed a tiling.*

Proof. Let us assume that the extended (F) property holds. Let $(\mathbf{x}, a) \in \mathcal{S}_\sigma$. From Lemma 3, $z :=< \mathbf{x}, \mathbf{v}_\beta >\in \mathbb{Z}[\delta_\sigma(1), \cdots, \delta_\sigma(n)] \cap [0, \delta_\sigma(a))$. Let $d_{\sigma,a}(z) = (q_1, b_1, r_1) \cdots (q_k, b_k, r_k) \in \mathcal{F}_\mathcal{P}$ stand for its finite (σ, a)-expansion, according to the extended (F)-property. The face $(\mathbf{0}, b_k)$ belongs to $\mathcal{S}_\sigma$ following Proposition 2. Let $\mathbf{y} := \mathbf{M}^{-k}\mathbf{l}(q_k) + \mathbf{M}^{-k+1}\mathbf{l}(q_{k-1}) \cdots + M^{-1}\mathbf{l}(q_1)$. One checks that $(\mathbf{y}, a)$ belongs to $E_1^*(\sigma)^k(\mathbf{0}, b_k)$. By construction,

$$\langle \mathbf{y}, \mathbf{v}_\beta \rangle = \sum_{i=1}^{k} \delta_\sigma(q_i)\beta^{-i} = z =< \mathbf{x}, \mathbf{v}_\beta >$$

so that $\mathbf{y} = \mathbf{x}$ (both are integral points with the same projection on the discrete plane). We deduce that $(\mathbf{x}, a) \in \mathcal{S}_\sigma$. The converse follows similarly.

Let us assume that both the strong coincidence condition and the extended (F) property hold. Let k be fixed. According to Theorem 4 applied to σ^k, for every pair of faces $(\mathbf{x}, i), (\mathbf{y}, j) \in E_1^*(\sigma)^k((\mathbf{0}, 1) \cup \cdots \cup (\mathbf{0}, n))$, the tiles $\pi(\mathbf{x}) + \mathcal{R}_\sigma(i)$ and $\pi(\mathbf{y}) + \mathcal{R}_\sigma(j)$ are measurably disjoint. Since according the extended (F) property every point in the self-replicating substitution multiple tiling belongs to such an iterate, all the tiles are measurably disjoint and the multiple tiling is indeed a tiling. ∎

5. Equivalent tilings

Let us conclude this survey by alluding to the connections between the various multiple tilings that have been introduced in this paper and by reviewing some sufficient tiling conditions.

Different multiple tilings associated with unimodular primitive substitutions that have a Pisot dominant eigenvalue where introduced in the previous sections, namely, a self-replicating multiple tiling based on the Dumont-Thomas numeration system, a lattice multiple tiling, and lastly, a discrete multiple tiling; these two latter tilings are both defined in the irreducible case, that is, when the substitution satisfies the extra hypothesis to be a Pisot substitution. In this latter case, the discrete and the self-replicating multiple tilings coincide.

In the irreducible case, Ito and Rao prove in [IR06] that the lattice multiple tiling and the self-replicating multiple tiling are simultaneously tilings. The proof is based on the following construction: from the Rauzy fractal associated to a substitution on a d-letter alphabet, that is, a compact set in a $(d-1)$-dimensional space, one builds a compact set $\tilde{\mathcal{R}}_\sigma$ with nonempty interior in $\mathbb{R}^d$; this set is defined as the union of d cylinders with a transverse component along the expanding direction, based on each piece of the Rauzy fractal $\mathcal{R}_\sigma(a)$(living in the contracting space $\mathbb{H}_c$), of height equal to the size of the interval $[0, \delta_\sigma(a))$ in $\mathbb{R}$:

$$\tilde{\mathcal{R}}_\sigma = \bigcup_{a=1}^{d} \left(\mathcal{R}_\sigma(a) + [0, \delta_\sigma(a))\mathbf{u}_\beta\right).$$

This set is called the *Markov Rauzy fractal*; it is depicted in Fig. 10 in the Tribonacci case. According to Theorem 4, if the substitution σ satisfies the strong coincidence condition, then the pieces $\mathcal{R}_\sigma(a)$ are disjoint in measure in $\mathbb{H}_c$; this implies that the cylinders are also almost everywhere disjoint in measure in $\mathbb{R}^d$. One recovers in a natural way from the Markov multiple tiling the lattice and the self-replicating ones by intersecting it with a suitable hyperplane (see Fig. 11 in the Fibonacci case $1 \mapsto 12$, $2 \mapsto 1$). Consequently, as soon as one of those multiple tilings can be proved to be a tiling, then all the other multiple tilings are also indeed tilings [IR06].

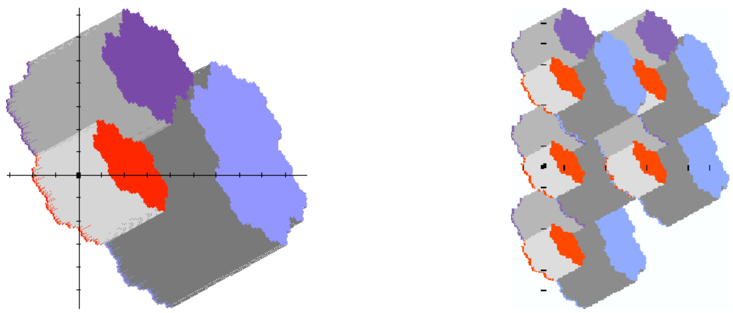

Figure 10: Markov Rauzy fractal; Markov multiple tiling.

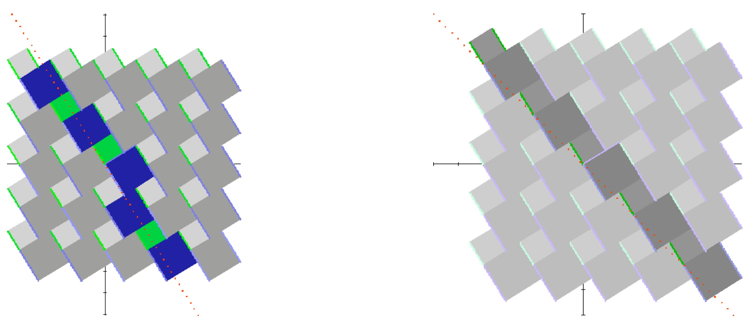

Figure 11: The Fibonacci Markov tiling: the self-replicating tiling lies in the intersection of the Markov tiling with the contracting direction; the lattice tiling is given by the projection on the contracting line of the pieces that cross the anti-diagonal line $x + y = 0$.

Central tiles associated with Pisot beta-shifts and substitutive dynamical systems provide efficient geometric representations of the corresponding dynamical systems, as illustrated in Section 3. One gets in particular a combinatorial necessary and sufficient condition for a substitutive unimodular system of Pisot type to be measure-theoretically isomorphic to its maximal translation factor [Sie04]. This has also consequences for the effective construction of Markov partitions for toral automorphisms, the main eigenvalue of which is a Pisot number [IO93, IO94, Pra99, Sie00]. Based on the approach of [KV98, VS93, SV98, Sid03a], an algebraic construction of symbolic representations of hyperbolic toral automorphisms as Markov partitions is similarly given in [Sch00, LS04], where homoclinic points are shown to play an essential role.

Further essential facts about this Markov Rauzy fractal are proved in [IR06]:

- The Markov Rauzy fractal provides a lattice multiple tiling of $\mathbb{R}^d$:

$$\bigcup_{\mathbf{z}\in\mathbb{Z}^d} \tilde{\mathcal{R}}_\sigma + \mathbf{z} = \mathbb{R}^d.$$

- If one intersects the contracting hyperplane $\mathbb{H}_c$ with the lattice multiple tiling of $\mathbb{R}^d$, one recovers the discrete multiple tiling:

$$\bigcup_{\mathbf{z}\in\mathbb{Z}^d} \pi(\tilde{\mathcal{R}}_\sigma + \mathbf{z}) \cap \mathbb{H}_c = \bigcup_{a\in\{1,\dots d\},\, \mathbf{z}\in\mathbb{Z}^d,\, \langle \mathbf{z},\mathbf{v}_\beta\rangle\in[0,\delta_\sigma(a))} \mathcal{R}_\sigma(a) + \pi(\mathbf{z}).$$

- The projections onto $\mathbb{H}_c$ of the pieces that cross the anti-diagonal hyperplane provide the lattice multiple tiling:

$$\bigcup_{\mathbf{z}\in\mathbb{Z}^d,\, \langle \mathbf{z},(1,\dots 1)\rangle=0} \pi(\tilde{\mathcal{R}}_\sigma + \mathbf{z}) = \mathcal{R}_\sigma + \sum_{1}^{d-1} \mathbb{Z}\pi(\mathbf{e}_i - \mathbf{e}_d).$$

Theorem 12 ([IR06]) *Let σ be a unimodular Pisot substitution. The thickness of a multiple tiling is defined as the integer k such that this covering is almost everywhere k-to-one. The self-replicating multiple tiling, the lattice multiple tiling, the discrete multiple tiling and the Markov multiple tiling have the same thickness. In particular, one of them is a tiling if and only if all the other ones are tilings.*

For each of these multiple tilings, some tiling conditions can be expressed according to the context in which the multiple tiling is defined (numeration, substitution, discrete geometry, symbolic dynamics...) Some of them are necessary and sufficient conditions, whereas the others are only sufficient conditions, but effective. We do not have space enough in the present paper to introduce all the formalism that is necessary to detail each of these conditions. Hence, the reader will find as a conclusion a small description of each of these conditions with the corresponding references.

- Super coincidence condition [IR06, BK04]: the discrete multiple tiling is a tiling if and only if the substitution satisfies a combinatorial condition called the super coincidence condition, which can be seen as a geometric generalization of the strong coincidence property.

- Balanced pairs [BR05]: the super coincidence condition is satisfied if and only if the balanced pair algorithm terminates.

- Geometry [IR06]: the discrete multiple tiling is a tiling if and only if the measure of at least one basic tile $\mathcal{R}_\sigma(i)$ is equal to the measure of the rhombus $\pi(\mathbf{0}, i)$.

- Discrete geometry [IR06]: the discrete multiple tiling is a tiling if and only if for every $i \in \mathcal{A}$, the sequence of the boundaries of the polygonals $X_n(i) := \pi(E_1^*(\sigma)^n(\mathbf{0}, i))$ tends to the boundary of the i-th basic tile $\mathcal{R}_\sigma(i)$ for the Hausdorff metric.

- Spectral theory [BK04, Sie04]: the lattice multiple tiling is a tiling if and only if the substitutive dynamical system (X_σ, S) has a purely discrete spectrum.

- Contact graphs [Sie04, Thu04]: the lattice multiple tiling [Sie04], respectively discrete multiple tiling [Thu04], is a tiling if and only if a finite graph describing the intersection of the pieces in the tiling is "small enough", that is, if it does not recognize the same language as the prefix-suffix automaton. Both conditions provide algorithms that are rather long but allow one to test whether a multiple tiling is a tiling (which is not the case for the balanced pair algorithm).

Let us stress the fact that most of the results mentioned in the present paper were obtained under the assumption that β is a Pisot unit. It is one of the main challenges to try to relax the Pisot unit hypothesis, and to be able to prepare the playground for the non-Pisot case on the one hand, in the flavor of [KV98, LS04] and for the non-unit case, on the other hand, according to the p-adic approach developed in [Sie03, BS05]. Let us mention [AFHI05], where a simple example of an automorphism of the free group on 4 generators, with an associated matrix that has 4 distinct complex eigenvalues, two of them of modulus larger than 1, and the other 2 of modulus smaller than 1 (non-Pisot case) is handled in details. Let us recall that substitutions are particular cases of free group morphisms, the main simplification being that we have no problem of cancellations.

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