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Some constructions for the higher-dimensional three-distance theorem

by

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This paper is dedicated to Robert Tijdeman on the occasion of his 75th birthday

1. Introduction. For a given real number α in $(0,1)$, let us place the points $\{0\}, \{\alpha\}, \{2\alpha\}, \dots, \{(N-1)\alpha\}$ on the unit circle, where $\{x\}$ denotes as usual the fractional part of x. These points partition the unit circle into N intervals having at most three lengths, one being the sum of the other two. This property is known as the *three-distance theorem* and can be seen as a geometric interpretation of good approximation properties of the Farey partial convergents in the continued fraction expansion of α. In the literature, this theorem is also called the Steinhaus theorem or the three-length, three-gap, or three-step theorem.

The three-distance theorem was initially conjectured by Steinhaus, first proved V. T. Sós [S58] and Surányi [Surá58], and then by Slater [Sla64], Świerczkowski [Ś59], Halton [Hal65]. A survey of the different approaches used by these authors is found for instance in [AB98, vR88, Sla67, Lan91]. More recent proofs have also been given in [vR88, Lan91], or in [MS17] relying on the properties of the space of two-dimensional Euclidean lattices. See also [Ble91, PSZ16] for the study of the limiting distribution of the gaps.

There exist numerous generalizations of the three-gap theorem. Let us quote for instance generalizations to groups [FS92], to some isometries of compact Riemannian manifolds [BS08], or to interval exchange transformations [Tah17]. Among the generalizations, there are two natural Diophantine frameworks that are dual, namely distance theorems for toral translations

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on the d-dimensional torus $\mathbb{T}^d$ (see e.g. [Che07, Che14, Vij08]) and distance theorems for linear forms in d variables on the one-dimensional torus $\mathbb{T}$. This is the framework of the present paper, where we focus on linear forms in two variables, and consider points $m\alpha + n\beta$, for $0 \le n, m < N$, in $\mathbb{T}$.

This generalization has been considered by Erdős (as recalled in [GS93]) and also in [Lia79, CG76, GS93, FH95, Che00, BHJ$^+$12, HM17]. See also [CGVZ02] for the number of so-called primitive gaps. In particular, the following is proved in [Che00]. Let $\alpha_1, \dots, \alpha_d \in \mathbb{T}$ ($d \ge 3$) and $2 \le n_1 \le \cdots \le n_d$ be integers. The set $\{\sum_{i=1}^d k_i\alpha_i : 0 \le k_i < n_i,\ i = 1, \dots, d\}$ divides $\mathbb{T}$ into intervals whose lengths take at most $\prod_{i=1}^{d-1} n_i + 3\prod_{i=1}^{d-2} n_i + 1$ values. When $d = 2$, the upper bound is $N + 3$ for the case of interest here ($m\alpha + n\beta$, for $0 \le n, m < N$), as proved in [GS93].

There are natural cases where it is known that the number of distances is bounded (with respect to N, for the points $m\alpha + n\beta$ with $0 \le n, m < N$). This is the case when $1, \alpha, \beta$ are rationally dependent (this has been proved by Holzman, as recalled in [GS93]). Badly approximable vectors (α, β) have also been proved by Boshernitzan and Dyson to produce a finite number of distances. For a proof, see [BHJ$^+$12]. Nevertheless, it is proved in [HM17] that the number of lengths is generically unbounded, with an approach via homogeneous dynamics based on the ergodic properties of the diagonal action on the space of lattices. However, no explicit examples of this generic situation have been known. The object of the present paper is to construct such examples.

Our main result is the following.

THEOREM 1.1. *Consider the set* $E_N(\alpha, \beta) := \{n\alpha + m\beta \in \mathbb{T} : 0 \le n, m < N\}$, *and let* $\Delta(E_N(\alpha, \beta))$ *stand for the set of distances between neighbor points of* $E_N(\alpha, \beta)$. *We provide effective constructions for the following existence results.*

(i) *There exist* (α, β) *with* $1, \alpha, \beta$ *rationally independent and* (α, β) *not badly approximable such that*

$$\forall N, \qquad \#\Delta(E_N(\alpha, \beta)) \le 7.$$

(ii) *There exist* (α, β) *with* $1, \alpha, \beta$ *rationally independent such that*

$$\limsup_{N\to\infty} \#\Delta(E_N(\alpha, \beta)) = \infty.$$

Our proof avoids the use of a higher-dimensional analogue of continued fractions. We rely on the (regular) continued fraction expansions of α and β and we combine several 'rectangular' levels of points of the form $n\alpha + m\beta$ for $0 \le n < N$ and $0 \le m < M$, where N or M is the denominator of a principal convergent of α or β.

We ask the question of the minimality of the number of lengths: is it possible to find (α, β) with $1, \alpha, \beta$ rationally independent such that $\#\Delta(E_N(\alpha, \beta)) \leq 6$ for all N?

As an application and motivation for this theorem, one deduces results on frequencies of square factors in two-dimensional Sturmian words, such as studied in [BV00, BT02]. Two-dimensional Sturmian words are defined as codings of $\mathbb{Z}^2$-actions by rotations on the one-dimensional torus $\mathbb{T}$. More precisely, let α, β, ρ be real numbers with $1, \alpha, \beta$ rationally independent and $0 < \alpha + \beta < 1$. A *two-dimensional Sturmian word* over the three-letter alphabet $\{1, 2, 3\}$ (with parameters α, β, ρ) is defined as a function $f : \mathbb{Z}^2 \to \{1, 2, 3\}$ with, for all $(m, n) \in \mathbb{Z}^2$, ($f(m, n) = i \Leftrightarrow m\alpha + n\beta + \rho \in I_i$ modulo 1), where either $I_1 = [0, \alpha)$, $I_2 = [\alpha, \alpha + \beta)$, $I_3 = [\alpha + \beta, 1)$, or $I_1 = (0, \alpha]$, $I_2 = (\alpha, \alpha + \beta]$, $I_3 = (\alpha + \beta, 1]$. According to [BV00], the frequencies of square factors of size N are equal to the lengths obtained by putting on $\mathbb{T}$ the points $-n\alpha - m\beta$ for $-1 \leq n \leq N - 1$, $0 \leq m \leq N$. One thus has a correspondence between lengths and frequencies, whereas gap theorems correspond to return words. Note that convergence toward frequencies (expressed in terms of balance properties) has been considered in [BT02]. More generally, for results of the same flavor for cut and project sets generalizing the Sturmian framework, see [HKWS16, HJKW17].

Contents of the paper. Let us briefly sketch the contents of this paper. Notation is introduced in Section 2 together with a basic lemma (Lemma 2.1) that allows one to express in a convenient way the clockwise neighbor of a point of the form $n\alpha + m\beta$. A construction providing pairs (α, β) with a bounded number of lengths is described in Section 3, while the case of an unbounded number of lengths is handled in Section 4: statement (i) of Theorem 1.1 is proved in Section 3, and (ii) in Section 4.

2. Preliminaries. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let α, β be real numbers in $(0, 1)$. *We assume throughout that* $1, \alpha, \beta$ *are rationally independent.*

For q, q' positive integers, we define

$$E_{q,q'}(\alpha, \beta) := \{n\alpha + m\beta \in \mathbb{T} : 0 \leq n < q,\ 0 \leq m < q'\},$$

and when $q = q'$, we use the notation $E_N(\alpha, \beta) := E_{q,q'}(\alpha, \beta)$ with $N := q = q'$. We furthermore consider

$$\mathcal{E}_{q,q'}(\alpha, \beta) := \{(n, m) : 0 \leq n < q,\ 0 \leq m < q'\}.$$

We will also use the shorthand notation $E_{q,q'}$, E_N and $\mathcal{E}_{q,q'}$.

Points in $E_{q,q'}(\alpha, \beta)$ are considered as positioned on the unit circle oriented clockwise endowed with the origin point 0. The point $n\alpha + m\beta$ is thus considered as positioned at distance $\{n\alpha + m\beta\}$ from 0. The point immediately after $n\alpha + m\beta$ clockwise on the unit circle, that is, its clockwise *neighbor*, is denoted as $\Phi_{q,q'}(n\alpha+m\beta)$, or $\Phi(n\alpha+m\beta)$ if there is no confusion. This thus defines a map $\Phi_{q,q'}$ on $E_{q,q'}$ called the *neighbor map*. For two points a,b in $\mathbb{T}$, the interval (a,b) in $\mathbb{T}$ corresponds to the interval considered clockwise on the unit circle with endpoints a and b. The set $E_{q,q'}$ thus partitions the unit circle into disjoint intervals $(n\alpha+m\beta, \Phi_{q,q'}(n\alpha+m\beta))$ for $(n,m)\in\mathcal{E}_{q,q'}$.

For a finite subset E of $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, we denote by $\Delta(E)$ the set of distances between neighbor points of E (again with distances being measured clockwise). For any (n,m) in $\mathcal{E}_{q,q'}$, $\Delta_{q,q'}(n,m)$ (or $\Delta(n,m)$ if there is no confusion) stands for the distance between $n\alpha+m\beta$ and $\Phi_{q,q'}(n\alpha+m\beta)$.

For any positive integer q, we define the nonnegative integer $|n|_q$ by

$$|n|_q \equiv n \pmod q \quad \text{and} \quad 0\le |n|_q < q.$$

We will consider the map $n\mapsto |n+r|_q$ for a given integer r. In particular, if $0\le r<q$, then $|n+r|_q = n+r$ if $0\le n<q-r$, and $|n+r|_q=n+r-q$ if $q-r\le n<q$.

Let q,q' be given coprime positive integers. Then, for any integers r,r' such that $\gcd(r,q)=1$, $\gcd(r',q')=1$, $0<|r|<q$, $0<|r'|<q'$, the map

$$\varphi_{q,q'}\colon \mathcal{E}_{q,q'}\to\mathcal{E}_{q,q'}, \qquad (n,m)\mapsto (|n+r|_q, |m+r'|_{q'}),$$

is a cyclic permutation of $\mathcal{E}_{q,q'}$. We will thus be able to describe the elements of $\mathcal{E}_{q,q'}$ as the elements of the orbit of $(0,0)$ under $\varphi_{q,q'}$. In particular, for each (n,m) with $0\le n<q$, $0\le m<q'$, there exists a unique k with $0\le k<qq'$ satisfying $(n,m)=(|kr|_q,|kr'|_{q'})$. Indeed, since $1,\alpha,\beta$ are rationally independent, the map $\phi_{q,q'}$ acting on $E_{q,q'}(\alpha,\beta)$ and defined by (1)

$$\phi_{q,q'}(n\alpha+m\beta) = \langle \varphi_{q,q'}(n,m),(\alpha,\beta)\rangle \tag{2.1}$$

is easily seen to be injective, and thus surjective.

Let $(a_i)_{i\ge1}$, $(a'_j)_{j\ge1}$ stand for the respective sequences of partial quotients of α and β in their continued fraction expansions, and denote by $(q_i)_{i\ge1}$, $(q'_j)_{j\ge1}$ the denominators of their principal convergents. Note that we will make a strong use of

$$q_k\|q_{k-1}\alpha\| + q_{k-1}\|q_k\alpha\| = 1. \tag{2.2}$$

Here we denote $\|t\|$ by the distance from $t\in\mathbb{R}$ to the nearest integer.

We now consider $E_{q_i,q'_j}(\alpha,\beta)=\{n\alpha+m\beta\in\mathbb{T} : 0\le n<q_i,\ 0\le m<q'_j\}$ for indices i,j such that $q'_j=b'q_i+1$ for some positive integer b'. Note that $b'q'_{j-1}$ is coprime to q'_j since b' and q'_{j-1} are coprime to q'_j. We take $r:=-(-1)^iq_{i-1}$ and $r':=(-1)^jb'q'_{j-1}$. We consider the following cyclic

(1) This map is well-defined since $1,\alpha,\beta$ are rationally independent.

permutations acting respectively on $\mathcal{E}_{q_i,q'_j}(\alpha,\beta)$ and $E_{q_i,q'_j}(\alpha,\beta)$:

$$\varphi_{q_i,q'_j}\colon n\alpha+m\beta \mapsto (|n-(-1)^i q_{i-1}|_{q_i}, |m+(-1)^j b' q'_{j-1}|_{q'_j}),$$
$$\phi_{q_i,q'_j}\colon n\alpha+m\beta \mapsto |n-(-1)^i q_{i-1}|_{q_i}\alpha+|m+(-1)^j b' q'_{j-1}|_{q'_j}\,\beta.$$

Lemma 2.1 below shows that, under assumption (2.3) below, the clockwise neighbor point $\Phi_{q_i,q'_j}(n\alpha+m\beta)$ of $n\alpha+m\beta$ in $E_{q_i,q'_j}(\alpha,\beta)$ is exactly $\phi(n\alpha+m\beta)$, where we use the shorthand notation $\phi=\phi_{q_i,q'_j}$. Lemma 2.1 will be applied in the proofs of both statements of Theorem 1.1. In particular, it will play a crucial role in Section 3 for the case of a bounded number of lengths. Indeed, in order to count the number of lengths for a square set of points $\mathcal{E}_N$, we consider several rectangular subsets of $\mathcal{E}_N$, i.e., several levels in E_N, with the points of E_{q_i,q'_j} corresponding to the first level. Further levels of points will then be inserted or removed. Note that Lemma 2.1 provides a case where there are only four possible lengths.

Lemma 2.1. *Let α,β be real numbers in $(0,1)$ such that $1,\alpha,\beta$ are rationally independent. Let $(q_i)_{i\geq 1}$, $(q'_j)_{j\geq 1}$ stand for the denominators of their principal convergents. Assume that for some $i,j\geq 1$,*

$$q'_j = b' q_i + 1$$

for some positive integer b'. Let

$$\phi(n\alpha+m\beta) := \begin{cases} |n+q_{i-1}|_{q_i}\,\alpha+|m-b'q'_{j-1}|_{q'_j}\beta & \text{if } i,j \text{ are odd},\\ |n+q_{i-1}|_{q_i}\,\alpha+|m+b'q'_{j-1}|_{q'_j}\beta & \text{if } i \text{ is odd, } j \text{ is even},\\ |n-q_{i-1}|_{q_i}\,\alpha+|m-b'q'_{j-1}|_{q'_j}\beta & \text{if } i \text{ is even, } j \text{ is odd},\\ |n-q_{i-1}|_{q_i}\,\alpha+|m+b'q'_{j-1}|_{q'_j}\beta & \text{if } i,j \text{ are even}.\end{cases}$$

Then ϕ is a permutation of $E_{q_i,q'_j}(\alpha,\beta)$.

Under the further assumption that

$$\|q'_j\beta\| < \|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\|, \tag{2.3}$$

the maps Φ and ϕ coincide, that is, the point $\Phi(n\alpha+m\beta)$ immediately after $n\alpha+m\beta$ clockwise on the unit circle, for $0\leq n<q_i$, $0\leq m<q'_j$, is $\phi(n\alpha+m\beta)$. Moreover, the distance (measured clockwise) $\Delta(n,m)$ between $n\alpha+m\beta$ and $\phi(n\alpha+m\beta)$, for $0\leq n<q_i$, $0\leq m<q'_j$, takes one of the following values:

$$\begin{aligned}&\|q_{i-1}\alpha\|-b'\|q'_{j-1}\beta\|, && \|q_{i-1}\alpha\|-b'\|q'_{j-1}\beta\|-\|q'_j\beta\|,\\ &\|q_{i-1}\alpha\|-b'\|q'_{j-1}\beta\|+\|q_i\alpha\|, && \|q_{i-1}\alpha\|-b'\|q'_{j-1}\beta\|+\|q_i\alpha\|-\|q'_j\beta\|.\end{aligned}$$

More precisely, if i,j are odd, then $\Delta(n,m)$ equals:

$$\begin{array}{ll}
\|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\|, & 0 \le n < q_i - q_{i-1},\ b'q'_{j-1} \le m < q'_j, \\
\|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\| - \|q'_j\beta\|, & 0 \le n < q_i - q_{i-1},\ 0 \le m < b'q'_{j-1}, \\
\|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\| + \|q_i\alpha\|, & q_i - q_{i-1} \le n < q_i,\ b'q'_{j-1} \le m < q'_j, \\
\|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\| + \|q_i\alpha\| - \|q'_j\beta\|, & q_i - q_{i-1} \le n < q_i,\ 0 \le m < b'q'_{j-1},
\end{array}$$

and if i, j are even, then $\Delta(n,m)$ equals:

$$\begin{array}{ll}
\|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\|, & q_{i-1} \le n < q_i,\ 0 \le m < q'_j - b'q'_{j-1}, \\
\|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\| - \|q'_j\beta\|, & q_{i-1} \le n < q_i,\ q'_j - b'q'_{j-1} \le m < q'_j, \\
\|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\| + \|q_i\alpha\|, & 0 \le n < q_{i-1},\ 0 \le m < q'_j - b'q'_{j-1}, \\
\|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\| + \|q_i\alpha\| - \|q'_j\beta\|, & 0 \le n < q_{i-1},\ q'_j - b'q'_{j-1} \le m < q'_j.
\end{array}$$

Similar formulas hold for the other cases.

Proof. Recall that $\gcd(b'q'_{j-1}, q'_j) = 1$. We first assume that i, j are odd. Then

$$q_{i-1}\alpha - p_{i-1} = \|q_{i-1}\alpha\|, \quad q'_{j-1}\beta - p'_{j-1} = \|q'_{j-1}\beta\|$$

and

$$q_i\alpha - p_i = -\|q_i\alpha\|, \quad q'_j\beta - p'_j = -\|q'_j\beta\|.$$

Therefore,

$$q_{i-1}\alpha - b'q'_{j-1}\beta = \|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\| + (p_{i-1} - b'p'_{j-1}).$$

It follows that

$$\begin{aligned}
&\phi(n\alpha + m\beta) - (n\alpha + m\beta) \\
&\quad = |n + q_{i-1}|_{q_i}\alpha + |m - b'q'_{j-1}|_{q'_j}\beta - (n\alpha + m\beta) \\
&\quad = \begin{cases}
q_{i-1}\alpha - b'q'_{j-1}\beta, & 0 \le n < q_i - q_{i-1},\ b'q'_{j-1} \le m < q'_j, \\
q_{i-1}\alpha - (b'q'_{j-1} - q'_j)\beta, & 0 \le n < q_i - q_{i-1},\ 0 \le m < b'q'_{j-1}, \\
(q_{i-1} - q_i)\alpha - b'q'_{j-1}\beta, & q_i - q_{i-1} \le n < q_i,\ b'q'_{j-1} \le m < q'_j, \\
(q_{i-1} - q_i)\alpha - (b'q'_{j-1} - q'_j)\beta, & q_i - q_{i-1} \le n < q_i,\ 0 \le m < b'q'_{j-1}.
\end{cases}
\end{aligned}$$

Let us assume that (2.3) holds. Let $\widetilde{\Delta}(n,m)$ stand for the distance (measured clockwise) between $n\alpha + m\beta$ and $\phi(n\alpha + m\beta)$ for $0 \le n < q_i$ and $0 \le m < q'_j$. Denote $D := \|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\|$. One has

$$\widetilde{\Delta}(n,m) = \begin{cases}
D, & 0 \le n < q_i - q_{i-1},\ b'q'_{j-1} \le m < q'_j, \\
D - \|q'_j\beta\|, & 0 \le n < q_i - q_{i-1},\ 0 \le m < b'q'_{j-1}, \\
D + \|q_i\alpha\|, & q_i - q_{i-1} \le n < q_i,\ b'q'_{j-1} \le m < q'_j, \\
D + \|q_i\alpha\| - \|q'_j\beta\|, & q_i - q_{i-1} \le n < q_i,\ 0 \le m < b'q'_{j-1}.
\end{cases}$$

By (2.3), for all four cases, the values of the right hand side are positive and less than 1.

We now assume that i is odd and j is even. Then $q_{i-1}\alpha + b'q'_{j-1}\beta = \varDelta + (bp_{i-1} + b'p'_{j-1})$. Similarly, we deduce that

$$\widetilde{\varDelta}(n,m) = \begin{cases} D, & 0 \le n < q_i - q_{i-1},\ 0 \le m < q'_j - b'q'_{j-1}, \\ D - \|q'_j\beta\|, & 0 \le n < q_i - q_{i-1},\ q'_j - b'q'_{j-1} \le m < q'_j, \\ D + \|q_i\alpha\|, & q_i - q_{i-1} \le n < q_i,\ 0 \le m < q'_j - b'q'_{j-1}, \\ D + \|q_i\alpha\| - \|q'_j\beta\|, & q_i - q_{i-1} \le n < q_i,\ q'_j - b'q'_{j-1} \le m < q'_j. \end{cases}$$

If i is even and j is odd, then

$$\widetilde{\varDelta}(n,m) = \begin{cases} D, & q_{i-1} \le n < q_i,\ b'q'_{j-1} \le m < q'_j, \\ D - \|q'_j\beta\|, & q_{i-1} \le n < q_i,\ 0 \le m < b'q'_{j-1}, \\ D + \|q_i\alpha\|, & 0 \le n < q_{i-1},\ b'q'_{j-1} \le m < q'_j, \\ D + \|q_i\alpha\| - \|q'_j\beta\|, & 0 \le n < q_{i-1},\ 0 \le m < b'q'_{j-1}. \end{cases}$$

Lastly, if i, j are even, then

$$\widetilde{\varDelta}(n,m) = \begin{cases} D, & q_{i-1} \le n < q_i,\ 0 \le m < q'_j - b'q'_{j-1}, \\ D - \|q'_j\beta\|, & q_{i-1} \le n < q_i,\ q'_j - b'q'_{j-1} \le m < q'_j, \\ D + \|q_i\alpha\|, & 0 \le n < q_{i-1},\ 0 \le m < q'_j - b'q'_{j-1}, \\ D + \|q_i\alpha\| - \|q'_j\beta\|, & 0 \le n < q_{i-1},\ q'_j - b'q'_{j-1} \le m < q'_j. \end{cases}$$

Hence, we conclude that, for the four cases obtained by considering the parity of i, j, one gets

$$\begin{aligned} \#\{(n,m) : \widetilde{\varDelta}(n,m) = D\} &= (q_i - q_{i-1})(q'_j - b'q'_{j-1}), \\ \#\{(n,m) : \widetilde{\varDelta}(n,m) = D - \|q'_j\beta\|\} &= (q_i - q_{i-1})b'q'_{j-1}, \\ \#\{(n,m) : \widetilde{\varDelta}(n,m) = D + \|q_i\alpha\|\} &= q_{i-1}(q'_j - b'q'_{j-1}), \\ \#\{(n,m) : \widetilde{\varDelta}(n,m) = D + \|q_i\alpha\| - \|q'_j\beta\|\} &= q_{i-1}b'q'_{j-1}. \end{aligned}$$

We now show that ϕ sends a point to its neighbor point in the clockwise direction, that is, ϕ and $\varPhi$ coincide. It is sufficient to notice that the $q_iq'_j$ intervals $\big((n\alpha + m\beta), \phi(n\alpha + m\beta)\big)$ of $\mathbb{T}$ never overlap. Indeed, the sum of their lengths, $\widetilde{\varDelta}(n,m)$, equals 1, as is shown below using (2.2):

$$\begin{aligned} 1 &= q'_j - b'q_i \\ &= (q_i\|q_{i-1}\alpha\| + q_{i-1}\|q_i\alpha\|)q'_j - b'(q'_j\|q'_{j-1}\beta\| + q'_{j-1}\|q'_j\beta\|)q_i \\ &= q_iq'_j(\|q_{i-1}\alpha\| - b'\|q'_{j-1}\beta\|) + q_{i-1}q'_j\|q_i\alpha\| - b'q'_{j-1}q_i\|q'_j\beta\| \\ &= \sum_{(m,n)\in\mathcal{E}_{q_i,q'_j}} \big(\phi(n\alpha + m\beta) - (n\alpha + m\beta)\big). \ \blacksquare \end{aligned}$$

Remark 2.2. The map φ_{q_i,q'_j} (associated with ϕ through (2.1)) is an exchange of four rectangles on $\mathcal{E}_{q_i,q'_j}$. For an illustration, see Figure 1 below.

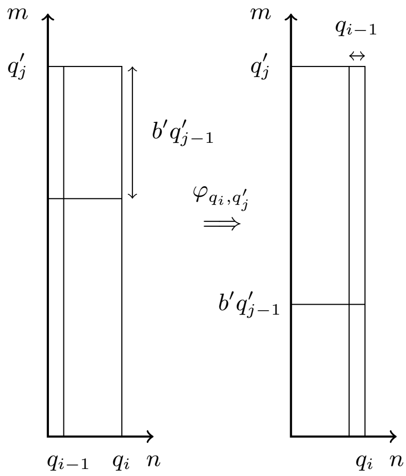

Fig. 1. The action of φ_{q_i,q'_j} on $\mathcal{E}_{q_i,q'_j}(\alpha,\beta)$ is an exchange of four subrectangles (here, i and j are assumed to be even).

Remark 2.3. According to [CGVZ02], a distance in $\Delta(E_{q,q'}(\alpha,\beta))$ is said to be *primitive* if it is not a sum of shorter lengths (not necessarily distinct). It is proved in [CGVZ02] that there are at most four primitive lengths for $E_{q,q'}(\alpha,\beta)$. The lengths given in Lemma 2.1 are primitive (under the assumption that $1,\alpha,\ \beta$ are rationally independent).

3. Bounded number of lengths. This section is devoted to the proof of Theorem 1.1(i). We provide a strategy for constructing examples of pairs (α,β) providing a low number of distances $\Delta(E_N(\alpha,\beta))$ for all N. We will rely on Lemma 2.1, and use the existence of positive integers b such that $q_i = bq'_j + 1$, as well as the existence of positive integers b' such that $q'_i = b'q_j + 1$ for suitable i,j, with α and β playing a symmetrical role.

Construction of the sequences of convergents $(q_k)_k$ and $(q'_k)_k$. We provide a construction of sequences $(q_k)_k$, $(q'_k)_k$ of convergents and sequences $(b_k)_k$, $(b'_k)_k$ such that the following holds, for all $k \geq 1$:

$$q'_k = b'_k q_k + 1, \qquad q_{k+1} = b_{k+1} q'_k + 1. \tag{3.1}$$

Recall that $q_{-1} = q'_{-1} = 0$ and $q_0 = q'_0 = 1$. We then start with $q_1 = 3$, $q'_1 = (q_1)^3 + 1 = 28$ with $b'_1 = 9$. Also $a_1 = q_1 = 3$, $a'_1 = q'_1 = 28$. Let

$$a_2 = ((q_1)^6 + q_0 - 1)b'_1 + (q_1)^5 = 3^8 + 3^5, \qquad q_2 = 3^9 + 3^6 + 1 = 3^6 q'_1 + 1.$$

We set $b_2 = 3^6 = (q_1)^6 + q_0 - 1$.

Assume now that for some index k, one has $q'_k = b'_k q_k + 1$. Choose $a_{k+1} = ((q_k)^6 + q_{k-1} - 1)b'_k + (q_k)^5$. Then

$$\begin{aligned}q_{k+1} &= a_{k+1}q_k + q_{k-1} = ((q_k)^6 + q_{k-1} - 1)b'_k q_k + (q_k)^6 + q_{k-1}\\ &= ((q_k)^6 + q_{k-1} - 1)(b'_k q_k + 1) + 1 = ((q_k)^6 + q_{k-1} - 1)q'_k + 1.\end{aligned}$$

Let $b_{k+1} = (q_k)^6 + q_{k-1} - 1$. Then $q_{k+1} = b_{k+1}q'_k + 1$. Next, we set $a'_{k+1} = ((q'_k)^6 + q'_{k-1} - 1)b_{k+1} + (q'_k)^5$. Then similarly

$$\begin{aligned}q'_{k+1} &= a'_{k+1}q'_k + q'_{k-1} = ((q'_k)^6 + q'_{k-1} - 1)b_{k+1}q'_k + (q'_k)^6 + q'_{k-1}\\ &= ((q'_k)^6 + q'_{k-1} - 1)(b_{k+1}q'_k + 1) + 1 = ((q'_k)^6 + q'_{k-1} - 1)q_{k+1} + 1\end{aligned}$$

and $b'_{k+1} = (q'_k)^6 + q'_{k-1} - 1$.

In summary, we inductively construct sequences $(q_k)_k$, $(q'_k)_k$ satisfying, for any $k \geq 1$,

$$\begin{aligned}a_{k+1} &= ((q_k)^6 + q_{k-1} - 1)b'_k + (q_k)^5,\\ a'_{k+1} &= ((q'_k)^6 + q'_{k-1} - 1)b_{k+1} + (q'_k)^5,\end{aligned} \tag{3.2}$$

$$b_{k+1} = (q_k)^6 + q_{k-1} - 1, \qquad b'_{k+1} = (q'_k)^6 + q'_{k-1} - 1. \tag{3.3}$$

Then, for any $k \geq 1$, (3.1) holds. Note that

$$q_{k+1} = b_{k+1}q'_k + 1 = b_{k+1}(b'_k q_k + 1) + 1 = b'_k(b_{k+1}q_k) + (b_{k+1} + 1). \tag{3.4}$$

Since

$$\begin{aligned}a_{k+1} &= \frac{q_{k+1} - q_{k-1}}{q_k} = \frac{b_{k+1}q'_k + 1 - q_{k-1}}{q_k}\\ &= \frac{((q_k)^6 + q_{k-1} - 1)q'_k + 1 - q_{k-1}}{q_k} \geq (q_k)^5 q'_k,\end{aligned}$$

we have

$$q_{k-1}\|q_k\alpha\| < \frac{1}{a_{k+1}a_k} < \frac{1}{2q'_k} < \|q'_{k-1}\beta\| < \|q'_{k-1}\beta\| + q'_{k-1}\|q'_k\beta\|.$$

Therefore,

$$\begin{aligned}\|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\| &> \frac{1 - q_{k-1}\|q_k\alpha\|}{q_k} - \frac{b'_k}{q'_k}\\ &= \frac{q'_k - b'_k q_k}{q_k q'_k} - \frac{q_{k-1}}{q_k}\|q_k\alpha\| = \frac{1}{q_k q'_k} - \frac{q_{k-1}}{q_k}\|q_k\alpha\|\\ &= \frac{q_{k+1}\|q_k\alpha\| + q_k\|q_{k+1}\alpha\|}{q_k q'_k} - \frac{q_{k-1}}{q_k}\|q_k\alpha\|\\ &= \left(\frac{b_{k+1}}{q_k} + \frac{1}{q_k q'_k} - \frac{q_{k-1}}{q_k}\right)\|q_k\alpha\| + \frac{\|q_{k+1}\alpha\|}{q'_k}\\ &= \left((q_k)^5 - \frac{1}{q_k} + \frac{1}{q_k q'_k}\right)\|q_k\alpha\| + \frac{\|q_{k+1}\alpha\|}{q'_k} > 0.\end{aligned} \tag{3.5}$$

We also claim that

$$q_k < b'_k < (q_k)^3, \qquad q'_k < b_{k+1} < (q'_k)^3. \tag{3.6}$$

Indeed, if $q_k < b'_k < (q_k)^3$, then using (3.3) we have

$$q'_k = b'_k q_k + 1 < q_k^4 + 1 < b_{k+1} < (q_k^2+1)^3 < (b'_k q_k + 1)^3 = (q'_k)^3.$$

The choice of q_1, q'_1, b_1 with $q_1 < b'_1 < q_1^3$ concludes the proof of the claim.

Rational independence of $1, \alpha, \beta$. Suppose that $1, \alpha, \beta$ are rationally dependent. Then there exist integers n_0, n_1, n_2 satisfying $n_0 + n_1\alpha + n_2\beta = 0$. Since α, β are both irrational, one has $n_1, n_2 \neq 0$. Then, for large k such that

$$|n_1| < q_{k+1}/q'_k \quad \text{and} \quad |n_2| < b'_{k+1}/2 < b'_{k+1} q_{k+1} \|q_k\alpha\| < q'_{k+1}\|q_k\alpha\|,$$

we have

$$\|n_1 q'_k \alpha\| = \|n_2 q'_k \beta\| \leq |n_2|\, \|q'_k\beta\| < |n_2|/q'_{k+1} < \|q_k\alpha\|.$$

This contradicts $\|n\alpha\| > \|q_k\alpha\|$ for any $1 \leq n < q_{k+1}$ (see for instance [Lan95, Chapter 1, Theorem 6]).

Let us check now that (α, β) is not badly approximable. Recall that an irrational vector (α, β) is said to be *badly approximable* if there exists $C > 0$ such that

$$\|n\alpha + m\beta\| > \frac{C}{|(n,m)|^2}$$

for any nonzero pair of integers (n, m). For the example constructed in this section, if $(n, m) = (q_k, 0)$, then by (3.4) one gets

$$\|q_k\alpha + 0\beta\| = \|q_k\alpha\| < \frac{1}{q_{k+1}} < \frac{1}{b'_k b_{k+1} q_k} < \frac{1}{(q_k)^7} = \frac{1}{(q_k)^5}\, \frac{1}{|(q_k, 0)|^2}.$$

Therefore, (α, β) is not badly approximable.

Organization of the proof. We first assume $q_k < N \leq q'_k$ and k is even, and provide all the details for this case. The case of k odd, and then the case $q_k < N \leq q'_k$, will be briefly discussed at the end of the proof.

We thus assume $q_k < N \leq q'_k$ and k is even (see Figure 2). Note that $q'_k = b'_k q_k + 1 > q_k$. The proof will be divided into three steps.

- We first describe the lengths in $E_{q_k, q'_k}(\alpha, \beta)$. There are four lengths according to Lemma 2.1.
- Then, we deduce the description of the lengths in $E_{q_k, N}(\alpha, \beta)$ from the description in $E_{q_k, q'_k}(\alpha, \beta)$. We reduce the set of points (n, m) under consideration in this step. Dynamically, this will correspond to inducing the map ϕ_{q_k, q'_k} (or similarly φ_{q_k, q'_k}). We will go from four lengths to six lengths.
- Lastly, the description of the lengths in $E_N(\alpha, \beta)$ will be deduced from the description in $E_{q_k, N}(\alpha, \beta)$ by performing an 'exduction' step with points (n, m) being inserted, creating a seventh length.

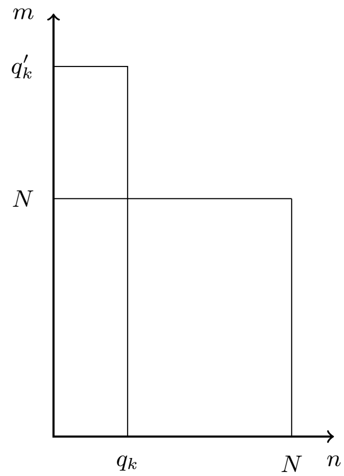

Fig. 2. The sets $\mathcal{E}_{q_k,q'_k}$, $\mathcal{E}_{q_k,N}$ and $\mathcal{E}_N$

Distribution of the points of $E_{q_k,q'_k}(\alpha,\beta)$. We apply Lemma 2.1 for $E_{q_k,q'_k}(\alpha,\beta)$ by using the fact that $q'_k = b'_k q_k + 1$. With the notation of the lemma, we have $i = j = k$, $b = b'_k$. We are in the case of i, j even, since k is even. Observe that assumption (2.3) holds, namely $\|q'_k\beta\| < \|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\|$. This comes from (3.5), applied twice to get the first two inequalities below:

$$\|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\| > \|q_k\alpha\| > b'_{k+1}\|q'_k\beta\| \geq \|q'_k\beta\|.$$

By Lemma 2.1, the neighbor map Φ_{q_k,q'_k} on $E_{q_k,q'_k}(\alpha,\beta)$ satisfies

$$\Phi_{q_k,q'_k} : n\alpha + m\beta \mapsto |n - q_{k-1}|_{q_k}\alpha + |m + b'_k q'_{k-1}|_{q'_k}\beta,$$

and

$$\begin{aligned}\Delta_{q_k,q'_k}&(n,m)\\ &= \begin{cases} \|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\|, & q_{k-1} \leq n,\ m < q'_k - b'_k q'_{k-1},\\ \|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\| - \|q'_k\beta\|, & q_{k-1} \leq n,\ q'_k - b'_k q'_{k-1} \leq m,\\ \|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\| + \|q_k\alpha\|, & n < q_{k-1},\ m < q'_k - b'_k q'_{k-1},\\ \|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\| + \|q_k\alpha\| - \|q'_k\beta\|, & n < q_{k-1},\ q'_k - b'_k q'_{k-1} \leq m. \end{cases}\end{aligned}$$

Define the map φ_{q_k,q'_k} on $\mathcal{E}_{q_k,q'_k}$ by $\varphi_{q_k,q'_k}(n,m) = (|n-q_{k-1}|_{q_k}, |m+b'_k q'_{k-1}|_{q'_k})$. Its action is shown in Figure 3 (left) as an exchange of four subrectangles. Recall that φ_{q_k,q'_k} and Φ_{q_k,q'_k} are related by (2.1).

Distribution of the points of $E_{q_k,N}(\alpha,\beta)$ for $q_k < N \leq q'_k$. We obtain $\Phi_{q_k,N}$ on $E_{q_k,N}(\alpha,\beta)$ by iterating the map Φ_{q_k,q'_k}. Indeed, since $E_{q_k,N}(\alpha,\beta)$ is a subset of $E_{q_k,q'_k}(\alpha,\beta)$, the neighbor map $\Phi_{q_k,N}$ on $E_{q_k,N}(\alpha,\beta)$ is the

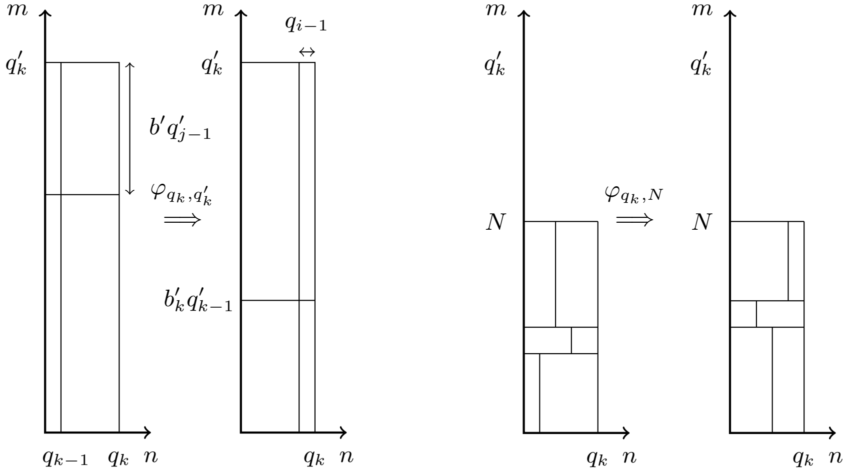

Fig. 3. The action of φ_{q_k,q'_k} on $\mathcal{E}_{q_k,q'_k}(\alpha,\beta)$ is an exchange of four subrectangles (left), and the action of $\varphi_{q_k,N}$ on $\mathcal{E}_{q_k,N}(\alpha,\beta)$ is an exchange of six subrectangles (right).

induced map of Φ_{q_k,q'_k} on $E_{q_k,N}(\alpha,\beta)$, i.e.,

$$\Phi_{q_k,N}(x) = (\Phi_{q_k,q'_k})^{\tau(x)}(x),$$

where $\tau(x) = \min\{\ell \geq 1 : (\Phi_{q_k,q'_k})^{\ell}(x) \in E_{q_k,N}(\alpha,\beta)\}$ is the first return time to $E_{q_k,N}(\alpha,\beta)$ of the map Φ_{q_k,q'_k}. This induction step will create two more subrectangles, that is, φ_{q_k,q'_k} acts as an exchange of four subrectangles, while $\varphi_{q_k,N}$ acts as an exchange of six subrectangles (see Figure 3).

The following lemma expresses the fact that the return time τ takes three values. Note that the statement below does not depend on the parity of k.

Lemma 3.1. *Let τ be the first return time to $E_{q_k,N}(\alpha,\beta)$ of the map Φ_{q_k,q'_k}. There exist τ_1,τ_2,τ_3, N_1,N_2,N_3 such that, for each $n\alpha+m\beta \in E_{q_k,N}(\alpha,\beta)$,*

$$\tau(n\alpha+m\beta) = \begin{cases} \tau_1 & \text{if } 0 \leq m < N_1, \\ \tau_1+\tau_2 & \text{if } N_1 \leq m < N_2, \\ \tau_2 & \text{if } N_2 \leq m < N. \end{cases}$$

Moreover, there exist nonnegative integers d_1, d_2 such that

$$\begin{aligned} [0,N_1) + \tau_1 b'_k q'_{k-1} &= [N-N_1,N) + d_1 q'_k, \\ [N_1,N_2) + (\tau_1+\tau_2) b'_k q'_{k-1} &= [N-N_2,N-N_1) + (d_1+d_2) q'_k, \\ [N_2,N) + \tau_2 b'_k q'_{k-1} &= [0,N-N_2) + d_2 q'_k. \end{aligned}$$

Proof. We prove the lemma for k even, but the same argument works for k odd. Recall that, for even k, we have

$$\Phi_{q_k,q'_k}(n\alpha+m\beta) = |n-q_{k-1}|_{q_k}\alpha + |m+b'_k q'_{k-1}|_{q'_k}\beta.$$

Thus

$$(\Phi_{q_k,q'_k})^{\ell}(n\alpha+m\beta) \in E_{q_k,N}(\alpha,\beta) \text{ if and only if } 0 \le |m+\ell b'_k q'_{k-1}|_{q'_k} < N.$$

Let

$$\bar\tau(m) = \min\{\ell \ge 1 : 0 \le |m+\ell b'_k q'_{k-1}|_{q'_k} < N\}.$$

The discrete version of the three-gap problem (see e.g. [Sla67]) applied to the translation by $b'_k q'_{k-1}$ modulo N provides the existence of τ_1, τ_2, τ_3, N_1, N_2, N_3 such that

$$\bar\tau(m) = \begin{cases} \tau_1 & \text{if } 0 \le m < N_1, \\ \tau_1+\tau_2 & \text{if } N_1 \le m < N_2, \\ \tau_2 & \text{if } N_2 \le m < N, \end{cases}$$

as well as the existence of nonnegative integers d_1, d_2 satisfying

$$[0,N_1) + \tau_1 b'_k q'_{k-1} = [N-N_1, N) + d_1 q'_k,$$
$$[N_2,N) + \tau_2 b'_k q'_{k-1} = [0, N-N_2) + d_2 q'_k.$$

Clearly,

$$[N_1,N_2) + (\tau_1+\tau_2) b'_k q'_{k-1} = [N-N_2, N-N_1) + (d_1+d_2) q'_k.$$

Lemma 3.1 is thus a direct consequence of the discrete three-gap problem. ∎

Therefore, for k even, we deduce from Lemma 3.1 that

$$\Phi_{q_k,N}(n\alpha+m\beta) = (\Phi_{q_k,q'_k})^{\tau(n\alpha+m\beta)}(n\alpha+m\beta) \tag{3.7}$$
$$= \begin{cases} |n-\tau_1 q_{k-1}|_{q_k}\alpha + (m+\tau_1 b'_k q'_{k-1} - d_1 q'_k)\beta, & 0 \le m < N_1, \\ |n-(\tau_1+\tau_2) q_{k-1}|_{q_k}\alpha & \\ \quad +(m+(\tau_1+\tau_2) b'_k q'_{k-1} - (d_1+d_2) q'_k)\beta, & N_1 \le m < N_2, \\ |n-\tau_2 q_{k-1}|_{q_k}\alpha + (m+\tau_2 b'_k q'_{k-1} - d_2 q'_k)\beta, & N_2 \le m < N. \end{cases}$$

Let h_1, h_2, h_3 be nonnegative integers satisfying

$$\tau_1 q_{k-1} = h_1 q_k + r_1, \quad \tau_2 q_{k-1} = h_2 q_k + r_2, \quad (\tau_1+\tau_2) q_{k-1} = h_3 q_k + r_3$$

with $0 \le r_1, r_2, r_3 < q_k$. Each of the three cases splits into two subcases according to whether or not n is smaller than r_i, for $i = 1,2,3$. Thus,

$$\Delta_{q_k,N}(n,m) = \begin{cases} \Delta_1 + \|q_k\alpha\| & \text{if } 0 \le n < r_1,\ 0 \le m < N_1, \\ \Delta_1 & \text{if } r_1 \le n < q_k,\ 0 \le m < N_1, \\ \Delta_3 + \|q_k\alpha\| & \text{if } 0 \le n < r_3,\ N_1 \le m < N_2, \\ \Delta_3 & \text{if } r_3 \le n < q_k,\ N_1 \le m < N_2, \\ \Delta_2 + \|q_k\alpha\| & \text{if } 0 \le n < r_2,\ N_2 \le m < N, \\ \Delta_2 & \text{if } r_2 \le n < q_k,\ N_2 \le m < N, \end{cases}$$

where

$$\begin{aligned}
\Delta_1 &= \tau_1(\|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\|) - d_1\|q'_k\beta\| + h_1\|q_k\alpha\|,\\
\Delta_2 &= \tau_2(\|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\|) - d_2\|q'_k\beta\| + h_2\|q_k\alpha\|,\\
\Delta_3 &= (\tau_1+\tau_2)(\|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\|) - (d_1+d_2)\|q'_k\beta\| + h_3\|q_k\alpha\|.
\end{aligned}$$

Indeed, if, for example, $0 \le m < N_1$, then, by (3.7),

$$\begin{aligned}
&\Phi_{q_k,N}(n\alpha + m\beta) - (n\alpha+m\beta)\\
&\qquad = |n-\tau_1 q_{k-1}|_{q_k}\alpha + (m+\tau_1 b'_k q'_{k-1} - d_1 q'_k)\beta - (n\alpha+m\beta)\\
&\qquad = (|n-r_1|_{q_k} - n)\alpha + (\tau_1 b'_k q'_{k-1} - d_1 q'_k)\beta\\
&\qquad = \begin{cases} (-r_1+q_k)\alpha + (\tau_1 b'_k q'_{k-1} - d_1 q'_k)\beta & \text{if } n < r_1,\\ -r_1\alpha + (\tau_1 b'_k q'_{k-1} - d_1 q'_k)\beta & \text{if } n \ge r_1,\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
-r_1\alpha + (\tau_1 b'_k q'_{k-1} - d_1 q'_k)\beta &= (h_1 q_k - \tau_1 q_{k-1})\alpha + (\tau_1 b'_k q'_{k-1} - d_1 q'_k)\beta\\
&= \tau_1(\|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\|) - d_1\|q'_k\beta\| + h_1\|q_k\alpha\| = \Delta_1.
\end{aligned}$$

The action of $\varphi_{q_k,N}(n,m) = (\varphi_{q_k,q'_k})^{\tau(m)}(n,m)$ on $\mathcal{E}_{q_k,N}$ is illustrated in Figure 3.

From $E_{q_k,N}(\alpha,\beta)$ to $E_N(\alpha,\beta)$. Let $N = aq_k + R$ with $a \ge 1$ and $1 \le R \le q_k$ (recall that $q_k < N \le q'_k$). Since $E_{q_k,N}$ is a subset of E_{q_k,q'_k}, we have

$$\min \Delta(E_{q_k,N}) \ge \min\Delta(E_{q_k,q'_k}) = \|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\| - \|q'_k\beta\|.$$

Using (3.5) and (3.6), we find that

$$\begin{aligned}
\min\Delta(E_{q_k,N}) &\ge \|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\| - \|q'_k\beta\|\\
&> \left((q_k)^5 - \frac{1}{q_k}\right)\|q_k\alpha\| - \frac{\|q_k\alpha\|}{b'_{k+1}} > ((q_k)^3+1)\|q_k\alpha\|\\
&> \frac{b'_k q_k + 1}{q_k}\|q_k\alpha\| = \frac{q'_k}{q_k}\|q_k\alpha\| \ge \frac{N}{q_k}\|q_k\alpha\| > a\|q_k\alpha\|.
\end{aligned}\tag{3.8}$$

We claim that

$$\Phi_N(n\alpha+m\beta) = \begin{cases} (n+q_k)\alpha + m\beta & \text{if } 0 \le n < N - q_k,\\ \Phi_{q_k,N}(|n|_{q_k}\alpha + m\beta) & \text{if } N-q_k \le n < N.\end{cases}\tag{3.9}$$

Proof of (3.9). If $0 \le n < R = N - aq_k$, then the points $(n+q_k)\alpha + m\beta$, $(n+2q_k)\alpha + m\beta$, $\dots$, $(n+aq_k)\alpha + m\beta$ are between $n\alpha+m\beta$ and $\Phi_{q_k,N}(n\alpha+m\beta)$, as shown in Figure 4. Therefore, $\Delta_N(n+cq_k, m) = \|q_k\alpha\|$

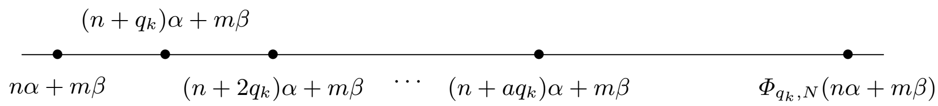

Fig. 4. Illustration of the proof of (3.9) when $0 \le n < R$

for $0 \le c \le a-1$, and $\Delta_N(n+aq_k, m) = \Delta_{q_k,N}(n,m) - a\|q_k\alpha\|$, which is positive by (3.8).

If $R \le n < q_k$, then the points $(n+q_k)\alpha + m\beta$, $(n+2q_k)\alpha + m\beta$, $\ldots$, $(n+(a-1)q_k)\alpha + m\beta$ lie between $n\alpha + m\beta$ and $\Phi_{q_k,N}(n\alpha+m\beta)$. In this case, the gaps between two adjacent points of $E_N(\alpha,\beta)$ are given by $\Delta_N(n+cq_k, m) = \|q_k\alpha\|$ for $0 \le c \le a-2$, and $\Delta_N(n+(a-1)q_k, m) = \Delta_{q_k,N}(n,m) - (a-1)\|q_k\alpha\|$, which is positive by (3.8). ■

Therefore, using (3.7), we deduce that

$$\Phi_N(n\alpha+m\beta)$$
$$= \begin{cases} (n+q_k)\alpha+m\beta, & 0 \le n < N-q_k, \\ |n-\tau_1 q_{k-1}|_{q_k}\alpha + (m+\tau_1 b'_k q'_{k-1} - d_1 q'_k)\beta, & n \ge N-q_k,\ 0 \le m < N_1, \\ |n-(\tau_1+\tau_2)q_{k-1}|_{q_k}\alpha & \\ \quad +(m+(\tau_1+\tau_2)b'_k q'_{k-1} - (d_1+d_2)q'_k)\beta, & n \ge N-q_k,\ N_1 \le m < N_2, \\ |n-\tau_2 q_{k-1}|_{q_k}\alpha + (m+\tau_2 b'_k q'_{k-1} - d_2 q'_k)\beta, & n \ge N-q_k,\ N_2 \le m < N. \end{cases}$$

Let $\bar h_1, \bar h_2, \bar h_3$ be nonnegative integers satisfying

$$N-\tau_1 q_{k-1} = \bar h_1 q_k + \bar r_1,\ N-\tau_2 q_{k-1} = \bar h_2 q_k + \bar r_2,\ N-(\tau_1+\tau_2)q_{k-1} = \bar h_3 q_k + \bar r_3$$

with $0 \le \bar r_1, \bar r_2, \bar r_3 < q_k$. Then

$$\Delta_N(n,m) = \begin{cases} \|q_k\alpha\|, & 0 \le n < N-q_k, \\ \Delta_1 + \|q_k\alpha\|, & N-q_k \le n < N-\bar r_1,\ 0 \le m < N_1, \\ \Delta_1, & N-\bar r_1 \le n < N,\ 0 \le m < N_1, \\ \Delta_3 + \|q_k\alpha\|, & N-q_k \le n < N-r_3,\ N_1 \le m < N_2, \\ \Delta_3, & N-\bar r_3 \le n < q_k,\ N_1 \le m < N_2, \\ \Delta_2 + \|q_k\alpha\|, & N-q_k \le n < N-\bar r_2,\ N_2 \le m < N, \\ \Delta_2, & N-\bar r_2 \le n < N,\ N_2 \le m < N, \end{cases}$$

where

$$\Delta_1 = \tau_1(\|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\|) - d_1\|q'_k\beta\| - \bar h_1\|q_k\alpha\|,$$
$$\Delta_2 = \tau_2(\|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\|) - d_2\|q'_k\beta\| - \bar h_2\|q_k\alpha\|,$$
$$\Delta_3 = (\tau_1+\tau_2)(\|q_{k-1}\alpha\| - b'_k\|q'_{k-1}\beta\|) - (d_1+d_2)\|q'_k\beta\| - \bar h_3\|q_k\alpha\|.$$

The action of $\varphi_N(n,m)$ on $\mathcal{E}_N$ is illustrated in Figure 5.

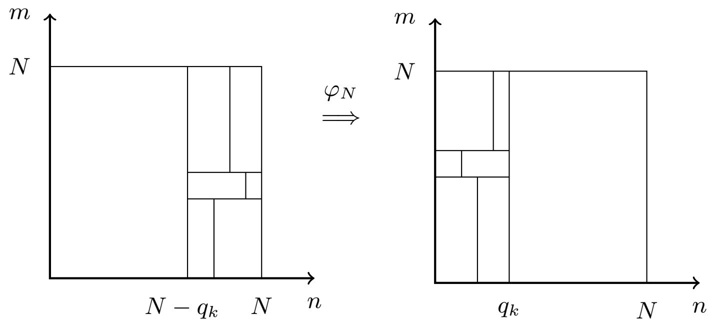

Fig. 5. The action of φ_N on $\mathcal{E}_N$ is an exchange of seven subrectangles.

End of the proof. The case $q_k < N \le q'_k$ with k even has thus been handled. In the case of k odd, $\Delta_{q_k,q'_k}(n,m)$ still takes four values, as discussed in Lemma 2.1, namely

$$\begin{aligned} &\|q_{k-1}\alpha\| - b'_k\|q_{k-1}\beta\|, && \|q_{k-1}\alpha\| - b'_k\|q_{k-1}\beta\| - \|q'_k\beta\|,\\ &\|q_{k-1}\alpha\| - b'_k\|q_{k-1}\beta\| + \|q_k\alpha\|, && \|q_{k-1}\alpha\| - b'_k\|q_{k-1}\beta\| + \|q_k\alpha\| - \|q'_k\beta\|. \end{aligned}$$

We also have at most six values for $\Delta(E_{q_k,N}(\alpha,\beta))$ which are obtained by considering the induced map of Φ_{q_k,q'_k}. Observe that $\Delta_{q_k,q'_k}(n,m)$ takes the same values as in the case $q_k < N \le q'_k$ with k even. It follows from (3.9) that there are seven values for $\Delta(E_N(\alpha,\beta))$.

Lastly, the case $q'_k < N \le q_{k+1}$ is similarly deduced by induction from the case $E_{q'_k,q_{k+1}}(\alpha,\beta)$. This ends the proof of Theorem 1.1(i).

Remark 3.2. Observe that, in continuation of Remark 2.3, there are here also four primitive lengths.

4. Unbounded number of lengths. This section is devoted to the proof of statement (ii) of Theorem 1.1. The strategy works as follows: one wants to regularly get indices k for which $q_k = q'_k + 1$. This will imply that q_k and q'_k are coprime and that they have the same size. This will allow us in particular to consider mainly the first level $E_{q_k,q'_k}(\alpha,\beta)$ (and in fact even $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$). Now, we provide a construction of α and β for which $q_{4k-3}+1 = q'_{4k-3}$ and $q_{4k-2}-1 = q'_{4k-2}$, for all k. Furthermore, we will have $a'_{4k+1} = 1$.

Construction of the sequences of convergents $(q_i)_i$ and $(q'_j)_j$. We consider irrationals α and β in $(0,1)$ with respective sequences of partial quotients $(a_i)_i$ and $(a'_j)_j$ satisfying

$$a_1 = 2, \quad a_2 = 2 \quad \text{and} \quad a'_1 = 3, \quad a'_2 = 1.$$

Then

$$q_0 = 1,\ q_1 = 2,\ q_2 = 5, \qquad q'_0 = 1,\ q'_1 = 3,\ q'_2 = 4.$$

We now inductively define $(a_i)_i, (a'_j)_j$ as follows. Suppose that

$$q_{4k-3} + 1 = q'_{4k-3}, \qquad q_{4k-2} - 1 = q'_{4k-2} \qquad \text{for } k \geq 1.$$

Furthermore, let $R_k := q_{4k-3} + 1 = q'_{4k-3}$ and $Q_k := q_{4k-2} - 1 = q'_{4k-2}$. Set

$$a_{4k-1} := 1, \quad a_{4k} := 3, \quad a_{4k+1} := 2Q_k + R_k - 1, \qquad a_{4k+2} := 6Q_k + 4R_k,$$
$$a'_{4k-1} := 2, \quad a'_{4k} := 4Q_k + 3R_k - 2, \quad a'_{4k+1} := 1, \quad a'_{4k+2} := 6Q_k + 4R_k - 1.$$

Then

$$q_{4k-1} = Q_k + R_k, \qquad q'_{4k-1} = 2Q_k + R_k,$$
$$q_{4k} = 4Q_k + 3R_k + 1, \qquad q'_{4k} = 8Q_k^2 + (10R_k - 3)Q_k + 3R_k^2 - 2R_k,$$

and

$$q_{4k+1} = R_{k+1} - 1, \quad q'_{4k+1} = R_{k+1}, \quad q_{4k+2} = Q_{k+1} + 1, \quad q'_{4k+2} = Q_{k+1},$$

where we put inductively

$$R_{k+1} = 8Q_k^2 + (10R_k - 1)Q_k + 3R_k^2 - R_k,$$
$$Q_{k+1} = 48Q_k^3 + (96R_k - 6)Q_k^2 + (43R_k^2 - 5R_k - 2)Q_k + 12R_k^3 - 4R_k^2 - R_k.$$

Rational independence of $1, \alpha, \beta$. Suppose that $1, \alpha, \beta$ are rationally dependent. Then there exist integers n_0, n_1, n_2 satisfying $n_0 + n_1\alpha + n_2\beta = 0$. Since α, β are both irrational, $n_1, n_2 \neq 0$. Then we have

$$\|n_1 q'_{4k+1}\alpha\| = \|n_2 q'_{4k+1}\beta\| \leq |n_2|\,\|q'_{4k+1}\beta\| < \frac{|n_2|}{q'_{4k+2}} < \frac{|n_2|}{a'_{4k+2}q'_{4k+1}}.$$

Thus, there exists an integer p satisfying

$$\left|\alpha - \frac{p}{n_1 q'_{4k+1}}\right| < \frac{|n_2|}{|n_1| a'_{4k+2}(q'_{4k+1})^2}. \tag{4.1}$$

Choose k large enough for $a'_{4k+2} > 2|n_1|\,|n_2|$ to hold. Then, by Legendre's theorem (see e.g. [Bug04, Theorem 1.8]), one gets $p/(n_1 q'_{4k+1}) = p_s/q_s$ for some positive integer s. Since $q_{4k+2} = q'_{4k+2} + 1 > a'_{4k+2}q'_{4k+1} > |n_1| q'_{4k+1}$, we get $s \leq 4k+1$. Also from $q'_{4k+1} = q_{4k+1} + 1$, we get $s \neq 4k+1$. If we assume $s \leq 4k$, then

$$\left|\alpha - \frac{p_s}{q_s}\right| \geq \left|\alpha - \frac{p_{4k}}{q_{4k}}\right| \geq \frac{1}{2q_{4k}q_{4k+1}} > \frac{1}{2(q'_{4k+1})^2} > \frac{|n_1|\,|n_2|}{a'_{4k+2}(q'_{4k+1})^2},$$

which contradicts (4.1).

Organization of the proof. We will work mainly with the points of the first level provided by $E_{q_{4k+1}, q'_{4k+1}}(\alpha, \beta)$. This will be sufficient to derive infinitely many lengths for the points in E_N with $N = q_{4k+1} = q'_{4k+1} - 1$.

The study of the first level will be divided into Lemma 4.1 and Proposition 4.2. The main difficulty here is that the map ϕ of Lemma 2.1 provides points that can be either to the right, or to the left of a given point (assumption (2.3) does not hold).

Distribution of the points of $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$**.** We now consider points of the first level provided by $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$. Recall that $q'_{4k+1} = q_{4k+1}+1$. With the notation of Lemma 2.1, put $i=j=4k+1$. Observe that $b'=1$. We consider

$$\delta_k := \|q'_{4k-1}\beta\| - \|q_{4k}\alpha\|.$$

According to Lemma 4.1 below, one has $\delta_k > 0$.

Note that there are more than the four lengths of Lemma 2.1 since assumption (2.3) is not satisfied. Indeed, one has $-\|q_{4k}\alpha\| + \|q'_{4k-1}\beta\| = \delta_k > 0$, which contradicts $\|q'_{4k+1}\beta\| < \|q_{4k}\alpha\| - \|q'_{4k}\beta\|$, by noticing that $\|q'_{4k-1}\beta\| = \|q'_{4k}\beta\| + \|q'_{4k+1}\beta\|$, since $a'_{4k+1} = 1$. However, even though (2.3) is not satisfied, Lemma 2.1 provides a convenient expression ϕ for the neighbor map, which will be used in the proof of Proposition 4.2 below, showing that there are at most 12 lengths.

Lemma 4.1. *For all k,*

$$0 < 2\delta_k a_{4k+1} < \|q_{4k+1}\alpha\| < \|q'_{4k+1}\beta\|.$$

Proof. We have

$$\|q_{k-1}\alpha\| = \frac{1}{q_k + q_{k-1}\frac{\|q_k\alpha\|}{\|q_{k-1}\alpha\|}} = \cfrac{1}{q_k + \cfrac{q_{k-1}}{a_{k+1} + \cfrac{1}{a_{k+2}+\ddots}}}.$$

Hence

$$\|q_{4k}\alpha\| = \frac{1}{R_{k+1} - 1 + \frac{4Q_k+3R_k+1}{6Q_k+4R_k+s}} = \frac{1}{R_{k+1} - \frac{1}{3} + \frac{R_k+3-2s}{3(6Q_k+4R_k+s)}},$$

$$\|q'_{4k-1}\beta\| = \frac{1}{R_{k+1} - 2Q_k - R_k + \frac{2Q_k+R_k}{1+\frac{1}{6Q_k+4R_k-1+s'}}} = \frac{1}{R_{k+1} - \frac{1}{3} + \frac{R_k+s'}{3(6Q_k+4R_k+s')}},$$

where

$$s := \cfrac{1}{a_{4k+3} + \cfrac{1}{a_{4k+4}+\ddots}}, \qquad s' := \cfrac{1}{a'_{4k+3} + \cfrac{1}{a'_{4k+4}+\ddots}}$$

satisfy

$$\frac{6Q_{k+1}+3R_{k+1}+1}{8Q_{k+1}+4R_{k+1}+1}=\frac{3a_{4k+5}+4}{4a_{4k+5}+5}<s<\frac{3a_{4k+5}+1}{4a_{4k+5}+1}=\frac{6Q_{k+1}+3R_{k+1}-2}{8Q_{k+1}+4R_{k+1}-3},$$

$$\frac{4Q_{k+1}+3R_{k+1}-2}{8Q_{k+1}+6R_{k+1}-3}=\frac{a'_{4k+4}}{2a'_{4k+4}+1}<s'<\frac{a'_{4k+4}+1}{2a'_{4k+4}+3}=\frac{4Q_{k+1}+3R_{k+1}-1}{8Q_{k+1}+6R_{k+1}-1}.$$

Then

$$\delta_k=\frac{\frac{R_k+3-2s}{3(6Q_k+4R_k+s)}-\frac{R_k+s'}{3(6Q_k+4R_k+s')}}{\big(R_{k+1}-\frac{1}{3}+\frac{R_k+s'}{3(6Q_k+4R_k+s')}\big)\big(R_{k+1}-\frac{1}{3}+\frac{R_k+3-2s}{3(6Q_k+4R_k+s)}\big)}.$$

By elementary computation we get

$$\frac{1}{3(6Q_k+5R_k)R_{k+1}^2}<\delta_k<\frac{1}{3(6Q_k+4R_k)(R_{k+1}-1/3)^2}.$$

Also,

$$\|q_{4k+1}\alpha\|=\frac{1}{Q_{k+1}+1+(R_{k+1}-1)s},\qquad \|q'_{4k+1}\beta\|=\frac{1}{Q_{k+1}+R_{k+1}s'},$$

thus

$$0<2\delta_k a_{4k+1}<\frac{1}{3(R_{k+1}-1/3)^2}<\frac{1}{Q_{k+1}+R_{k+1}}$$
$$<\|q_{4k+1}\alpha\|<\|q'_{4k+1}\beta\|<\frac{1}{Q_{k+1}}. \blacksquare$$

PROPOSITION 4.2. *Let α,β be given by the construction above. Consider points of the first level provided by $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$. The neighbor map $\Phi=\Phi_{q_{4k+1},q'_{4k+1}}$ satisfies the following.*

(1) *If* $q_{4k}\le n<q_{4k+1}$ *and* $q'_{4k-1}\le m<q'_{4k+1}$, *then*
$$\Phi(n\alpha+m\beta)=(n-q_{4k})\alpha+(m-q'_{4k-1})\beta,\qquad \Delta(n,m)=\delta_k.$$

(2) *If* $0\le n<q_{4k-1}$ *and* $q'_{4k+1}-a_{4k+1}q'_{4k-1}\le m<q'_{4k+1}$, *then*
$$\Phi(n\alpha+m\beta)=(n+a_{4k+1}q_{4k})\alpha+(m+a_{4k+1}q'_{4k-1}-q'_{4k+1})\beta,$$
$$\Delta(n,m)=\|q'_{4k+1}\beta\|-a_{4k+1}\delta_k.$$

(3) *If* $q_{4k-1}\le n<q_{4k}$ *and* $q'_{4k+1}-(a_{4k+1}-1)q'_{4k-1}\le m<q'_{4k+1}$, *then*
$$\Phi(n\alpha+m\beta)=(n+(a_{4k+1}-1)q_{4k})\alpha+(m+(a_{4k+1}-1)q'_{4k-1}-q'_{4k+1})\beta,$$
$$\Delta(n,m)=\|q'_{4k+1}\beta\|-(a_{4k+1}-1)\delta_k.$$

(4) *If* $q_{4k-1}\le n<2q_{4k-1}$ *and* $q'_{4k+1}-a_{4k+1}q'_{4k-1}\le m<q'_{4k+1}-(a_{4k+1}-1)q'_{4k-1}$, *then*
$$\Phi(n\alpha+m\beta)=(n+2a_{4k+1}q_{4k}-q_{4k+1})\alpha+(m+2a_{4k+1}q'_{4k-1}-q'_{4k+1})\beta,$$
$$\Delta(n,m)=\|q_{4k+1}\alpha\|+\|q'_{4k+1}\beta\|-2a_{4k+1}\delta_k.$$

(5) *If* $2q_{4k-1} \le n < q_{4k}$ *and* $q'_{4k+1} - a_{4k+1}q'_{4k-1} \le m < q'_{4k+1} - (a_{4k+1}-1)q'_{4k-1}$, *then*

$$\begin{aligned}\Phi(n\alpha + m\beta) &= (n + (2a_{4k+1}-1)q_{4k} - q_{4k+1})\alpha \\ &\quad + (m + (2a_{4k+1}-1)q'_{4k-1} - q'_{4k+1})\beta, \\ \Delta(n,m) &= \|q_{4k+1}\alpha\| + \|q'_{4k+1}\beta\| - (2a_{4k+1}-1)\delta_k.\end{aligned}$$

(6) *If* $0 \le n < q_{4k-1}$ *and* $q'_{4k+1} - (a_{4k+1}+1)q'_{4k-1} \le m < q'_{4k+1} - a_{4k+1}q'_{4k-1}$, *then*

$$\begin{aligned}\Phi(n\alpha + m\beta) &= (n + 2a_{4k+1}q_{4k} - q_{4k+1})\alpha + (m + 2a_{4k+1}q'_{4k-1} - q'_{4k+1})\beta, \\ \Delta(n,m) &= \|q_{4k+1}\alpha\| + \|q'_{4k+1}\beta\| - 2a_{4k+1}\delta_k.\end{aligned}$$

(7) *If* $q_{4k-1} \le n < q_{4k}$ *and* $q'_{4k+1} - (a_{4k+1}+1)q'_{4k-1} \le m < q'_{4k+1} - a_{4k+1}q'_{4k-1}$, *then*

$$\begin{aligned}\Phi(n\alpha + m\beta) &= (n + a_{4k+1}q_{4k} - q_{4k+1})\alpha + (m + a_{4k+1}q'_{4k-1})\beta, \\ \Delta(n,m) &= \|q_{4k+1}\alpha\| - a_{4k+1}\delta_k.\end{aligned}$$

(8) *If* $0 \le n < q_{4k}$ *and* $q'_{4k+1} - (a_{4k+1}+c+1)q'_{4k-1} \le m < q'_{4k+1} - (a_{4k+1}+c)q'_{4k-1}$, *where* $1 \le c \le a_{4k+1}-1$, *then*

$$\begin{aligned}\Phi(n\alpha + m\beta) &= (n + (a_{4k+1}+c)q_{4k} - q_{4k+1})\alpha + (m + (a_{4k+1}+c)q'_{4k-1})\beta, \\ \Delta(n,m) &= \|q_{4k+1}\alpha\| - (a_{4k+1}+c)\delta_k.\end{aligned}$$

(9) *If* $0 \le n < 2q_{4k-1}$ *and* $0 \le m < q'_{4k+1} - 2a_{4k+1}q'_{4k-1}$, *then*

$$\begin{aligned}\Phi(n\alpha + m\beta) &= (n + 2a_{4k+1}q_{4k} - q_{4k+1})\alpha + (m + 2a_{4k+1}q'_{4k-1})\beta, \\ \Delta(n,m) &= \|q_{4k+1}\alpha\| - 2a_{4k+1}\delta_k.\end{aligned}$$

(10) *If* $2q_{4k-1} \le n < q_{4k}$ *and* $0 \le m < q'_{4k+1} - (2a_{4k+1}-1)q'_{4k-1}$, *then*

$$\begin{aligned}\Phi(n\alpha + m\beta) &= (n + (2a_{4k+1}-1)q_{4k} - q_{4k+1})\alpha + (m + (2a_{4k+1}-1)q'_{4k-1})\beta, \\ \Delta(n,m) &= \|q_{4k+1}\alpha\| - (2a_{4k+1}-1)\delta_k.\end{aligned}$$

(11) *If* $(c-1)q_{4k} + 2q_{4k-1} \le n < cq_{4k} + 2q_{4k-1}$ *and* $0 \le m < q'_{4k-1}$, *where* $1 \le c \le a_{4k+1}-1$, *then*

$$\begin{aligned}\Phi(n\alpha + m\beta) &= (n + (2a_{4k+1}-c)q_{4k} - q_{4k+1})\alpha + (m + (2a_{4k+1}-c)q'_{4k-1})\beta, \\ \Delta(n,m) &= \|q_{4k+1}\alpha\| - (2a_{4k+1}-c)\delta_k.\end{aligned}$$

(12) *If* $(a_{4k+1}-1)q_{4k} + 2q_{4k-1} \le n < q_{4k+1}$ *and* $0 \le m < q'_{4k-1}$, *then*

$$\begin{aligned}\Phi(n\alpha + m\beta) &= (n + a_{4k+1}q_{4k} - q_{4k+1})\alpha + (m + a_{4k+1}q'_{4k-1})\beta, \\ \Delta(n,m) &= \|q_{4k+1}\alpha\| - a_{4k+1}\delta_k.\end{aligned}$$

Remark 4.3. Observe that there are overlapping regions between points corresponding to cases (8) and (10) (when $c = a_{4k+1}-1$), and between cases (10) and (11) (when $c = 1$).

Proof of Proposition 4.2. Let $\overline{\Phi}$ and $\overline{\Delta}$ stand for the functions defined in the statement of the proposition. We want to prove that $\overline{\Phi}$ coincides with Φ on $E_{q_{4k+1},q'_{4k+1}}$, and that similarly $\overline{\Delta}$ coincides with Δ. The proof works as for Lemma 2.1: we will show that the sum of the lengths provided by $\overline{\Delta}$ (with multiplicities) equals 1.

According to Lemma 4.1, one checks that the lengths $\overline{\Delta}$ are all nonnegative. The intervals for the pairs (n,m) in Proposition 4.2 are also well-defined. Indeed, one checks that $q_{4k}-2q_{4k-1}>0$ by noticing that $a_{4k}=3$, and also $q'_{4k+1}-(2a_{4k+1}+1)q'_{4k-1}>0$.

One has $q'_{4k+1}=q_{4k+1}+1$. With the notation of Lemma 2.1, one has $b'=1$. Since $q'_{4k-1}=q'_{4k+1}-a'_{4k+1}q'_{4k}=q'_{4k+1}-q'_{4k}$, we have

$$|m-q'_{4k}|_{q'_{4k+1}}=|m+q'_{4k-1}|_{q'_{4k+1}}.$$

As in Lemma 2.1, we consider the cyclic permutation ϕ on $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$ defined, for all $(n,m)\in\mathcal{E}_{q_{4k+1},q'_{4k+1}}$, by

$$\begin{aligned}\phi(n\alpha+m\beta)&=|n+q_{4k}|_{q_{4k+1}}\alpha+|m-q'_{4k}|_{q'_{4k+1}}\beta\\&=|n+q_{4k}|_{q_{4k+1}}\alpha+|m+q'_{4k-1}|_{q'_{4k+1}}\beta.\end{aligned}$$

We first provide some dynamical insight on the way the 12 lengths in Proposition 4.2 have been obtained. As stressed before, assumption (2.3) is not satisfied, and the neighbor map Φ is not equal to ϕ. In fact, there are points x for which $\phi(x)$ is obtained from x by performing a clockwise jump of δ_k, but there are also points x for which $\phi(x)$ is located in the anticlockwise direction ($\phi(x)<x$), with x being the clockwise neighbor of $\phi(x)$. However, the map Φ can be recovered by performing suitable inductions of the map ϕ on the set of points for which $\phi(x)>x$. Let

$$G:=\{n\alpha+m\beta : q_{4k+1}-q_{4k}\le n<q_{4k+1} \text{ or } q'_{4k+1}-q'_{4k-1}=q'_{4k}\le m<q'_{4k+1}\}.$$

One has $G\subset E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$. Then G is the set of points such that $\phi(x)>x$, that is, $\phi(x)$ is obtained from x by performing a clockwise jump of $\delta_k=\|q'_{4k-1}\beta\|-\|q_{4k}\alpha\|>0$. Elements $n\alpha+m\beta$ in $\phi(G)$ are such that $0\le n<q_{4k}$ or $0\le m<q'_{4k-1}$. This is the complement of the set of (n,m) corresponding to case (1). Let F_G be defined on $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$ as the first entering time of ϕ to G, that is,

$$F_G(n\alpha+m\beta):=\min\{\ell\ge 0:\phi^\ell(n\alpha+m\beta)\in G\}.$$

Also, define S_G as the second entering time of ϕ to G:

$$\begin{aligned}S_G(n\alpha+m\beta):=&\min\{\ell\ge F_G(n\alpha+m\beta)+1:\phi^\ell(n\alpha+m\beta)\in G\}\\&=F_G(\phi^{F_G(n\alpha+m\beta)+1}(n\alpha+m\beta))+F_G(n\alpha+m\beta).\end{aligned}$$

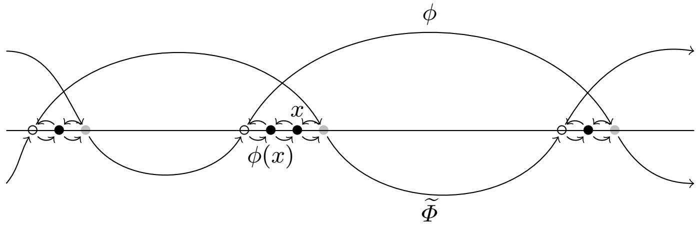

Fig. 6. The points marked by $\circ$ are elements of G. Elements of $\phi(G)$ are marked in light gray.

We need to consider the second entering time to recover an element located in the clockwise direction.

Let us now define a map $\widetilde{\Phi}$ on $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$ as follows (see Figure 6):

$$\widetilde{\Phi}(n\alpha+m\beta) := \begin{cases} \phi^{-1}(n\alpha+m\beta) & \text{if } n\alpha+m\beta \notin \phi(G), \\ \phi^{S_G(n\alpha+m\beta)}(n\alpha+m\beta) & \text{if } n\alpha+m\beta \in \phi(G). \end{cases}$$

The map $\widetilde{\Phi}$ is a cyclic permutation on $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$. This is illustrated by the skyscraper tower construction of Figure 7 (see for instance [Pet89, p. 40]). One can check that $\widetilde{\Phi}$ coincides with the function $\overline{\Phi}$ on $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$. We will not use this fact in the proof but, as said before, it aims at providing some insight on the organisation of the cases that occur in the statement of Proposition 4.2.

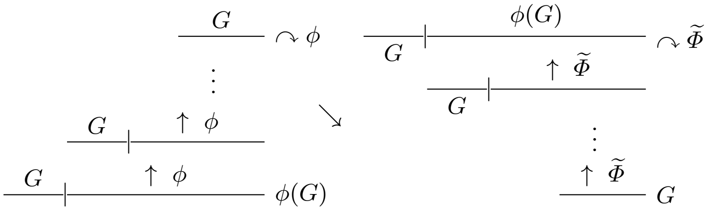

Fig. 7. Each k-level of the tower is moved to the level of index $-k$, with the indices of tower on the left being positive, and negative on the right. The actions on the rooftops are ϕ and $\widetilde{\Phi}$, respectively.

We now come back to the proof of Proposition 4.2. Let us count the number of points (n,m) taking the same value $\overline{\Delta}$. There are

(a) $(q_{4k+1}-q_{4k})q'_{4k}$ points such that $\overline{\Delta}(n,m)=\delta_k$ (case (1));
(b) $q_{4k-1}a_{4k+1}q'_{4k-1}$ points such that $\overline{\Delta}(n,m) = \|q'_{4k+1}\beta\| - a_{4k+1}\delta_k$ (case (2));
(c) $(q_{4k}-q_{4k-1})(a_{4k+1}-1)q'_{4k-1}$ points such that $\overline{\Delta}(n,m) = \|q'_{4k+1}\beta\| - (a_{4k+1}-1)\delta_k$ (case (3));
(d) $2q_{4k-1}q'_{4k-1}$ points such that $\overline{\Delta}(n,m) = \|q_{4k+1}\alpha\|+\|q'_{4k+1}\beta\|-2a_{4k+1}\delta_k$ (cases (4) and (6));

(e) $(q_{4k}-2q_{4k-1})q'_{4k-1}$ points such that $\overline{\Delta}(n,m)=\|q_{4k+1}\alpha\|+\|q'_{4k+1}\beta\|-(2a_{4k+1}-1)\delta_k$ (case (5));
(f) $2(q_{4k}-q_{4k-1})q'_{4k-1}$ points such that $\overline{\Delta}(n,m)=\|q_{4k+1}\alpha\|-a_{4k+1}\delta_k$ (cases (7) and (12));
(g) $2q_{4k}q'_{4k-1}$ points such that $\overline{\Delta}(n,m)=\|q_{4k+1}\alpha\|-(a_{4k+1}+c)\delta_k$ for $1\le c\le a_{4k+1}-1$ (cases (8) and (11));
(h) another $(q_{4k}-2q_{4k-1})(q'_{4k+1}-(2a_{4k+1}+1)q'_{4k-1})$ points such that $\overline{\Delta}(n,m)=\|q_{4k+1}\alpha\|-(2a_{4k+1}-1)\delta_k$ (those points correspond to case (10), and case (8) with $c=a_{4k+1}-1$, but we do not take into account case (11) with $c=1$);
(i) $2q_{4k-1}(q'_{4k+1}-2a_{4k+1}q'_{4k-1})$ points such that $\overline{\Delta}(n,m)=\|q_{4k+1}\alpha\|-2a_{4k+1}\delta_k$ (case (9)).

As noticed in Remark 4.3, there are overlaps between cases (8) and (10) ($c=a_{4k+1}-1$) and between cases (10) and (11) ($c=1$). There are $(q_{4k}-2q_{4k-1})q'_{4k-1}$ points in both intersections, thus in (h) the total number of points is

$$\begin{aligned}(q_{4k}-2q_{4k-1})(q'_{4k+1}-(2a_{4k+1}-1)q'_{4k-1})-2(q_{4k}-2q_{4k-1})q'_{4k-1}\\ =(q_{4k}-2q_{4k-1})(q'_{4k+1}-(2a_{4k+1}+1)q'_{4k-1}).\end{aligned}$$

We denote the sum of all the lengths $\overline{\Delta}$ of the intervals given in the statement of the proposition by

$$S:=\sum_{0\le n\le q_{4k+1},\,0\le m<q'_{4k+1}}\overline{\Delta}(n,m)=S_0+S_1+S_2+S_3,$$

where S_0 corresponds to case (a), S_1 to cases (b) and (c), S_2 to cases (d) and (e), and S_3 to the other cases. This yields

$$\begin{aligned}S_0 &:= (q_{4k+1}-q_{4k})q'_{4k}\delta_k,\\ S_1 &:= q_{4k-1}a_{4k+1}q'_{4k-1}(\|q'_{4k+1}\beta\|-a_{4k+1}\delta_k)\\ &\quad+(q_{4k}-q_{4k-1})(a_{4k+1}-1)q'_{4k-1}(\|q'_{4k+1}\beta\|-(a_{4k+1}-1)\delta_k),\\ S_2 &:= 2q_{4k-1}q'_{4k-1}(\|q_{4k+1}\alpha\|+\|q'_{4k+1}\beta\|-2a_{4k+1}\delta_k)\\ &\quad+(q_{4k}-2q_{4k-1})q'_{4k-1}(\|q_{4k+1}\alpha\|+\|q'_{4k+1}\beta\|-(2a_{4k+1}-1)\delta_k).\end{aligned}$$

Let us prove that $S=1$. Since the sum of the lengths $\overline{\Delta}$ for case (g) is

$$\begin{aligned}&2q_{4k}q'_{4k-1}\sum_{c=1}^{a_{4k+1}-1}(\|q_{4k+1}\alpha\|-(a_{4k+1}+c)\delta_k)\\ &=2q_{4k}q'_{4k-1}(a_{4k+1}-1)(\|q_{4k+1}\alpha\|-a_{4k+1}\delta_k)-2q_{4k}q'_{4k-1}\frac{(a_{4k+1}-1)a_{4k+1}\delta_k}{2}\\ &=2q_{4k}q'_{4k-1}(a_{4k+1}-1)\|q_{4k+1}\alpha\|-3q_{4k}q'_{4k-1}(a_{4k+1}-1)a_{4k+1}\delta_k,\end{aligned}$$

we get

$$
\begin{aligned}
S_3 := {} & 2(q_{4k}-q_{4k-1})q'_{4k-1}(\|q_{4k+1}\alpha\| - a_{4k+1}\delta_k) \\
& + 2q_{4k}q'_{4k-1}(a_{4k+1}-1)\|q_{4k+1}\alpha\| - 3q_{4k}q'_{4k-1}a_{4k+1}(a_{4k+1}-1)\delta_k \\
& + (q_{4k}-2q_{4k-1})(q'_{4k+1}-(2a_{4k+1}+1)q'_{4k-1})(\|q_{4k+1}\alpha\| - (2a_{4k+1}-1)\delta_k) \\
& + 2q_{4k-1}(q'_{4k+1}-2a_{4k+1}q'_{4k-1})(\|q_{4k+1}\alpha\| - 2a_{4k+1}\delta_k).
\end{aligned}
$$

Further,

$$
\begin{aligned}
S_1 &= (q_{4k+1}-q_{4k})q'_{4k-1}\|q'_{4k+1}\beta\| \\
&\qquad - ((a_{4k+1}-1)(q_{4k+1}-q_{4k}) + a_{4k+1}q_{4k-1})q'_{4k-1}\delta_k, \\
S_2 &= q_{4k}q'_{4k-1}(\|q_{4k+1}\alpha\| + \|q'_{4k+1}\beta\|) - (2q_{4k+1}-q_{4k})q'_{4k-1}\delta_k, \\
S_3 &= q_{4k}(q'_{4k+1}-q'_{4k-1})\|q_{4k+1}\alpha\| \\
&\qquad + ((a_{4k+1}+1)(q_{4k+1}+q_{4k-1}) - q_{4k})q'_{4k-1}\delta_k - (2q_{4k+1}-q_{4k})q'_{4k+1}\delta_k.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S &= q_{4k}q'_{4k+1}\|q_{4k+1}\alpha\| + q_{4k+1}q'_{4k-1}\|q'_{4k+1}\beta\| \\
&\qquad + (q_{4k+1}-q_{4k})q'_{4k}\delta_k + (q_{4k+1}-q_{4k})q'_{4k-1}\delta_k - (2q_{4k+1}-q_{4k})q'_{4k+1}\delta_k \\
&= q_{4k}q'_{4k+1}\|q_{4k+1}\alpha\| + q_{4k+1}q'_{4k-1}\|q'_{4k+1}\beta\| - q_{4k+1}q'_{4k+1}\delta_k \\
&= q_{4k}q'_{4k+1}\|q_{4k+1}\alpha\| + q_{4k+1}q'_{4k-1}\|q'_{4k+1}\beta\| \\
&\qquad - q_{4k+1}q'_{4k+1}(\|q'_{4k}\beta\| + \|q'_{4k+1}\beta\| - \|q_{4k}\alpha\|) \\
&= q'_{4k+1}(q_{4k}\|q_{4k+1}\alpha\| + q_{4k+1}\|q_{4k}\alpha\|) - q_{4k+1}(q'_{4k}\|q'_{4k+1}\beta\| + q'_{4k+1}\|q'_{4k}\beta\|) \\
&= q'_{4k+1} - q_{4k+1} = 1.
\end{aligned}
$$

Hence, the intervals $(n\alpha+m\beta, \overline{\Phi}(n\alpha+m\beta))$ never overlap (as intervals of $\mathbb{T}$), which implies that $\overline{\Phi}(n\alpha+m\beta)$ is the neighbor point of $n\alpha+m\beta$, that is, $\overline{\Phi} = \Phi$, which ends the proof of Proposition 4.2. ∎

End of the proof. According to cases (11) and (12) of Proposition 4.2, one has, for $n = cq_{4k} + 2q_{4k-1}$, $0 \le c \le a_{4k+1}-1$ and $m = 0$,

$$
\begin{aligned}
& \Phi_{q_{4k+1},q'_{4k+1}}((cq_{4k}+2q_{4k-1})\alpha + 0\beta) \\
&\qquad = ((2a_{4k+1}-1)q_{4k} - q_{4k+1} + 2q_{4k-1})\alpha + (2a_{4k+1}-c-1)q'_{4k-1}\beta \\
&\qquad = (q_{4k+1}-q_{4k})\alpha + (2a_{4k+1}-c-1)q'_{4k-1}\beta.
\end{aligned}
$$

Let $N = q_{4k+1} = q'_{4k+1} - 1$. For each $0 \le c \le a_{4k+1}-1$, the following pair of points belongs to $E_N(\alpha,\beta)$:

$$
(cq_{4k}+2q_{4k-1})\alpha, \qquad (q_{4k+1}-q_{4k})\alpha + (2a_{4k+1}-c-1)q'_{4k-1}\beta.
$$

Since $E_N(\alpha,\beta) \subset E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$ and the pairs above are adjacent points of $E_{q_{4k+1},q'_{4k+1}}(\alpha,\beta)$, for each $0 \le c \le a_{4k+1}-1$ we have

$$\begin{aligned}(q_{4k+1}-q_{4k})\alpha &+ (2a_{4k+1}-c-1)q'_{4k-1}\beta - (cq_{4k}+2q_{4k-1})\alpha \\ &= ((2a_{4k+1}-c-1)q_{4k}-q_{4k+1})\alpha + (2a_{4k+1}-c-1)q'_{4k-1}\beta \\ &= \|q_{4k+1}\alpha\| - (2a_{4k+1}-c-1)\delta_k \in \Delta(E_N(\alpha,\beta)).\end{aligned}$$

Since the sequence $(a_{4k+1})_k$ of partial quotients goes to infinity, we conclude that

$$\limsup_{N\to\infty} \#\Delta E_N(\alpha,\beta) = \infty,$$

which completes the proof of Theorem 1.1(ii).

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References

[ADG+16] F. Adiceam, D. Damanik, F. Gähler, U. Grimm, A. Haynes, A. Julien, A. Navas, L. Sadun, and B. Weiss, *Open problems and conjectures related to the theory of mathematical quasicrystals*, Arnold Math. J. 2 (2016), 579–592.

[AB98] P. Alessandri and V. Berthé, *Three distance theorems and combinatorics on words*, Enseign. Math. (2) 44 (1998), 103–132.

[BT02] V. Berthé and R. Tijdeman, *Balance properties of multi-dimensional words*, Theoret. Comput. Sci. 273 (2002), 197–224.

[BV00] V. Berthé and L. Vuillon, *Tilings and rotations on the torus: a two-dimensional generalization of Sturmian sequences*, Discrete Math. 223 (2000), 27–53.

[BS08] I. Biringer and B. Schmidt, *The three gap theorem and Riemannian geometry*, Geom. Dedicata 136 (2008), 175–190.

[Ble91] P. M. Bleher, *The energy level spacing for two harmonic oscillators with generic ratio of frequencies*, J. Statist. Phys. 63 (1991), 261–283.

[BHJ+12] P. M. Bleher, Y. Homma, L. L. Ji, R. K. W. Roeder, and J. D. Shen, *Nearest neighbor distances on a circle: multidimensional case*, J. Statist. Phys. 146 (2012), 446–465.

[Bug04] Y. Bugeaud, *Approximation by Algebraic Numbers*, Cambridge Tracts in Math. 160, Cambridge Univ. Press, Cambridge, 2004.

[Che00] N. Chevallier, *Three distance theorem and grid graph*, Discrete Math. 223 (2000), 355–362.

[Che07] N. Chevallier, *Cyclic groups and the three distance theorem*, Canad. J. Math. 59 (2007), 503–552.

[Che14] N. Chevallier, *Stepped hyperplane and extension of the three distance theorem*, in: Ergodic Theory and Dynamical Systems, De Gruyter Proc. Math., De Gruyter, Berlin, 2014, 81–92.

[CG76] F. R. K. Chung and R. L. Graham, *On the set of distances determined by the union of arithmetic progressions*, Ars Combin. 1 (1976), 57–76.

[CGVZ02] C. Cobeli, G. Groza, M. Vâjâitu, and A. Zaharescu, *Generalization of a theorem of Steinhaus*, Colloq. Math. 92 (2002), 257–266.

[FH95] A. S. Fraenkel and R. Holzman, *Gap problems for integer part and fractional part sequences*, J. Number Theory 50 (1995), 66–86.

[FS92] E. Fried and V. T. Sós, *A generalization of the three-distance theorem for groups*, Algebra Universalis 29 (1992), 136–149.

[GS93] J. F. Geelen and R. J. Simpson, *A two-dimensional Steinhaus theorem*, Australas. J. Combin. 8 (1993), 169–197.

[Hal65] J. H. Halton, *The distribution of the sequence* $\{n\xi\}$ $(n = 0, 1, 2, \dots)$, Proc. Cambridge Philos. Soc. 61 (1965), 665–670.

[HJKW17] A. Haynes, A. Julien, H. Koivusalo, and J. Walton, *Statistics of patterns in typical cut and project sets*, arXiv:1702.04041 (2017).

[HKWS16] A. Haynes, H. Koivusalo, J. Walton, and L. Sadun, *Gaps problems and frequencies of patches in cut and project sets*, Math. Proc. Cambridge Philos. Soc. 161 (2016), 65–85.

[HM17] A. Haynes and J. Marklof, *Higher dimensional Steinhaus and Slater problems via homogeneous dynamics*, arXiv:1707.04094 (2017).

[Lan95] S. Lang, *Introduction to Diophantine Approximations*, 2nd ed., Springer, New York (1995).

[Lan91] M. Langevin, *Stimulateur cardiaque et suites de Farey*, Period. Math. Hungar. 23 (1991), 75–86.

[Lia79] F. M. Liang, *A short proof of the 3d distance theorem*, Discrete Math. 28 (1979), 325–326.

[MS17] J. Marklof and A. Strömbergsson, *The three gap theorem and the space of lattices*, Amer. Math. Monthly 124 (2017), 741–745.

[Pet89] K. Petersen, *Ergodic Theory*, Cambridge Stud. Adv. Math. 2, Cambridge Univ. Press, Cambridge, 1989.

[PSZ16] G. Polanco, D. Schultz, and A. Zaharescu, *Continuous distributions arising from the three gap theorem*, Int. J. Number Theory 12 (2016), 1743–1764.

[Sla64] N. B. Slater, *Distribution problems and physical applications*, Compos. Math. 16 (1964), 176–183.

[Sla67] N. B. Slater, *Gaps and steps for the sequence* $n\theta$ *mod 1*, Proc. Cambridge Philos. Soc. 63 (1967), 1115–1123.

[S58] V. T. Sós, *On the distribution mod 1 of the sequence* $n\alpha$, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 1 (1958), 127–134.

[Surá58] J. Surányi, *Über die Anordnung der Vielfachen einer reellen Zahl mod 1*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 1 (1958), 107–111.

[Ś59] S. Świerczkowski, *On successive settings of an arc on the circumference of a circle*, Fund. Math. 46 (1959), 187–189.

[Tah17] D. Taha, *The three gaps theorems, interval exchange transformations, and zippered rectangles*, arXiv:1708.04380 (2017).

[vR88] T. van Ravenstein, *The three gap theorem (Steinhaus conjecture)*, J. Austral. Math. Soc. Ser. A 45 (1988), 360–370.

[Vij08] S. Vijay, *Eleven Euclidean distances are enough*, J. Number Theory 128 (2008), 1655–1661.

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Abstract (will appear on the journal's web site only)

For a given real number α, let us place the fractional parts of the points $0, \alpha, 2\alpha, \ldots, (N-1)\alpha$ on the unit circle. These points partition the unit circle into intervals having at most three lengths, one being the sum of the other two. This is the three-distance theorem. We consider a two-dimensional version of the three-distance theorem obtained by placing on the unit circle the points $n\alpha + m\beta$ for $0 \leq n, m < N$. We provide examples of pairs of real numbers (α, β), with $1, \alpha, \beta$ rationally independent, for which there are finitely many lengths between successive points (and in fact, seven lengths), with (α, β) not badly approximable, as well as examples for which there are infinitely many lengths.