Multidimensional Euclidean algorithms, numeration and substitutions

V. Berthé
Laboratoire d'Informatique Algorithmique : Fondements et Applications,
Université Paris Diderot, Paris 7 - Case 7014
F-75205 Paris Cedex 13 , France
berthe@liafa.jussieu.fr

Abstract: The aim of this survey is to discuss multidimensional continued fraction and Euclidean algorithms from the viewpoint of numeration systems, substitutions, and the symbolic dynamical systems they generate. We will mainly focus on two types of multidimensional algorithms, namely, unimodular Markovian ones which include the most classical ones like e.g. Jacobi-Perron algorithm, and algorithms issued from lattice reduction. We will discuss these algorithms motivated by the study of substitutive dynamical systems, symbolic dynamical systems with low complexity function, and discrete geometry.

Keywords: multidimensional continued fractions, Euclidean algorithms, numeration systems, lattice reduction, symbolic dynamical systems, substitutions.

1 Introduction

The aim of this survey is to discuss multidimensional continued fraction and Euclidean algorithms from the viewpoint of numeration systems, substitutions, and the symbolic dynamical systems they generate. Let us note first that continued fraction algorithms enter in a natural way in the framework of numeration, when considering numeration in a wide sense as the art of representation of numbers (integers, rational, real, complex numbers, vectors, etc.). This is also closely related to the viewpoint of numeration dynamics, developed by M. Keane, and of arithmetic dynamics, such as described in the survey [129]: arithmetic dynamics deals with arithmetic expansions and codings of dynamical systems that preserve their arithmetic structure.

Let us give now a flavor of what is meant here by substitutive viewpoint on continued fractions. A substitution is a simple and basic object in word combinatorics and in symbolic dynamics (i.e., the study of discrete dynamical systems obtained by working with infinite sequences of symbols endowed with the shift). A substitution is a non-erasing morphism of the free monoid: it replaces letters by words. For more on substitutions and the symbolic dynamical systems they generate, see [112, 111]. As an example of the interactions between Euclid's algorithm and substitutions, let us focus on the family of Sturmian sequences (see also Section 3.1). Sturmian sequences are infinite words with values in a two-letter alphabet that are obtained as codings of irrational rotations acting on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with respect to a particular two-set partition of the one-dimensional torus $\mathbb{T}$. For a thorough description of Sturmian sequences, see [25] and Chap. 6 in [111]. Sturmian sequences can be perfectly understood thanks to some representation involving substitutions and Euclid's algorithm: by expanding α as a continued fraction, we represent it as an infinite product of square matrices of size two with nonnegative integer entries; each of these matrices can be seen as the abelianized matrix of a substitution, with the action of a substitution being considered as a combinatorial interpretation of a step (additive or multiplicative) of Euclid's algorithm. A Sturmian sequence is then proved to be generated as an infinite composition of these substitutions. This composition is governed by the Ostrowski numeration which uses as a numeration scale the convergents of a given real number $\alpha \in (0, 1)$, such as described e.g. in [30]. The Sturmian framework is a salient example of the relations between numeration, continued fractions and dynamics. We will use it as a guideline and as a motivation for a possible generalization throughout this survey.

More generally, the connection between word combinatorics and multidimensional continued fractions is particularly striking within the so-called S-adic framework. A sequence is said to be S-adic if it is generated by an infinite composition of a finite number of substitutions. This covers various families of infinite words with a rich dynamical behavior such as Sturmian sequences. In order to understand the geometric and symbolic nature of the dynamical systems they generate, we are mainly interested in the two following problems: first, finding geometric interpretations of these systems, and secondly, developing multidimensional continued fraction algorithms that rule their S-adic expansion. This belongs to the so-called *Rauzy program*, such as detailed in the survey [28]. This program can be sketched as follows: find generalizations of the interaction between Sturmian sequences and rotations which would naturally generate (simultaneous) approximation algorithms. As an example, see [113] where a continued fraction expansion associated with interval exchanges is discussed. Our approach is complementary to that of [28]. Instead of discussing continued fraction algorithms that are issued from the study of some classic families of infinite words allowing to perform Rauzy's program, we try to bring some elements of answer to the following question: which types of generalizations of continued fraction algorithms can be of some use in symbolic dynamics and in discrete geometry?

We have no claim to being exhaustive in our exposition of generalizations of Euclidean and continued fraction algorithms. We have chosen to spotlight here representative aspects of the theory in connection with substitutions and numeration systems. For historical aspects, the reader is referred to the classic references [24, 137, 34]. For ergodic aspects, see [125, 127].

Let us sketch the contents of this survey paper. Section 2 includes basic introductory material on substitutions, symbolic dynamical systems, and the question of their geometric representation. Section 3 aims at introducing first motivations for the introduction of suitable generalized Euclidean's algorithms, namely the study of discrete lines and the S-adic approach. Multidimensional continued fractions are discussed in Section 4 in full generality. We detail the case of unimodular Markovian algorithms in Section 5 and the case of algorithms based on lattice reduction in Section 6. This paper ends with a discussion on possible applications of these algorithms to symbolic dynamics and discrete geometry.

2 Substitutive dynamical systems

2.1 Substitutions on words and symbolic dynamical systems

We consider a finite set of *letters* $\mathcal{A}$, called *alphabet*. A (finite) *word* is an element of the free monoid $\mathcal{A}^*$ generated by $\mathcal{A}$. A *substitution* σ over the alphabet $\mathcal{A}$ is a non-erasing endomorphism of the free monoid $\mathcal{A}^*$ (non-erasing means that the image of any letter is not equal to the empty word but contains at least one letter). For $i \in \mathcal{A}$ and for $w \in \mathcal{A}^*$, let $|w|_i$ stand for the number of occurrences of the letter i in the word w. Let d stand for the cardinality of $\mathcal{A}$. The map

$$\mathbf{l}\colon \mathcal{A}^* \to \mathbb{N}^d,\ w \mapsto (|w|_1, |w|_2, \cdots, |w|_d)$$

is called the *abelianization map*. This map is also referred to as the *Parikh mapping*. Let σ be a substitution. Its *incidence matrix* (also called *abelianized matrix*) $M_\sigma = (m_{i,j})_{1\le i,j\le d}$ is defined as the square matrix with entries $m_{i,j} = |\sigma(j)|_i$ for all i, j. We say that σ is *unimodular* if $\det(M_\sigma) = \pm 1$. A substitution is said *primitive* if there exists a power of its incidence matrix whose entries are all positive.

Let S denote the following map defined on $\mathcal{A}^{\mathbb{N}}$, called the (one-sided) *shift*:

$$S((u_n)_{n\in\mathbb{N}}) = (u_{n+1})_{n\in\mathbb{N}}. \tag{1}$$

We endow the set $\mathcal{A}^{\mathbb{N}}$ with the the following metrics: for $x, y \in \mathcal{A}^{\mathbb{N}}$

$$d(x, y) = (1 + \inf\{k \ge 0;\ x_k \ne y_k\})^{-1}.$$

It is a compact space. Two sequences are close to each other if their first terms coincide.

Let us see now how to associate with a substitution a symbolic dynamical system, defined as a closed shift invariant subset of $\mathcal{A}^{\mathbb{N}}$. Let σ be a primitive substitution over

$\mathcal{A}$. Let $u \in \mathcal{A}^{\mathbb{N}}$ be such that $\sigma^k(u) = u$ for some $k \geq 1$. Such an infinite word exists by primitivity of σ. Indeed, there exist a letter a and a positive integer k such that $\sigma^k(a)$ begins with a; consider as first letter of u this letter a; take for u $\lim_{n\to\infty} \sigma^{kn}(a)$. Let $\overline{\mathcal{O}(u)}$ be the positive orbit closure of the sequence u under the action of the shift S, i.e., the closure of the set $\mathcal{O}(u) = \{S^n(u) \mid n \geq 0\}$. The substitutive symbolic dynamical system (X_σ, S) generated by σ is defined as $X_\sigma := \overline{\mathcal{O}(u)}$. One easily checks by primitivity that (X_σ, S) does not depend on the choice of the infinite sequence u fixed by some power of σ. For more details, see [112].

For analogue notions of substitutions and associated dynamical systems defined on tilings and point sets, acting as inflation/subdivision rules, see the surveys [134, 118, 110].

2.2 Geometric representations

One fundamental question concerning substitutive dynamical systems (X_σ, S) deals with the possibility of giving to them a geometric representation. By geometric representation, one considers here dynamical systems of a geometric nature that are either topologically or measure-theoretically isomorphic to (X_σ, S). In particular, one looks for conditions under which it is possible to give a geometric representation of a substitutive dynamical system as a translation on the torus, or on a locally compact abelian group. This latter question can be reformulated in spectral terms: which are the substitutions whose associated dynamical system has discrete spectrum? For more details, see e.g. [143, 111, 112]. Note that measure-theoretic discrete spectrum and topological discrete spectrum are proved to be equivalent for primitive substitutive dynamical systems [66].

A substitution is said *Pisot irreducible* if the characteristic polynomial of its incidence matrix is the minimal polynomial of a Pisot number, that is, an algebraic integer β whose conjugates (distinct from β) have modulus smaller than 1. It is widely believed that Pisot irreducible substitutions have purely discrete spectrum. For more details, see e.g. [111], Chap. 7 and [22]. See also in the same vein [119] whose main concern is Pisot automorphisms of the torus (instead of substitutions). Consider as a first example the Fibonacci substitution $\sigma\colon a \mapsto ab,\ b \mapsto a$; (X_σ, S) is measure-theoretically isomorphic to $(\mathbb{R}/\mathbb{Z}, R_{\frac{1+\sqrt{5}}{2}})$. For more details see e.g. Chap. 5 in [111]. Furthermore, two-letter Pisot substitutions are known to have discrete spectrum [21, 65, 67]. See also [22, 32, 75] for recent results on Pisot substitutive dynamical systems.

One strategy for providing geometric representations has been developed by Rauzy and can be considered as a part of Rauzy's program mentioned in the introduction. This approach has been developed in the case of the Tribonacci substitution $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ in [114]. It is a primitive, unimodular and Pisot irreducible substitution. Its characteristic polynomial is $X^3 - X^2 - X - 1$ and its dominant eigenvalue $\beta > 1$ is a Pisot number.

Theorem 1 ([114]) *Let σ be the Tribonacci substitution $\sigma\colon 1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1$.*

Let $R_\beta\colon \mathbb{T}^2 \to \mathbb{T}^2$, $x \mapsto x + (1/\beta, 1/\beta^2)$. The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the toral translation $(\mathbb{T}^2, R_\beta)$.

The proof makes use of the fact that the Tribonacci sequence $\sigma^\infty(1) = \lim_{n\to\infty} \sigma^n(1)$ codes the orbit of the point 0 under the action of the translation R_β with respect to a particular partition of $\mathbb{T}^2$. In order to get this partition, one constructs a so-called *Rauzy fractal* as follows, according to [114]. One first represents $(u_n)_{n\in\mathbb{N}} = \sigma^\infty(1)$ as a broken line via the abelianization map $\mathbf{l}$. The vertices of this broken line belong to $\mathbb{Z}^3$ and are of the form $\mathbf{l}(u_0 \cdots u_n)$ for $n \in \mathbb{N}$. We then project the vertices of this broken line according to the eigenspaces of the incidence matrix M_σ, that is, along its expanding line onto its contracting plane. The corresponding projection is denoted by π_c. The Rauzy fractal associated with σ is then obtained by taking the closure of this set of points, i.e., as

$$\mathcal{R}_\sigma := \overline{\{\pi_c \circ \mathbf{l}(u_0 \cdots u_n) \mid n \in \mathbb{N}\}}.$$

We then divide $\mathcal{R}_\sigma$ into the three pieces defined for $i = 1, 2, 3$ as

$$\mathcal{R}_\sigma(i) := \overline{\{\pi_c \circ \mathbf{l}(u_0 \cdots u_n) \mid u_n = i,\ n \in \mathbb{N}\}}.$$

Theorem 1 can be reformulated as follows: the Rauzy fractal $\mathcal{R}_\sigma$ is a fundamental domain of $\mathbb{T}^2$ and $\sigma^\infty(1)$ codes the orbit of the point 0 under the action of the translation R_β with respect to the particular partition $(\mathcal{R}_\sigma(i))_{i=1,2,3}$ of the fundamental domain $\mathcal{R}_\sigma$ of $\mathbb{T}^2$.

Rauzy fractals were first introduced in [114] in the case of the Tribonacci substitution, and then in [138], in the case of the β-numeration associated with the Tribonacci number. Rauzy fractals can more generally be associated with Pisot substitutions (see [101, 102, 133, 15, 37, 38, 22, 75, 130, 131] and the surveys [32, 111]), as well as with Pisot β-shifts under the name of *central tiles* (see [6, 7, 8]).

Let us make several comments concerning Theorem 1. First, a statement generalizing Theorem 1 is conjectured to hold for any Pisot irreducible substitution; note that the corresponding parameters would be algebraic, since they are given by eigenvalues and eigenvectors of the incidence matrix of the substitution. Secondly, the broken line obtained by applying the abelianization map $\mathbf{l}$ to the prefixes of an infinite word u fixed by σ can be considered as a discrete line; hence, having a generalization of Theorem 1 can be of some interest from a discrete geometry viewpoint. Thirdly, the symbolic coding provided by u and σ allows one to recover arithmetic information concerning the toral translation R_β associated with σ, as illustrated in [116, 1]. In particular, the subtiles $\mathcal{R}_\sigma(i)$ of the Rauzy fractal are bounded remainder sets for R_β. We recall that a subset A of $\mathbb{T}^d$ with (Lebesgue) measure $\mu(A)$ is said to be a *bounded remainder set* for the translation $R_\alpha : x \mapsto x + \alpha$ ($\alpha \in \mathbb{T}^d$) if there exists $C > 0$ such that $\forall N,\quad |\mathrm{Card}\{0 \le n < N \mid n\alpha \in A\} - N\mu(A)| \le C$. For a detailed discussion on the possible choice of atoms in the partition in Theorem 1, see [28].

2.3 Numeration and substitutions: The Dumont-Thomas numeration system

In order to get suitable geometric representations of dynamical systems, there exists a natural numeration system that plays a nontrivial role, namely, the Dumont-Thomas numeration system introduced in [46, 47]. For more details on the relations with Rauzy fractals, see [37, 38, 32, 19]. Given a primitive substitution σ and an infinite word u that satisfies $\sigma(u) = u$, this numeration provides a representation of prefixes of u, based on the greedy algorithm.

Let us try to get a flavor of the way this numeration system works. Let $u = (u_n)_{n\in\mathbb{N}}$ be such that $\sigma(u) = u$. The idea is to decompose prefixes $u_0 \cdots u_{N-1}$ of u into concatenations of images by powers of σ of a finite number of words belonging to some set $\mathcal{D}_\sigma$. The powers of σ will play the role of a basis in a classical number system, and the set $\mathcal{D}_\sigma$ can be seen as a set of digits. Since $\sigma(u) = u$, there exists L such that

$$\sigma(u_0 \cdots u_{L-1}) \leq u_0 \cdots u_{N-1} < \sigma(u_0 \cdots u_L),$$

that is, $\sigma(u_0 \cdots u_{L-1})$ is a prefix of $u_0 \cdots u_{N-1}$ (with maybe equality), and $u_0 \cdots u_{N-1}$ is a proper prefix of $\sigma(u_0 \cdots u_L)$. There thus exists a proper prefix p of $\sigma(u_L)$ such that $u_0 \cdots u_{N-1} = \sigma(u_0 \cdots u_{L-1})\, p$ with $\sigma(u_L) = p\, u_N\, s$. By iterating this decomposition, one obtains for every N

$$u_0 \cdots u_{N-1} = \sigma^K(p_K)\sigma^{K-1}(p_{K-1}) \cdots \sigma(p_1)p_0,$$

where the p_i belong to the finite set of words $\mathcal{D}_\sigma$ made of the proper prefixes of the images of the letters by σ.

To obtain a numeration system on natural integers one takes the lengths of these words, i.e., $N = |u_0 \cdots u_{N-1}| = |\sigma^K(p_K)| + |\sigma^{K-1}(p_{K-1})| + \cdots + |\sigma(p_1)| + |p_0|$. This numeration also extends to real numbers by providing generalized radix expansions of positive real numbers, with digits belonging to a finite subset of the number field $\mathbb{Q}(\beta)$, where β is the Perron–Frobenius eigenvalue of σ, i.e., the dominant eigenvalue of the incidence of the primitive substitution σ.

As an example, one checks that every prefix w of the Tribonacci word u can be uniquely expanded as $w = \sigma^n(p_n)\sigma^{n-1}(p_{n-1}) \cdots p_0$, where the words p_i are equal to the empty word or to the letter 1. It is easily seen that one never gets three digits equal to 1 in a row. If σ is a constant length substitution of length q, then one recovers the q-adic numeration. If σ is a β-substitution for a Parry number β, then one recovers the β-numeration. For more details, see [46, 47].

3 How to reach nonalgebraic parameters?

We now have set up all the elements required for the study of substitutive dynamical systems: in order to get geometric representations, one introduces arithmetic tools such as Dumont-Thomas numeration that allows a thorough description of their associated Rauzy fractal. Nevertheless, so far we have only considered iterations of a single substitution. We have seen in Section 2.2 that this yields arithmetic results concerning algebraic parameters: these parameters are expressed in terms of eigenvectors and eigenvalues of the incidence matrix of the substitution. We now want to be able to reach nonalgebraic parameters. In particular, we would like to be able to define Rauzy fractals for any toral translation on the torus $\mathbb{T}^d$, for $d \geq 1$, in order to get a statement generalizing Theorem 1. Our motivations run from dynamical systems, through arithmetics, to discrete geometry, this would allow us to associate with any direction a broken line. Indeed, Rauzy fractals do not only produce geometric representations of substitutive dynamical systems, but have also very interesting Diophantine applications. We refer to [71, 3] for representative examples.

3.1 Numbers, sequences and lattices: dynamical representation of discrete lines

There is a situation where we can reach nonalgebraic parameters, namely, for translations on the one-dimensional torus. Instead of working with infinite words generated by the iteration of a substitution, we consider Sturmian sequences. From a discrete geometry viewpoint, Sturmian sequences are codings of standard arithmetic discrete lines via the Freeman code according to the terminology of [117]. More precisely, a Sturmian sequence $s_{\alpha,\rho}$ is a coding over a two-letter alphabet of the orbit of the point ρ of the one-dimensional torus $\mathbb{T}^1$ under the action of the irrational rotation $R_\alpha : \ x \mapsto x + \alpha$ (compare with Theorem 1). Let α be an irrational number in $(0, 1)$. Let $0 < \alpha < 1$ and $0 \leq \rho \leq 1$. Let $R_\alpha : \ \mathbb{T}^1 \to \mathbb{T}^1$ be the rotation of angle α. We first introduce two partitions of $\mathbb{T}^1$ as follows:

$$\underline{I}_1 = [0, 1-\alpha), \quad \underline{I}_2 = [1-\alpha, 1), \quad \overline{I}_1 = (0, 1-\alpha], \quad \overline{I}_2 = (1-\alpha, 1];$$

we then define respectively the two following infinite words by

$$\underline{s}_{\alpha,\rho} = \begin{cases} 1 & \text{if } R_\alpha^n(\rho) \in \underline{I}_1, \\ 2 & \text{if } R_\alpha^n(\rho) \in \underline{I}_2, \end{cases}$$

$$\overline{s}_{\alpha,\rho} = \begin{cases} 1 & \text{if } R_\alpha^n(\rho) \in \overline{I}_1, \\ 2 & \text{if } R_\alpha^n(\rho) \in \overline{I}_2. \end{cases}$$

Moreover, Sturmian sequences have a very simple combinatorial description. Sturmian sequences are exactly those one-sided infinite sequences with complexity $p(n) = n + 1$, i.e., having $n + 1$ factors of length n for every n (see [103, 43]). A very detailed description of these results can be found in [25], see also [111].

As a consequence, Sturmian sequences admit one isolated letter, i.e., 00 and 11 cannot be both factors of a given Sturmian sequence: they have 3 factors of length 2. More precisely, let $u \in \{0,1\}^{\mathbb{N}}$ be a Sturmian sequence of slope α. Exactly one of the words ii ($i \in \{0,1\}$) is a factor of u. Hence, there is a unique sequence u' such that $u = S^b(\sigma_i(u'))$, where $b = 0$ if u does not begin in i and $b = 1$ otherwise (recall that S stands for the shift, see (1)), and

$$\sigma_0 \colon 0 \mapsto 0,\ \sigma_0 \colon 1 \mapsto 10,\ \sigma_1 \colon 0 \mapsto 01,\ \sigma_1 \colon 1 \mapsto 1.$$

Another way of recovering this "desubstitution" process is to perform an induction, that is, to work with the first return map of the rotation R_α on a suitable subinterval of the unit interval. For more details, see [30, 28]. What is particularly interesting here is that one checks that the sequence u' is again a Sturmian sequence, but with a different angle α'. Indeed, the substitutions σ_i can be seen as transformations acting on bases of the lattice $\mathbb{Z}^2$ via their incidence matrix. Since a Sturmian sequence is a coding of a discrete line in the lattice $\mathbb{Z}^2$, u' is again a coding of a discrete line but in a different lattice. By reiterating this process, one thus deduces that a Sturmian sequence u can be written as an infinite composition of a finite number of substitutions. If one is only interested in a description of the symbolic dynamical system that u generates, (that is, to the set of its factors), then one checks that it coincides with the symbolic dynamical system generated by the infinite sequence

$$\lim_{n\to+\infty} \sigma_0^{a_1}\sigma_1^{a_2}\cdots\sigma_{2n}^{a_{2n}}\sigma_{2n+1}^{a_{2n+1}}(0).$$

Such a representation is called an *S-adic expansion.*

The incidence matrices of the substitutions σ_i, for $i = 0$ and 1, are equal to $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$, respectively. They correspond to the matrices that perform the additive steps of Euclid's algorithm $(a,b) \mapsto (a_1,b_1) = (a-b,b)$, i.e., $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right]$. The action of Euclid's algorithm is translated here in symbolic terms as the action of substitutions, with the digits a_n being the partial quotients in the continued fraction expansion of the angle α. We thus have realized the action of the set of square invertible matrices with nonnegative entries on the bases of the lattice $\mathbb{Z}^2$ as a noncommutative action by substitutions: we associate with a square invertible integer matrix M in a noncanonical way a substitution whose incidence matrix is M.

If one wants a description of the sequence u itself, the expansion involves not only regular continued fractions but also Ostrowski numeration system (see Section 3.2 below), as well as the set of substitutions

$$\sigma_0 \colon 0 \mapsto 0,\ \sigma_0 \colon 1 \mapsto 10,\ \sigma_1 \colon 0 \mapsto 01,\ \sigma_1 \colon 1 \mapsto 1$$

$$\sigma_0' \colon 0 \mapsto 0,\ \sigma_0' \colon 1 \mapsto 01,\ \sigma_1' \colon 0 \mapsto 10,\ \sigma_1' \colon 1 \mapsto 1.$$

We thus get

$$u = \lim_{n\to+\infty} (\sigma')_0^{a_1-c_1} \circ \sigma_0^{c_1} \circ (\sigma')_1^{a_2-c_2} \circ \sigma_1^{c_2} \circ \cdots \circ (\sigma')_{k-1}^{a_n-c_n} \circ \sigma_{n-1}^{c_n}(0),$$

where the coefficients c_n are produced by Ostrowski numeration system, introduced in the next section. For more details, see [30] and Chap. 6 in [111].

More generally, for results on the connections between continued fractions and Sturmian sequences, and in particular, applications of the mirror formula (i.e., $\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \cdots, a_1]$) in word combinatorics, see the survey [2].

3.2 Ostrowski's numeration system

Ostrowski numeration system is based on the numeration scale given by the sequence of denominators in the continued fraction expansion of a given real number (see [106]). It is a generalization of the Zeckendorf representation [144] (which involves Fibonacci numbers and the golden ratio). Let $\alpha \in (0,1)$ be an irrational number. Let $\alpha = [0; a_1, a_2, \ldots, a_n, \ldots]$ be its continued fraction expansion with convergents p_n/q_n. Every integer N can be expanded uniquely in the form $N = \sum_{k=1}^{m} b_k q_{k-1}$, where the b_k's are nonnegative integers, $0 \leq b_1 \leq a_1 - 1$, $0 \leq b_k \leq a_k$ for $k \geq 2$, and $b_k = 0$ if $b_{k+1} = a_{k+1}$.

Ostrowski's representation of integers can be extended to real numbers. The base is given by the sequence $(\theta_n)_{n\geq 0}$, where $\theta_n = q_n\alpha - p_n$. Every real number $-\alpha \leq \beta < 1 - \alpha$ can be expanded uniquely in the form $\beta = \sum_{k=1}^{+\infty} c_k \theta_{k-1}$, where the c_k's are nonnegative integers, $0 \leq c_1 \leq a_1 - 1$, $0 \leq c_k \leq a_k$ for $k \geq 2$, $c_k = 0$ if $c_{k+1} = a_{k+1}$, and $c_k \neq a_k$ for infinitely many odd integers.

For more on the connections between Sturmian sequences and Ostrowski numeration, see [26], Chap. 6 in [111], and also [14] which is devoted to the so-called scenery flow. According to [136, 72, 70, 73, 142], the digits in Ostrowski's numeration are produced by introducing a skew product of the continued fraction transformation, which allows their metrical study, performed in [70, 73]. We will come back to this in Section 5.2. Note that the odometer associated with Ostrowski's numeration (in the sense of [60]) is metrically isomorphic to a rotation on $\mathbb{T}^1$ (see for instance [48, 142] and more recently [20] and the references therein). This result can be considered as a generalization of Theorem 1.

3.3 S-adic expansions and multidimensional continued fraction expansions

We have seen that a Sturmian sequence can be written as an infinite composition of a finite number of substitutions (called S-adic expansion) and that its S-adic expansion can be described thanks to the continued fraction expansion of its slope. S-adic sequences generalize in a natural way substitutive sequences. The aim of this section is to sustain the idea that they are the right framework for trying to get suitable codings of toral translations in the flavor of Theorem 1, with the sequence of coefficients in the S-adic expansion being produced by a multidimensional continued fraction algorithm.

We expect furthermore such infinite sequences coding toral translations to have an at most linear number of factors of a given length (they are said to be of *linear complexity*). Indeed, each substitution in the S-adic expansion is considered as a step in a multidimensional continued fraction algorithm. Usually, the corresponding substitutions have integer transvection or permutation matrices as incidence matrix. We recall that a *transvection matrix* is a matrix M of the form $M = Id + \lambda E_{ij}$, where Id stands for the identity matrix, $1 \leq i, j \leq d$, $i \neq j$, $\lambda \in \mathbb{R}$, and E_{ij} is the square matrix having all entries equal to 0 and the entry of index (i, j) equal to 1. It is said positive if $\lambda > 0$ and it is said to be an integer transvection if $\lambda \in \mathbb{Z}$. Such matrices often preserve a linear growth for the complexity function. Furthermore, we would like to recover from the dynamical and combinatorial properties of these infinite sequences, arithmetical information on the parameters underlying translation on the torus. This will be easier if these coding sequences have low complexity, in terms of numbers of factors.

There exist several well-studied families of words that could produce codings of translations of the torus, but they all have quadratic complexity. For instance, billiard words are defined as codings of trajectories of billiards in a cube; they are shown to have quadratic complexity (see [16, 23]). Let us quote also [42] where a construction method is considered which produces step by step a broken line whose vertices belong to $\mathbb{Z}^3$ that approximates a given direction by choosing at each step the closest point. It is proved in [42] that such a broken line can be obtained by selecting integer points by shifting a polygonal window along the line. The complexity is here again quadratic. In both cases one is unable to associate with these infinite sequences a suitable continued fraction algorithm or an S-adic representation.

Expecting codings with linear complexity implies that the atoms of the partition that we are looking for when trying to generalize Theorem 1 are not just boxes of $\mathbb{T}^d$. Let us sustain this assertion with the Sturmian case. Another way of describing the desubstitution process allowing to recover the S-adic expansion of a Sturmian sequence is based on the notion of induction. According to [51, 115] if the induced map (i.e., the first return map) of a translation on a set A is still a translation, then this set A is a bounded remainder set. Intervals that are bounded remainder sets for the translation $R_\alpha \colon x \mapsto x + \alpha$ with $\alpha \in \mathbb{R}$ of $\mathbb{T}^1$ are known to be of length in $\alpha\mathbb{Z} + \mathbb{Z}$ [82]. Intervals $\underline{I}_i$ and $\overline{I}_i$, $i = 1, 2$, according to which Sturmian sequences of angle α code R_α are thus bounded remainder sets. This seems to indicate that, in the more elementary generalizations of the Sturmian/rotation interaction, the atoms of a coding of a translation should be preferably chosen as bounded remainder sets; this would allow one to reiterate the desubstitution/induction process. Hence they cannot be boxes in $\mathbb{T}^d$: indeed, there are no nontrivial rectangles which are bounded remainder sets for ergodic translations on the torus [98]. Furthermore, codings of translations with respect to boxes do not have linear complexity [135]. In order to get suitable atoms for the partition, we thus need to get specific constructions in the flavor of the one obtained for the Rauzy fractal in the Tribonacci case: we are looking for infinite words whose abelianized prefixes provide good integer approximations of a line in $\mathbb{R}^{d+1}$ which will give us a translation in $\mathbb{T}^d$ after some projectivization process. This will be the object of the next sections to discuss existing algorithms for simultaneous approximation.

Several combinatorial questions can be formulated in an efficient way in this S-

adic/continued fraction framework. Given an S-adic sequence, one can ask whether this sequence is substitutive, that is, whether it is a letter-to-letter projection of a fixed point of a substitution. Substitutive Sturmian sequences correspond to quadratic angles (for more details, see e.g. [30]). This result can be considered as a version of Galois' theorem for continued fraction expansions.

Convergence issues (and Diophantine approximation properties) for a multidimensional continued fractions algorithm underlying a family of infinite words correspond to the question of convergence toward frequencies of factors, which can also be expressed in measure-theoretic terms (in particular if one has unique ergodicity). The study of S-adic words thus leads to numerous questions that are of a combinatorial, arithmetic or else dynamical nature. Among them, the so-called S-adic conjecture aims at finding a characterization of infinite words having linear complexity in S-adic terms. For more details, see Chap. 11 in [111]. Note that infinite words having an at most linear number of factors of a given length are known to be S-adic, if they are furthermore assumed to be minimal [52]. For more on S-adic sequences, see e.g. [49, 50] and Chap. 11 in [111].

We have seen that S-adic expansions and multidimensional continued fraction algorithms are strongly related. More generally, generalizations of Euclid's algorithm intervene in a natural way in several problems issued from word combinatorics. As an example, let us quote Fine and Wilf's theorem. This theorem gives a condition on the length of the periods a finite word can have. More precisely, if w is a word having periods p and q with length greater than or equal to $p + q - \gcd(p, q)$, then w has period $\gcd(p, q)$. Assume now p and q coprime. The family of words with length $p + q - 2$ that are p and q periodic is particularly interesting. Such extremal words (with respect to Fine and Wilf's theorem) are known to be particular factors of Sturmian sequences, and their study involves Euclid's algorithm. For more details, see [25] and the references therein. There exist two natural types of generalizations of Fine and Wilf's theorem, either by extending the size of the alphabet [40], or by considering multidimensional words [132]. Extremal words for these generalizations can also be described in terms of multidimensional continued fraction algorithms. In particular, in the former case, an algorithm in the flavor of the fully subtractive algorithm (see [127] and Section 5) allows the construction of extremal words [139, 140].

4 Multidimensional continued fractions

4.1 Simultaneous approximations

Continued fractions are known to provide best approximations of a given real number in $(0, 1)$ (see e.g. [83]). The question is now to find similar algorithms yielding simultaneous rational approximations with same denominator, and of good quality, of d-uples of positive real numbers . Consider in particular the dimension $d = 2$ case. Given a pair of real numbers $(\alpha_1, \alpha_2) \in (0, 1)^2$, one looks for three sequences of nonnegative integers

$(p_n, q_n, r_n)_{n\in\mathbb{N}}$ such that

$$\lim p_n/q_n = \alpha_1, \lim r_n/q_n = \alpha_2,$$

with a good quality of rational approximation of (α_1, α_2). Dual problems consist in looking for small values of linear forms and small linear relations, and in detecting rational dependencies. Usual norms that are considered are the sup and the Euclidean norm.

Geometrically, this corresponds to look for approximations of a line in $\mathbb{R}^{d+1}$ by points in $\mathbb{Z}^{d+1}$, or in a dual way of a hyperplane in $\mathbb{R}^{d+1}$ by points in $\mathbb{Z}^{d+1}$. In arithmetic and dynamical terms, the underlying dynamical systems will be in the first case a translation on the torus

$$\mathrm{R}_\alpha \colon \mathbb{T}^{d+1} \to \mathbb{T}^{d+1},\ x = (x_1, \dots, x_{d+1}) \mapsto x + (\alpha_1, \dots, \alpha_{d+1}),$$

or a $\mathbb{Z}^{d+1}$-action of $\mathbb{T}^1$

$$(m_1, \cdots, m_{d+1}).(x, y) = m_1\alpha_1 + \cdots + m_{d+1}\alpha_{d+1}.$$

It remains to make more precise what is meant here by “good” quality of rational approximation. This notion first depends on a choice of a norm. Given a norm $|| \quad ||$ on $\mathbb{R}^2$, the quality of the approximation is measured by

$$\begin{aligned}\tfrac{1}{q_n}|||q_n\alpha||| &= \tfrac{1}{q_n}|||q_n(\alpha_1, \dots, \alpha_d)||| \\ &= \tfrac{1}{q_n} \min\left\{||(q_n\alpha_1 - p_1, \dots, q_n\alpha_d - p_d)|| \mid (p_1, \dots, p_d) \in \mathbb{Z}^d\right\}.\end{aligned}$$

Secondly, the quality of approximation can be measured with respect to Dirichlet’s theorem, i.e., $|||q\alpha|||$ has to be compared with $q^{-1/d}$. Let us recall Dirichlet’s theorem, which corresponds to the choice of the sup norm, and which is obtained as a direct application of the pigeonhole principle (see e.g. [63]).

Theorem 2 (Dirichlet’s theorem) *For any $(\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ and any Q, there exists a positive integer q with $q \le Q^d$ and integers p_i such that*

$$\max_{1\le i\le d} |q\alpha_i - p_i| < \frac{1}{Q}.$$

One thus deduces immediately that the system of inequalities

$$|\frac{p_i}{q} - \alpha_i| < \frac{1}{q^{1+\frac{1}{d}}}, \quad \text{for } i = 1, 2, \dots, d$$

admits infinitely many integer solutions.

This exponent is optimal as shown in [108], see also [39, 120]. In particular, we cannot hope to get a quality better than $0(q_n^{-\frac{3}{2}})$ when α_1, α_2 belong to a real cubic number field with $1, \alpha_1, \alpha_2$ linearly independent over $\mathbb{Q}$.

The proof of Dirichlet's theorem provides the existence of "good" approximations. We thus can make an exhaustive search but this is not an efficient algorithmic method. We do not know one algorithm giving the best quality in general. Algorithms based on lattice reduction theory will be discussed in Section 6: they combine efficiency and quality.

We focus in the present paper on simultaneous approximations but similar results can be discussed by involving in a dual way minimization of linear forms, and more generally of several linear forms (in particular for the algorithms of Section 6).

4.2 Noncanonicity of higherdimensional continued fractions algorithms

The aim of this section is to present several facts sustaining the claim that there is no canonical multidimensional continued fraction algorithm.

Regular continued fractions rely on Euclid's algorithm: starting with two numbers, one subtracts the smallest from the largest. If we start with at least three numbers, it is not clear to decide which operation has to be performed on these numbers in order to get something analogous to Euclid's algorithm, hence the diversity and multiplicity of existing generalizations. See Section 5 for an illustration.

Moreover, most of the one-dimensional continued fraction algorithms are closely related to the regular one. See for instance [85] which relies on the method of singularization; this method can be used to understand the relations between several one-dimensional continued fractions algorithms. This is mainly due to the algebraic structure of $SL(2,\mathbb{N})$. For $d \geq 2$, let $SL(d,\mathbb{N})$ denote the set of matrices of determinant 1 with nonnegative integer coefficients. This set endowed with the multiplication is a monoid, whose identity element is the identity matrix I_d. Recall that if p_n/q_n stands for the nth convergent of a given real number α in its regular continued fraction expansion, the so-called *unimodularity property* holds, namely

$$\det \begin{bmatrix} p_{n+1} & q_{n+1} \\ p_n & q_n \end{bmatrix} = (-1)^n \tag{2}$$

and the beginning of the continued fraction expansion of α can be recovered from the unique decomposition of $(-1)^n \left[\begin{smallmatrix} p_{n+1} & q_{n+1} \\ p_n & q_n \end{smallmatrix}\right]$ in the free monoid $SL(2,\mathbb{N})$. Indeed, the algebraic structure of $SL(2,\mathbb{N})$ is particularly simple: $SL(2,\mathbb{N})$ is a free and finitely generated monoid; it admits as generators $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$; any matrix in $SL(2,\mathbb{N})$ admits a unique decomposition in terms of these two matrices. This decomposition is a matricial translation of Euclid's algorithm and it corresponds to the continued fraction expansion.

For $d = 3$, the situation is completely different. First, $SL(3,\mathbb{N})$ is not free anymore, as shown by the following even permutation matrices whose third power is equal to the identity: $\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]$. But the main difference comes from the fact that $SL(3,\mathbb{N})$ is not finitely generated. Consider indeed the family of matrices

$$M_n := \begin{bmatrix} 1 & 0 & n \\ 1 & n-1 & 0 \\ 1 & 1 & n-1 \end{bmatrix}.$$

According to Chap. 12 in [111] these matrices are undecomposable for $n \geq 3$, i.e., they are not equal to an even permutation matrix, and, for any pair of matrices A, B in $SL(3, \mathbb{N})$ such that $M_n = AB$, A or B is an even permutation matrix. Note that even permutation matrices are exactly the matrices that admit an inverse in $SL(3, \mathbb{N})$.

Lastly, a rational number p/q is said to be a *best approximation* of the real number α if every p'/q' with $1 \leq q' \leq q$, $p/q \neq p'/q'$ satisfies

$$|q\alpha - p| < |q'\alpha - p'|.$$

Convergents in the continued fraction expansion of α and best approximations are known to coincide [39, 83]. Nevertheless, this notion is not so satisfying in higher dimension for defining continued fractions since first it depends on the norm [91], and second, we lose the unimodularity property (2). For more details, see [91, 92]. More precisely, let α be an irrational number, $|| \, ||$ be a given norm in $\mathbb{R}^d$ and $||| \, |||$ stand for the distance to the nearest integer. The sequence of best approximations of α with respect to the norm $|| \, ||$ is defined as the increasing sequence of nonnegative integers $(q_n)_{n\in\mathbb{N}}$ that satisfies

$$|||q_n(\alpha_1, \cdots, \alpha_d)||| < |||q(\alpha_1, \cdots, \alpha_d)|||$$

for any q with $1 \leq q < q_n$. The existence of an infinite sequence of best approximations can be derived in a classic way from Dirichlet's theorem or from Minkowski's first theorem (see e.g. [34]). Best approximations are shown to fail to be unimodular in [92] from which the following is quoted: "The absence of an exact higher-dimensional analogue for the continued fraction algorithm is reflected in the failure of property (i) in all higher dimensions. [...] higher-dimensional analogues of the continued fraction algorithm must include other approximations than just the best simultaneous approximations with respect to a fixed norm $|| \, ||$." Property (i) refers to the unimodularity property (2). More precisely, consider the square matrix M_n of size $d+1$ whose rows are given by successive best approximations vectors, i.e.,

$$M_n = \begin{bmatrix} \mathbf{v}_n \\ \mathbf{v}_{n+1} \\ \vdots \\ \mathbf{v}_{n+d} \end{bmatrix}$$

where $\mathbf{v}_n = (p_1^{(n)}, \cdots, p_d^{(n)}, q_n)$ is the best approximation integer vector with last entry q_n that provides $|||q_n(\alpha_1, \cdots, \alpha_d)|||$. Let D_n stand for the determinant of this matrix. It is proved in [92] that for any norm in dimension $d \geq 2$, there exists $\alpha \in \mathbb{R}^d$, with $\dim_{\mathbb{Q}}[1, \alpha_1, \cdots, \alpha_d] = d + 1$, such that for any positive integer N, there exists n for which $D_n = D_{n+1} = \cdots = D_{n+N} = 0$. Arbitrarily large determinants can even occur in dimension $d = 2$ with the sup norm.

All these reasons indicate that there is no canonical generalization of continued fractions to higher dimensions. Several approaches are possible that we describe below, by focusing on unimodular algorithms, that is, on algorithms satisfying the unimodularity property (2). We will not consider here multidimensional continued fractions based on Klein polyhedra and sails such as developed in [9, 88, 84, 10]. For more on the subject and its history, see e.g. [89, 90] and the references in [81].

4.3 Unimodular algorithms

We follow here the formalism introduced in [96] which allows enough generality to cover most classical unimodular types of algorithms, such as discussed in [137, 34, 127]. These algorithms belong mainly to two classes of algorithms, the Markovian ones ("without memory"), discussed in Section 5, and the algorithms based on reduction algorithms, that will be reviewed in Section 6.

According to [96], a d-dimensional unimodular continued fraction algorithm associated with $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ produces a sequence of matrices $(A^{(n)})_{n\in\mathbb{N}}$ with values in $GL(d+1, \mathbb{Z})$ as well as an initial matrix $P^{(0)}$ in $GL(d+1, \mathbb{Z})$. Matrices $A^{(n)}$ play the role of partial quotients. We then consider the matrices of $GL(d+1, \mathbb{Z})$ defined for all n as

$$P^{(n)} := A^{(n)} \cdots A^{(1)} P^{(0)} = \begin{bmatrix} p_{1,1}^{(n)} & \cdots p_{1,d}^{(n)} & q_1^{(n)} \\ & \cdots & \\ p_{d+1,1}^{(n)} & \cdots p_{d+1,d}^{(n)} & q_{d+1}^{(n)} \end{bmatrix}.$$

Matrices $P^{(n)}$ play the role of convergent matrices.

Usually an algorithm producing the sequence of matrices $(A^{(n)})_{n\in\mathbb{N}}$ can be defined in dynamical terms. Let $X \subset \mathbb{R}^d$ and $X_t \subset X$. Elements of X_t are called *terminal.* A d-dimensional *continued fraction map* over X is a map $T : X \to X$ such that $T(X_t) \subset X_t$ and, for any $\alpha \in X$, there is $A(\alpha) \in GL(d, \mathbb{Z})$ satisfying:

$$\alpha = A(\alpha)\, T(\alpha).$$

The associated *continued fraction algorithm* consists in iteratively applying the map T on a vector $\alpha \in X$. This yields the following sequence of matrices, called the *continued fraction expansion* of α: $(A(T^n(\alpha)))_{n\geq 1}$. This expansion is said to be *finite* if there is n such that $T^n(\alpha) \in X_t$, infinite otherwise.

An algorithm is said to be *additive* if all the matrices belong to a finite set. An algorithm is said to be *positive* if all the matrices are nonnegative. An algorithm is said to be *Markovian* if the map T is piecewise continuous; usually it is piecewisely an homography. Algorithms described in Section 5 are examples of Markovian algorithms, contrarily to the algorithms of Section 6. Markovian algorithms are also said to be "without memory". Indeed, the $(n+1)$th step of the algorithm only depends on the map T and on the value $T^n(\alpha)$.

The rows of the convergent matrices are meant to provide simultaneous approximations, i.e., one considers

$$(\frac{p_{j,1}^{(n)}}{q_j^{(n)}}, \cdots, \frac{p_{j,d}^{(n)}}{q_j^{(n)}}).$$

The integers $q_j^{(n)}$ play the role of nth convergents, and the vector $(p_{j,1}^{(n)}, \cdots, p_{j,d}^{(n)}, q_j^{(n)})$ is called an nth convergent vector.

In more geometric terms, this can be expressed as follows. One wants to approximate a given vectorial line directed by the nonzero vector $\ell = (\ell_1, \cdots, \ell_{d+1})$ in $\mathbb{R}^{d+1}$ by a sequence $(\mathbf{b}^{(n)})_{n\in\mathbb{N}}$ of integer lattice bases of $\mathbb{Z}^{d+1}$. The lattice bases generate cones (that are usually nested) that "tend" toward the line directed by ℓ. One recovers simultaneous rational approximations by setting $\alpha_i = \ell_i/\ell_{d+1}$ for $i = 1, \cdots, d$. Usually, one way to go from $\alpha \in \mathbb{R}^d$ to $\ell \in \mathbb{R}^{d+1}$ consists in setting $\ell_{d+1} = 1$, and working with entries $\ell_i \in [0,1]^d$ for $1 \leq i \leq d$, or in working with the simplex $\sum_{i=1}^{d+1} \ell_i = 1$, with $\ell_i \geq 0$ for all i. In [34] the algorithms are designed in such a way that for every n, ℓ belongs to the positive cone generated by the vectors $\mathbf{b}_i^{(n)}$, $i = 1, \ldots, d+1$, i.e., in

$$\{ \sum_{1\leq i\leq d+1} \lambda_i \mathbf{b}_i^{(n)} \mid \lambda_i \geq 0,\ \forall i = 1, \cdots, d+1\}.$$

We then expect that this sequence of bases of lattices that is produced converges toward the line generated by ℓ. Not all the existing algorithms enter this framework. See the discussion in [58] for instance. Furthermore, [34] adds extra assumptions on the allowed operations on the bases at each step n; they are of elementary types (they correspond to integer transvections): for every n there exist $i \neq j$ (with i, j depending on n) and $c_n \in \mathbb{N}$ such that

$$\mathbf{b}_i^{(n+1)} = \mathbf{b}_i^{(n)} + c_n \mathbf{b}_j^{(n)},\ \mathbf{b}_k^{(n+1)} = \mathbf{b}_k^{(n)} \text{ for } k \neq i. \tag{3}$$

This restriction is not a severe one and most of the algorithms discussed in the present survey enter this framework, by allowing also permutation rules between the vectors. Algorithms for which the choice of the coefficients i, j and c_i only depend on the cofactors of ℓ with respect to $\mathbf{b}^{(n)}$, i.e., the integers $a_i^{(n)}$ such that

$$\ell = a_1^{(n)} \mathbf{b}_1^{(n)} + \cdots + a_{d+1}^{(n)} \mathbf{b}_{d+1}^{(n)},$$

are called *vectorial* in [34]. They are Markovian algorithms.

As underlined in [34], "All continued fraction algorithms which have been proposed since the beginning (Jacobi, 1868), and up to about 1970 belong to this class. [...]. A great disadvantage is that the expansions of vectorial algorithms often converge too slowly or not at all." Nevertheless they are easier to study from an ergodic viewpoint for instance. In particular, the existence of an ergodic absolutely continuous invariant measure allows to understand the way the digits are distributed. They enter the category of fibred systems developed in [125].

Additive vs. multiplicative steps There are two types of steps that can be performed, small ones or maximal ones. Let us take as an illustration Euclid's algorithm: starting from two nonnegative numbers a and b, one subtracts the smallest one from the largest one. If one performs only one subtraction at each step, one obtains the so-called *additive* version of Euclid's algorithm. If one performs in one step as much subtractions as possible (i.e., if $0 \leq b \leq a$, a is replaced by $a - [a/b]b$), one gets a *multiplicative* algorithm. In dynamical terms, one considers either the Gauss map ($x \mapsto \{1/x\}$) or the Farey map ($x \mapsto \frac{x}{1-x}$ for $0 \leq x \leq 1/2$, and $x \mapsto \frac{1-x}{x}$ for $1/2 \leq x \leq 1$). Note that the Gauss map is known to have a finite ergodic invariant measure, which is not the case of the Farey map. The terminology division vs. subtractive algorithm is also used: see e.g. [34] where an algorithm is said to be subtractive if $c_n = 1$ in (3), and additive if c_n is chosen as the maximal possible number allowing the line to stay within the positive cone generated by the convergent vectors $\mathbf{b}_i^{(n)}$, for $i = 1, \cdots, d$. Additive and multiplicative versions for a same type of rule can lead to very distinct behaviors. See for instance the example of Selmer's algorithm quoted in [34] which is shown not to be able to be accelerated in a multiplicative form: the rule is to subtract the smallest nonzero entry from the largest one; in case of equality, we take the entry with smallest index; one checks that any triple of coprime positive integers leads to $(1, 1, 1)$; start from $(5, 4, 2)$; one checks that no multiplicative rule allows one to reach $(1, 1, 1)$ from $(5, 4, 2)$. This comes from the fact that the group of matrices generated by positive transvections and permutations is not commutative.

Convergence There are mainly two types of convergence. The convergence is said to be *weak* if the convergent vectors tend in angle toward the approximated line directed by ℓ, and *strong* if they tend in distance toward this line. One can also consider uniform convergence if one works with all the convergent vectors; otherwise, one only considers the convergent vector realizing the smallest distance. For more details, see [34, 127, 87]. In particular, as soon as one has weak convergence, a continued fraction algorithm allows one to approximate real vectors by sequences of rational vectors. Topological convergence corresponds to the fact that the natural partition of the underlying dynamical system is a generator. For a comparison of these notions of convergence for multidimensional continued fraction algorithms, see [87] with special focus on the notion of topological convergence. Note that for a unimodular multidimensional continued fraction algorithm, if the coordinates of ℓ are rationally independent, then the convergence cannot be uniformly strong. See for instance [127], Lemma 30 in Chap. 14. This allows to disprove for instance the strong convergence of Jacobi-Perron algorithm.

What is expected? We are given $(\alpha_1, \cdots, \alpha_d) \in \mathbb{R}_+^d$ which produces a sequence of bases $(\mathbf{b}^{(n)})_{n \in \mathbb{N}}$ of $\mathbb{Z}^{d+1}$and thus sequences of convergent vectors that yield simultaneous rational approximations. From an arithmetic viewpoint, a multidimensional continued fraction algorithm is expected to detect linear relations between $1, \alpha_1, \cdots, \alpha_d$, to give algebraic characterizations of periodic expansions, to have "good" properties of convergence, and to provide "good" rational approximations. Furthermore, one could hope to determine thanks to such an algorithm fundamental units (e.g., of a cubic number field), and to solve Diophantine equations. From a dynamical viewpoint, we also would like to have reasonable ergodic properties (concerning ergodic invariant measures, realizations of the natural extension, entropy, Lyapounov exponents, etc.), to be able to control the almost everywhere behavior like the a.e. speed of convergence, the distribution of the digits, to understand the "depth" and the number of executions of the algorithm if the parameters are rational, and to be able to perform a dynamical analysis according to the scheme discussed in [141].

Computational complexity In [94] various computational complexity results concerning simultaneous Diophantine approximation problems are considered. When the dimension d is fixed, algorithms are given which find a good approximation q with $1 \leq q \leq N$ for a given N with respect to a specified accuracy, or which find all best approximations in $[1, \cdots, N]$ in polynomial-time. Note that the following problem of decision is proved to be NP-hard: we are given a vector $\alpha \in \mathbb{Q}^d$, a positive integer N and an accuracy s_1/s_2; is there an integer Q with $1 \leq Q \leq N$ such that $|||Q\alpha||| \leq s_1/s_2$? (the distance to the nearest integer is expressed here with respect to the sup norm). Furthermore, Lagarias suggests in [94] that "the problem of locating best (sup norm) simultaneous approximations is harder than that of locating good simultaneous approximations". See also in the same flavor [64] concerning the problem of finding integer relations, and [18].

5 Markovian continued fraction algorithms: a zoo of algorithms

5.1 General description

We focus here on unimodular Markovian multidimensional continued fraction algorithms, according to the terminology introduced in Section 4.3. We recall the most classical ones which have lead to well-studied multi-dimensional continued fraction algorithms such as discussed in [34, 127]. In order to stress the simple rules that govern them, we express them in dimension $d + 1 = 3$. We thus start with parameters $(u_1, u_2, u_3) \in \mathbb{R}^3_+$. We have to decide which number has to be subtracted, and with respect to which number it has to be done. Usually numbers u_1, u_2, u_3 are sorted in increasing (or decreasing) order. We stress the subtraction rule but it is usually preceded and followed by a sorting operation.

- *Jacobi-Perron:* let $0 < u_1, u_2 \leq u_3$; one subtracts the first entry as often as we can from the other two ones

$$(u_1, u_2, u_3) \mapsto (u_2 - [\frac{u_2}{u_1}]u_1, u_3 - [\frac{u_3}{u_1}]u_1, u_1);$$

- *Brun:* we subtract the second largest entry from the largest one; for instance, if $0 \leq u_1 < u_2 < u_3$,
$$(u_1, u_2, u_3) \mapsto (u_1, u_2, u_3 - u_2);$$
- *Poincaré:* we subtract the second largest entry from the largest one, and the smallest entry from the second largest one; for instance, if $0 \leq u_1 < u_2 < u_3$
$$(u_1, u_2, u_3) \mapsto (u_1, u_2 - u_1, u_3 - u_2);$$
- *Selmer:* we subtract the smallest positive entry from the largest one; for instance, if $0 < u_1 < u_2 < u_2$
$$(u_1, u_2, u_3) \mapsto (u_1, u_2, u_3 - u_1);$$
- *Fully subtractive:* we subtract the smallest positive entry from all the largest ones; for instance, if $0 < u_1 \leq u_2 \leq u_3$
$$(u_1, u_2, u_3) \mapsto (u_1, u_2 - u_1, u_3 - u_1).$$

5.2 Some comparison elements

The convergence and the ergodic properties of these algorithms can vary even within these simple rules which *a priori* look similar. Apply for instance Poincaré algorithm T_P to $(1/\varphi^2, 1/\varphi, 100)$ where $1/\varphi^2 + 1/\varphi = 1$ and $1/\varphi > 0$. The kth iteration of Poincaré algorithm produces

$$T_P^k(1/\varphi^2, 1/\varphi, 100) = (1/\varphi^{k+2}, 1/\varphi^{k+1}, 100 - \sum_{i=0}^{k} 1/\varphi^{i+1}).$$

The value of $100 - \sum_{i \geq 1} 1/\varphi^i$ is always larger than the values taken by the first two ccordinates of $T_P^k(1/\varphi^2, 1/\varphi, 100)$ for any k. Hence, there is no "mixing" between the three coordinates when applying Poincaré on these initial values, and

$$\lim_{k \to \infty} T_P^k(1/\varphi^2, 1/\varphi, 100) \neq (0, 0, 0).$$

For more details, see [105] for Poincaré algorithm. Similar intriguing issues occur in the study of the fully subtractive algorithm; they have been considered in [100, 86].

For a description of the ergodic properties of these algorithms, see [127]. See [17] which shows in a very efficient way how to determine the invariant measure thanks to the natural extension. For a thorough study of the Lyapounov exponents of the Jacobi-Perron algorithm (which also applies to Brun algorithm), see [35, 36]. In particular, the a.e. exponential (strong) convergence of Brun [57, 99, 122] and Jacobi-Perron algorithm [36] (see also [95, 126]) holds: there exists $\delta > 0$ s.t. for a.e. (α, β), there exists $n_0 = n_0(\alpha, \beta)$ s.t. for all $n \geq n_0$
$$|\alpha - p_n/q_n| < \frac{1}{q_n^{1+\delta}}, \ |\beta - r_n/q_n| < \frac{1}{q_n^{1+\delta}},$$

where p_n, q_n, r_n are given by Brun (resp. Jacobi-Perron) algorithm.

Jacobi-Perron algorithm vs. Ostrowski algorithm The linear form of Jacobi-Perron algorithm is defined on $X = \{(u_1, u_2, u_3) \in \mathbb{R}^3 | 0 < u_1, u_2 \leq u_3\}$ by

$$(u_1, u_2, u_3) \mapsto (u_2 - \lfloor u_2/u_1 \rfloor u_1, u_3 - \lfloor u_3/u_1 \rfloor u_1, u_1).$$

If we set

$$\alpha_1 := u_1/u_3, \ \alpha_2 := u_2/u_3,$$

we recover its projective version defined on $(0,1) \times (0,1)$ as

$$(\alpha_1, \alpha_2) \mapsto \left(\frac{\alpha_2}{\alpha_1} - \left\lfloor \frac{\alpha_2}{\alpha_1} \right\rfloor, \frac{1}{\alpha_1} - \left\lfloor \frac{1}{\alpha_1} \right\rfloor \right) = (\{\alpha_2/\alpha_1\}, \{1/\alpha_1\}).$$

Let us compare it with the Ostrowski's mapping

$$(\alpha_1, \alpha_2) \mapsto (\{1/\alpha_1\}, \{\alpha_2/\alpha_1\}).$$

Ostrowski's mapping can not be considered as a multidimensional continued fraction algorithm, as e.g. illustrated by its ergodic study (see [72, 70, 73]). This algorithm is indeed a skew product of the Gauss map: the first coordinate is exactly the Gauss map and expands α_1 in continued fraction, whereas its second coordinate produces the digits of the Ostrowski expansion of α_2 with respect to the continued fraction expansion of α_1.

Jacobi-Perron algorithm vs. Brun algorithm Note that Brun algorithm is also called modified Jacobi-Perron algorithm: the modified Jacobi-Perron algorithm introduced by E. V. Podsypanin in [109] is a two-point extension of Brun algorithm. Both algorithms (Brun and Jacobi-Perron) are known to have an invariant ergodic probability measure equivalent to the Lebesgue measure (see for instance [124] and [127]). However, this measure is not known explicitly for Jacobi-Perron (the density of the measure is shown to be a piecewise analytical function in [35]), whereas it is known explicitly for Brun [17, 57]. Let us stress the difference between Brun and Jacobi-Perron's rule such as defined in Section 5.1. Brun algorithm is a space-ordering algorithm according to the terminology introduced in [62]. (Note that it is called ordered Jacobi-Perron in [61].) Furthermore, each step of Brun algorithm produces only one partial quotient. This helps in computing the natural extension and the invariant measure of Brun algorithm (see e.g. [17]). Contrary to Brun algorithm, the role played by u_1 and u_2 is not determined by a comparison between both parameters in Jacobi-Perron case; this might explain the fact that an explicit expression of the natural extension of this algorithm is still not known. Nevertheless, the framework of S-expansions and the so-called techniques of Insertion and Singularization (see [69]) allow one to relate both algorithms as shown in [121]; see also [123].

6 Lattice reduction algorithms and effective simultaneous rational approximations

Lattice reduction methods induce a second particularly fruitful way of exhibiting good simultaneous approximations or small values for linear forms. Algorithms based on lattice reduction theory are based on the following idea: lattice reduction algorithms do not produce *a priori* the smallest vector of a lattice but a reasonably small vector, that is, a vector that is small enough for guarantying Diophantine approximation properties that can be compared with Dirichlet's quality up to an approximation factor exponential in the dimension. We thus can consider these algorithms as providing effective versions of Dirichlet's theorem, yielding a satisfying compromise between efficient computation and sharpness of the obtained bounds, that is, between algorithmic issues and Diophantine quality.

Lattice reduction is based on the following elementary basis transformations: they can be described in terms of size reduction (the vector $\mathbf{b}_i$ of the basis $(\mathbf{b}_1, \ldots, \mathbf{b}_{d+1})$ is replaced by $\mathbf{b}_i - \lambda \mathbf{b}_j$ with $1 \leq j < i$), and of exchange steps, also called swaps (one exchanges $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$). These operations are decided with respect to the Gram-Schmitdt orthogonalization of the basis $\mathbf{b}$. See [64] for an interesting discussion on the connections between the approximation algorithms given in [53] (see also [54]) and [97]. See also [54]. For more on the way lattice reduction provides best approximations of a real number, see p. 226 and p. 267 of [104], and for a survey on the overall strategy for getting constructive type results in Diophantine approximation based on LLL, see p. 222 of [104]. Nevertheless, note that even in dimension 2, when using Gauss algorithm whose efficiency has been largely proved, one has "little control on the convergent which is returned; in particular, this is *not* the largest convergent with denominator less than $2\sqrt{C/3}$", how quoted in [104] (p.226 Example 1); the bound $2\sqrt{C/3}$ comes from Theorem 7, Chap. 6 of [104].

Let us sketch the basic strategy underlying the use of lattice reduction in this framework. We follow here the seminal paper [97]. This yields a very fruitful compromise between the quality of approximation (a good approximation is deduced from a small vector) and the efficiency (this small vector is obtained in polynomial time). We are given $(\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ that one wants to approximate. One works here again in a $d+1$-dimensional space, but we will introduce a one-parameter family of lattices $(\Lambda_t)_{t>0}$ with parameter t tending to 0. Let t be a positive real number, and let Λ_t be the lattice generated by the columns of the matrix

$$M_t := \begin{bmatrix} 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -\alpha_d \\ 0 & \cdots & \cdots & 0 & t \end{bmatrix}.$$

Note that $\det(M_t) = t$, hence, the lattice Λ_t changes at each step of the algorithm. Let us stress the fact that this strategy differs from the one discussed in Section 5 where one worked with bases of the fixed lattice $\mathbb{Z}^{d+1}$.

We will take t small, the parameter Q of Dirichlet's theorem being connected to t as follows: $Q = t^{-\frac{1}{d+1}}$.

One of the main features of the LLL algorithm is that it produces in polynomial time a nonzero vector $\mathbf{b} = (b_1, \dots b_{d+1})$ of the lattice Λ_t such that

$$||\mathbf{b}||_2 \leq 2^{d/4} \det(M_t)^{1/(d+1)} = 2^{d/4} t^{1/(d+1)}. \tag{4}$$

Note that the geometry of numbers, and more precisely Minkowski's first theorem, guaranties the existence of a "small" nonzero vector $\mathbf{x} \in \Lambda_t$, i.e., such that

$$||\mathbf{x}|| \leq \sqrt{(d+1)(d+5)/4}\ (\mathrm{vol}(\Lambda_t))^{1/(d+1)} = \sqrt{(d+1)(d+5)/4}\ t^{1/(d+1)}. \tag{5}$$

Let $(\mathbf{e}_i)_{i=1,\dots,d+1}$ stand for the canonical basis of $\mathbb{Z}^{d+1}$. There exist integers $p_1, \dots, p_d, q$ such that

$$\begin{aligned} \mathbf{b} &= p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + \cdots + p_d\mathbf{e}_d + q(-\alpha_1\mathbf{e}_1 - \cdots - \alpha_d\mathbf{e}_d + t\mathbf{e}_{d+1}) \\ &= (p_1 - q\alpha_1)\mathbf{e}_1 + \cdots + (p_d - q\alpha_d)\mathbf{e}_d + qt\mathbf{e}_{d+1}. \end{aligned}$$

One deduces from (4) that

$$\forall i = 1, \dots, d, \quad |p_i - \alpha_i q| \leq 2^{d/4} t^{1/(d+1)}$$

and

$$qt \leq 2^{d/4} t^{1/(d+1)}, \text{ i.e., } t^{\frac{1}{d+1}} \leq \frac{2^{1/4}}{q^{1/d}}.$$

We deduce that for all i

$$|p_i - \alpha_i q| \leq \frac{2^{(d+1)/4}}{q^{1/d}},$$

with

$$|q| \leq 2^{d/4} t^{-d/(d+1)} = 2^{d/4} Q^d.$$

The quality of approximation is the quality that is expected (with respect to Dirichlet's theorem) up to a multiplicative factor $2^{(d+1)/4}$ which depends exponentially of the dimension. We could have used (5) which would have given a different multiplicative factor but the same quality ($q^{1/d}$). Nevertheless, the interest of a lattice reduction algorithm such as LLL is that the small vector that is used is found in polynomial time.

The question now is to be able to devise a continued fraction algorithm from this. One has *a priori* to recompute everything from the beginning when one changes t. For a dynamical version, see [54, 53, 93, 94, 78, 79, 80, 33]. Let us quote in particular [96] based on the Minkowski lattice reduction and on the notion of lexicographically Minkowski reduced basis. This is not effective and produces best approximations (which are known to be NP-hard to locate in an interval [94]). This study is extended in [59].

7 Back to substitutions

Let us come back to the connections between the multidimensional continued fraction algorithms discussed in the previous sections and substitutions.

7.1 Cubic number fields

The case of cubic numbers is a natural situation for this interaction to play a significant role. Let us first start with results involving cubic numbers. It is shown in [4] that the set of limit values for

$$\{q^{\frac{1}{2}}|||q\alpha_1|||, q^{\frac{1}{2}}|||q\alpha_2||| \mid q > 0\}$$

is a discrete set of curves (hyperbolic curves or ellipses), and in particular a union of homothetic ellipses centered at the origin, whenever (α_1, α_2) form with 1 the basis of a real cubic number field with $\mathbb{Q}(\alpha_2)$ having a complex embedding.

The particular case of (α_1, α_2) having a purely periodic expansion of length 1 with respect to Brun algorithm, i.e., $(\alpha_1, \alpha_2) = (\alpha, \alpha^2)$ where α is the real root of $X^3 + kX - 1$ for $k > 0$, is investigated in [71]. The nearest ellipse is shown to be given by the convergents produced by Brun algorithm. This does not hold anymore for periodic points with a longer period such as shown in [76]. These results are obtained by introducing substitutions associated with Brun algorithm, according to [74]. This result is connected with the following one quoted in [91] and obtained as a direct consequence of [4, 5]: if $1, \alpha_1, \alpha_2$ is a $\mathbb{Q}$-basis of a non-totally real cubic field, the best simultaneous approximations of (α_1, α_2) with respect to a given norm are a subset of a finite number of third-order linear recurrences with constant coefficients whose polynomial is given by the fundamental unit of $\mathbb{Q}(\alpha_1, \alpha_2)$. This result is made more precise by exhibiting a suitable Euclidean norm in [41]. See also in the same flavor [68] which uses fractal geometry, numeration systems and Rauzy fractals. Lastly, note that, when $d = 2$, the characteristic polynomial of the matrix associated with a periodic expansion under Jacobi-Perron algorithm is irreducible and its dominant eigenvalue is a Pisot number [45], see also [107]. Concerning Brun algorithm, see [128].

7.2 Discrete geometry and generalized substitutions

We have discussed in Section 3.1 the connections between Sturmian sequences, discrete lines, substitutions and Euclid's algorithm. Let us extend this discussion to a higher-dimensional framework. Recall that our motivation is to get suitable multidimensional continued fraction algorithms in discrete geometry for the study of discrete planes and lines, as well as algorithms producing Rauzy fractals associated with nonalgebraic parameters.

Consider a unimodular multidimensional continued fraction algorithm. The strategy we propose consists in giving a combinatorial interpretation of the matrices produced by such an algorithm by associating with them substitutions via their incidence matrix. Our main tool is a formalism which associates a generalized substitution of a geometric nature with a unimodular matrix, and which produces approximations of the Rauzy fractal.

We have seen in Section 2.2 that given a Pisot substitution σ, its Rauzy fractal is defined as the closure of the image by the projection π_c (on the contracting plane of the incidence matrix M_σ along its expanding direction) of the abelianized images of the prefixes of an infinite word u that satisfies $\sigma(u) = u$. Rauzy fractals have also been proved to be attractors of some graph-directed iterated function system (see e.g. [15, 133, 75]). There exists a very useful formalism introduced in [15] that provides an algebraic way to describe this equation with respect to the substitution σ, namely the notion of *generalized substitution*. Generalized substitutions can be considered as multidimensional substitutions of non-constant length acting on multidimensional words (see e.g. [15]). This formalism due to Arnoux and Ito [15] was inspired by the geometrical formalism of [74], whose aim was to provide explicit Markov partitions for hyperbolic automorphisms of the torus associated with particular morphisms of the free group. They have already proved their efficiency in the spectral study of Pisot substitutive dynamical systems [111] or else in discrete geometry (see [12, 13, 11, 27, 55, 56, 29]). With any usual unimodular substitution σ can be associated a generalized substitution $E_1^*(\sigma)$ (recall that a substitution is said to be unimodular if the determinant of its incidence matrix equals ± 1). The generalized substitution $E_1^*(\sigma)$ is defined as the dual map of a natural geometric realization of σ. It maps facets of unit cubes onto unions of facets of unit cubes.

One of the key properties of generalized substitutions is that they map standard arithmetic discrete planes onto standard arithmetic discrete planes according to [15, 55]. Arithmetic discrete planes are basic objects in discrete geometry. According to the formalism derived from [117], they are defined as follows: let $v \in \mathbb{R}^d$, $\mu, \omega \in \mathbb{R}$;

$$\mathcal{P}(v, \mu, \omega) = \{x \in \mathbb{Z}^d \mid 0 \leq \langle x, v\rangle + \mu < \omega\}.$$

The parameter μ is called the translation parameter, and ω is called the thickness. If $\omega = \max_i\{|v_i|\} = ||v||_\infty$, then $\mathcal{P}(v, \mu, \omega)$ is said naive. If $\omega = \sum_i |v_i| = ||v||_1$, then $\mathcal{P}(v, \mu, \omega)$ is said standard.

Let $\mathcal{U}$ be the union of the upper facets of the unit cube. It is proved in [15] that, by renormalizing by M_σ^n the projection π_c of the sets $E_1^*(\sigma)^n(\mathcal{U})$ and by taking the limit with respect to the Hausdorff metric, one recovers the Rauzy fractal associated with σ, i.e.,

$$\lim_{n\to+\infty} M_\sigma^n \pi_c(E_1^*(\sigma))^n(\mathcal{U}) = \mathcal{R}_\sigma.$$

See Figure 1 for an illustration.

Furthermore, one checks that the vertices of $\mathcal{U}$, as well as the vertices of its images by a generalized substitution $E_1^*(\sigma)$ belong to any standard arithmetic discrete plane with parameter $\mu = 0$. Generalized substitutions thus provide a generation method for arithmetic discrete planes with parameter $\mu = 0$ for some algebraic parameters v, as well

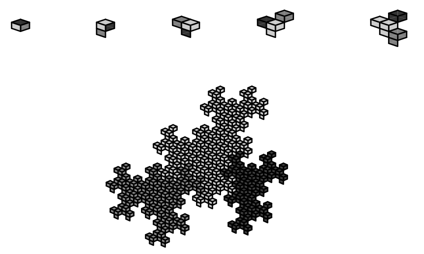

Figure 1: Iterations of a generalized substitution on the upper facets of the unit cube.

as a way to define Rauzy fractals with irrational parameters. Indeed, to generate non-algebraic discrete planes, one expands a given v with respect to a unimodular continued fraction algorithm such as Jacobi-Perron or Brun algorithm. We then can translate the expansion produced by Brun algorithm as a product of matrices in the formalism of generalized substitutions. A geometric version of Brun multidimensional continued fraction algorithm acting on discrete planes is given in [56, 29] in terms of generalized substitutions. If one wants to describe an arithmetic discrete plane with nonzero parameter μ, one then needs to involve a skew product of Brun algorithm in order to also expand μ: such a skew product will play the role of Ostrowski's skew product in the Sturmian case.

By using generalized substitutions associated with a given multidimensional continued fraction expansion, one thus can give an S-adic type of representation for discrete planes. For more details, see e.g. [12, 55, 56, 29]. This has applications to the study of local configurations, for the generation of discrete planes [31], and for the recognition of discrete planes: given a set of points in $\mathbb{Z}^3$, is it contained in an arithmetic discrete plane [56]? Lastly, let us quote the following problem concerning the connectedness of discrete planes when the thickness decreases. The question is to find the smallest thickness ω for which the plane $\mathcal{P}(v, \mu, \omega)$ is connected (either edge connected or vertex connected). The case of rational parameters has been solved in [77]. For the case of irrational parameters, see [44]. The method used in both papers relies on the use of the fully subtractive algorithm.

Let us conclude with the following open question: how to associate a Rauzy fractal with a nonperiodic Brun or a Jacobi-Perron expansion? If we know that the a.e. exponential convergence of Brun algorithm gives us convergence toward a Rauzy fractal, can we use a generalized Perron-Frobenius theorem to prove that its subtiles will be disjoint in measure? What about the tiling properties?

Acknowledgements

The author would like to thank P. Arnoux, A. Broise and the LAREDA project for many stimulating discussions, as well as the anonymous referee and T. Jolivet for their careful reading and their valuable comments.

References

[1] B. Adamczewski, *Répartition des suites $(n\alpha)_{n\in\mathbb{N}}$ et substitutions*, Acta Arith., 112 (2004), pp. 1–22.

[2] B. Adamczewski and J.-P. Allouche, *Reversals and palindromes in continued fractions*, Theoret. Comput. Sci., 380 (2007), pp. 220–237.

[3] B. Adamczewski, C. Frougny, A. Siegel, and W. Steiner, *Rational numbers with purely periodic beta-expansion*, Bull. London Math. Soc., 42 (2010), pp. 538–552.

[4] W. W. Adams, *Simultaneous asymptotic diophantine approximations to a basis of a real cubic number field*, J. Number Theory, 1 (1969), pp. 179–194.

[5] ——, *Simultaneous diophantine approximations and cubic irrationals*, Pacific J. Math., 30 (1969), pp. 1–14.

[6] S. Akiyama, *Pisot numbers and greedy algorithm*, in Number theory (Eger, 1996), de Gruyter, 1998, pp. 9–21.

[7] ——, *Self affine tilings and Pisot numeration systems*, in Number Theory and its Applications (Kyoto 1997), K. Győry and S. Kanemitsu, eds., Kluwer Acad. Publ., 1999, pp. 7–17.

[8] ——, *On the boundary of self affine tilings generated by Pisot numbers*, J. Math. Soc. Japan, 54 (2002), pp. 283–308.

[9] V. I. Arnold, *A-graded algebras and continued fractions*, Comm. Pure Appl. Math., 42 (1989), pp. 993–1000.

[10] ——, *Higher-dimensional continued fractions*, Regul. Chaotic Dyn., 3 (1998), pp. 10–17.

[11] P. Arnoux, V. Berthé, T. Fernique, and D. Jamet, *Functional stepped surfaces, flips and generalized substitutions*, Theoret. Comput. Sci., 380 (2007), pp. 251–267.

[12] P. Arnoux, V. Berthé, and S. Ito, *Discrete planes, $\mathbb{Z}^2$-actions, Jacobi-Perron algorithm and substitutions*, *Ann. Inst. Fourier*, 52 (2002), pp. 1001–1045.

[13] P. Arnoux, V. Berthé, and A. Siegel, *Two-dimensional iterated morphisms and discrete planes*, Theoret. Comput. Sci., 319 (2004), pp. 145–176.

[14] P. Arnoux and A. M. Fischer, *The scenery flow for geometric structures on the torus: the linear setting*, Chin. Ann. of Math., 22B (2001), pp. 1–44.

[15] P. Arnoux and S. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Belg. Math. Soc. Simon Stevin, 8 (2001), pp. 181–207.

[16] P. Arnoux, C. Mauduit, I. Shiokawa, and J.-I. Tamura, *Complexity of sequences defined by billiard in the cube*, Bull. Soc. Math. France, 122 (1994), pp. 1–12.

[17] P. Arnoux and A. Nogueira, *Mesures de Gauss pour des algorithmes de fractions continues multidimensionnelles*, Ann. Sci. École Norm. Sup., 26 (1993), pp. 645–664.

[18] L. Babai, B. Just, and F. Meyer auf der Heide, *On the limits of computations with the floor function*, Inform. and Comput., 78 (1988), pp. 99–107.

[19] G. Barat, V. Berthé, P. Liardet, and J. Thuswaldner, *Dynamical directions in numeration*, Ann. Inst. Fourier (Grenoble), 56 (2006), pp. 1987–2092.

[20] G. Barat and P. Liardet, *Dynamical systems originated in the Ostrowski alpha-expansion*, Ann. Univ. Sci. Budapest. Sect. Comput., 24 (2004), pp. 133–184.

[21] M. Barge and B. Diamond, *Coincidence for substitutions of Pisot type*, Bull. Soc. Math. France, 130 (2002), pp. 619–626.

[22] M. Barge and J. Kwapisz, *Geometric theory of unimodular Pisot substitutions*, Amer. J. Math., 128 (2006), pp. 1219–1282.

[23] Y. Baryshnikov, *Complexity of trajectories in rectangular billiards*, Commun. Math. Phys., 174 (1995), pp. 43–56.

[24] L. Bernstein, *The Jacobi-Perron algorithm—Its theory and application*, Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 207.

[25] J. Berstel and P. Séébold, *Sturmian words*, in Algebraic Combinatorics on Words, M. Lothaire, ed., vol. 90 of Encyclopedia of Mathematics and Its Applications, Cambridge University Press, 2002, pp. 45–110.

[26] V. Berthé, *Autour du système de numération d'Ostrowski*, Bull. Belg. Math. Soc. Simon Stevin, 8 (2001), pp. 209–239.

[27] V. Berthé, *Discrete geometry and symbolic dynamics.* Passare, Mikael (ed.), Complex analysis and digital geometry. Proceedings from the Kiselmanfest, Uppsala, Sweden, May 2006 on the occasion of Christer Kiselman's retirement. Uppsala: Univ. Uppsala. 81-110 (2009)., 2009.

[28] V. Berthé, S. Ferenczi, and L. Q. Zamboni, *Interactions between dynamics, arithmetics and combinatorics: the good, the bad, and the ugly*, in Algebraic and topological dynamics, vol. 385 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2005, pp. 333–364.

[29] V. Berthé and T. Fernique, *Brun expansions of stepped surfaces*, Discrete Mathematics, 311 (2011), pp. 521–543.

[30] V. Berthé, C. Holton, and L. Q. Zamboni, *Initial powers of Sturmian sequences*, Acta Arith., 122 (2006), pp. 315–347.

[31] V. Berthé, A. Lacasse, G. Paquin, and X. Provençal, *A study of Jacobi-Perron boundary words for the generation of discrete planes*, preprint, (2011).

[32] V. Berthé and A. Siegel, *Tilings associated with beta-numeration and substitutions*, Integers, 5 (2005), pp. A2, 46 pp. (electronic).

[33] W. Bosma and I. Smeets, *Finding simultaneous diophantine approximations with prescribed quality*, arxiv.org/pdf/1001.4455 (2010).

[34] A. J. Brentjes, *Multidimensional continued fraction algorithms*, Mathematisch Centrum, Amsterdam, 1981.

[35] A. Broise, *Fractions continues multidimensionnelles et lois stables*, Bull. Soc. Math. France, 124 (1996), pp. 97–139.

[36] A. Broise-Alamichel and Y. Guivarc'h, *Exposants caractéristiques de l'algorithme de Jacobi-Perron et de la transformation associée*, Ann. Inst. Fourier (Grenoble), 51 (2001), pp. 565–686.

[37] V. Canterini and A. Siegel, *Automate des préfixes-suffixes associé à une substitution primitive*, J. Théor. Nombres Bordeaux, 13 (2001), pp. 353–369.

[38] ——, *Geometric representation of substitutions of Pisot type*, Trans. Amer. Math. Soc., 353 (2001), pp. 5121–5144.

[39] J. W. S. Cassels, *An introduction to Diophantine approximation*, Hafner Publishing Co., New York, 1972. Facsimile reprint of the 1957 edition, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45.

[40] M. G. Castelli, F. Mignosi, and A. Restivo, *Fine and Wilf's theorem for three periods and a generalization of Sturmian words*, Theoret. Comput. Sci., 218 (1999), pp. 83–94. WORDS (Rouen, 1997).

[41] N. Chevallier, *Best simultaneous Diophantine approximations of some cubic algebraic numbers*, J. Théor. Nombres Bordeaux, 14 (2002), pp. 403–414.

[42] ——, *Coding of a translation of the two-dimensional torus*, Monatsh. Math., 157 (2009), pp. 101–130.

[43] E. M. Coven and G. A. Hedlund, *Sequences with minimal block growth*, Math. Systems Theory, 7 (1973), pp. 138–153.

[44] E. Domenjoud, D. Jamet, and J.-L. Toutant, *On the connecting thickness of arithmetical discrete planes*, in DGCI, 2009, pp. 362–372.

[45] E. Dubois, A. Farhane, and R. Paysant-Le Roux, *The Jacobi-Perron algorithm and Pisot numbers*, Acta Arithmetica, 111 (2004), pp. 269–275.

[46] J.-M. Dumont and A. Thomas, *Systèmes de numération et fonctions fractales relatifs aux substitutions*, Theoret. Comput. Sci., 65 (1989), pp. 153–169.

[47] ——, *Digital sum moments and substitutions*, Acta Arith., 64 (1993), pp. 205–225.

[48] Y. Dupain and V. T. Sós, *On the one-sided boundedness of discrepancy-function of the sequence* $\{n\alpha\}$, Acta Arith., 37 (1980), pp. 363–374.

[49] F. Durand, *Linearly recurrent subshifts have a finite number of non-periodic subshift factors*, Ergodic Theory Dynam. Systems, 20 (2000), pp. 1061–1078.

[50] ——, *Corrigendum and addendum to: "Linearly recurrent subshifts have a finite number of non-periodic subshift factors" [Ergodic Theory Dynam. Systems* **20** *(2000), no. 4, 1061–1078;*, Ergodic Theory Dynam. Systems, 23 (2003), pp. 663–669.

[51] S. Ferenczi, *Bounded remainder sets*, Acta Arith., 61 (1992), pp. 319–326.

[52] ——, *Rank and symbolic complexity*, Ergodic Theory Dynam. Systems, 16 (1996), pp. 663–682.

[53] H. R. P. Ferguson, *A noninductive* $\mathrm{GL}(n, \mathbf{Z})$ *algorithm that constructs integral linear relations for* n $\mathbf{Z}$*-linearly dependent real numbers*, J. Algorithms, 8 (1987), pp. 131–145.

[54] H. R. P. Ferguson and R. W. Forcade, *Generalization of the Euclidean algorithm for real numbers to all dimensions higher than two*, Bull. Amer. Math. Soc. (N.S.), 1 (1979), pp. 912–914.

[55] T. Fernique, *Multidimensional Sturmian sequences and generalized substitutions*, Int. J. Found. Comput. Sci., 17 (2006), pp. 575–600.

[56] ——, *Generation and recognition of digital planes using multi-dimensional continued fractions*, Pattern Recognition, 42 (2009), pp. 2229–2238.

[57] T. Fujita, S. Ito, M. Keane, and M. Ohtsuki, *On almost everywhere exponential convergence of the modified Jacobi-Perron algorithm: a corrected proof*, Ergodic Theory Dynam. Systems, 16 (1996), pp. 1345–1352.

[58] D. J. Grabiner, *Farey nets and multidimensional continued fractions*, Monatsh. Math., 114 (1992), pp. 35–61.

[59] D. J. Grabiner and J. C. Lagarias, *Cutting sequences for geodesic flow on the modular surface and continued fractions*, Monatsh. Math., 133 (2001), pp. 295–339.

[60] P. J. Grabner, P. Liardet, and R. F. Tichy, *Odometers and systems of numeration*, Acta Arith., 70 (1995), pp. 103–123.

[61] D. M. Hardcastle and K. Khanin, *On almost everywhere strong convergence of multi-dimensional continued fraction algorithms*, Ergodic Theory Dynam. Systems, 20 (2000), pp. 1711–1733.

[62] ——, *Continued fractions and the d-dimensional Gauss transformation*, Comm. Math. Phys., 215 (2001), pp. 487–515.

[63] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford Science Publications, 1979.

[64] J. Håstad, B. Just, J. C. Lagarias, and C.-P. Schnorr, *Polynomial time algorithms for finding integer relations among real numbers*, SIAM J. Comput., 18 (1989), pp. 859–881.

[65] M. Hollander and B. Solomyak, *Two-symbol Pisot substitutions have pure discrete spectrum*, Ergodic Theory Dynam. Systems, 23 (2003), pp. 533–540.

[66] B. Host, *Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable*, Ergodic Theory Dynam. Systems, 6 (1986), pp. 529–540.

[67] ——, *Représentation géométrique des substitutions sur 2 lettres.* Unpublished manuscript, 1992.

[68] P. Hubert and A. Messaoudi, *Best simultaneous Diophantine approximations of Pisot numbers and Rauzy fractals*, Acta Arith., 124 (2006), pp. 1–15.

[69] M. Iosifescu and C. Kraaikamp, *Metrical theory of continued fractions*, Dordrecht Kluwer Academic Publishers, 2002.

[70] S. Ito, *Some skew product transformations associated with continued fractions and their invariant measures*, Tokyo J. Math., 9 (1986), pp. 115–133.

[71] S. Ito, J. Fujii, H. Higashino, and S.-I. Yasutomi, *On simultaneous approximation to* (α, α^2) *with* $\alpha^3 + k\alpha - 1 = 0$, Journal Number Th., 99 (2003), pp. 255–283.

[72] S. Ito and H. Nakada, *On natural extensions of transformations related to Diophantine approximations*, in Number theory and combinatorics. Japan 1984 (Tokyo, Okayama and Kyoto, 1984), World Sci. Publishing, Singapore, 1985, pp. 185–207.

[73] ——, *Approximations of real numbers by the sequence* $\{n\alpha\}$ *and their metrical theory*, Acta Math. Hungar., 52 (1988), pp. 91–100.

[74] S. Ito and M. Ohtsuki, *Modified Jacobi-Perron algorithm and generating Markov partitions for special hyperbolic toral automorphisms*, Tokyo J. Math., 16 (1993), pp. 441–472.

[75] S. Ito and H. Rao, *Atomic surfaces, tilings and coincidences I. Irreducible case*, Israel J. Math., 153 (2006), pp. 129–156.

[76] S. ITO AND S.-I. YASUTOMI, *On simultaneous Diophantine approximation to periodic points related to modified Jacobi-Perron algorithm*, in Probability and number theory—Kanazawa 2005, vol. 49 of Adv. Stud. Pure Math., Math. Soc. Japan, Tokyo, 2007, pp. 171–184.

[77] D. JAMET AND J.-L. TOUTANT, *Minimal arithmetic thickness connecting discrete planes*, Discrete Applied Mathematics, 157 (2009), pp. 500–509.

[78] B. JUST, *Integer relations among algebraic numbers*, in Mathematical foundations of computer science 1989 (Porąbka-Kozubnik, 1989), vol. 379 of Lecture Notes in Comput. Sci., Springer, Berlin, 1989, pp. 314–320.

[79] ——, *Integer relations among algebraic numbers*, Math. Comp., 54 (1990), pp. 467–477.

[80] ——, *Generalizing the continued fraction algorithm to arbitrary dimensions*, SIAM J. Comput., 21 (1992), pp. 909–926.

[81] O. N. KARPENKOV, *Constructing multidimensional periodic continued fractions in the sense of Klein*, Math. Comp., 78 (2009), pp. 1687–1711.

[82] H. KESTEN, *On a conjecture of Erdős and Szüsz related to uniform distribution* mod 1, Acta Arith., 12 (1966/1967), pp. 193–212.

[83] A. Y. KHINTCHINE, *Continued fractions*, Translated by Peter Wynn, P. Noordhoff Ltd., Groningen, 1963.

[84] E. KORKINA, *La périodicité des fractions continues multidimensionnelles*, C. R. Acad. Sci. Paris Sér. I Math., 319 (1994), pp. 777–780.

[85] C. KRAAIKAMP, *A new class of continued fraction expansions*, Acta Arith., 57 (1991), pp. 1–39.

[86] C. KRAAIKAMP AND R. MEESTER, *Ergodic properties of a dynamical system arising from percolation theory*, Ergodic Theory Dynam. Systems, 15 (1995), pp. 653–661.

[87] ——, *Convergence of continued fraction type algorithms and generators*, Monatsh. Math., 125 (1998), pp. 1–14.

[88] G. LACHAUD, *Polyèdre d'Arnol′d et voile d'un cône simplicial: analogues du théorème de Lagrange*, C. R. Acad. Sci. Paris Sér. I Math., 317 (1993), pp. 711–716.

[89] ——, *Klein polygons and geometric diagrams*, in Number theory (Tiruchirapalli, 1996), vol. 210 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1998, pp. 365–372.

[90] ——, *Sails and Klein polyhedra*, in Number theory (Tiruchirapalli, 1996), vol. 210 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1998, pp. 373–385.

[91] J. C. Lagarias, *Best simultaneous Diophantine approximations. I. Growth rates of best approximation denominators*, Trans. Amer. Math. Soc., 272 (1982), pp. 545–554.

[92] ——, *Best simultaneous Diophantine approximations. II. Behavior of consecutive best approximations*, Pacific J. Math., 102 (1982), pp. 61–88.

[93] ——, *The computational complexity of simultaneous Diophantine approximation problems*, in 23rd annual symposium on foundations of computer science (Chicago, Ill., 1982), IEEE, New York, 1982, pp. 32–39.

[94] ——, *The computational complexity of simultaneous Diophantine approximation problems*, SIAM J. Comput., 14 (1985), pp. 196–209.

[95] ——, *The quality of the Diophantine approximations found by the Jacobi-Perron algorithm and related algorithms*, Monatsh. Math., 115 (1993), pp. 299–328.

[96] ——, *Geodesic multidimensional continued fractions*, Proc. London Math. Soc. (3), 69 (1994), pp. 464–488.

[97] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász, *Factoring polynomials with rational coefficients*, Math. Ann., 261 (1982), pp. 515–534.

[98] P. Liardet, *Regularities of distribution*, Compositio Mathematica, 61 (1987), pp. 267–293.

[99] R. Meester, *A simple proof of the exponential convergence of the modified Jacobi-Perron algorithm*, Ergodic Theory Dynam. Systems, 19 (1999), pp. 1077–1083.

[100] R. W. J. Meester and T. Nowicki, *Infinite clusters and critical values in two-dimensional circle percolation*, Israel J. Math., 68 (1989), pp. 63–81.

[101] A. Messaoudi, *Propriétés arithmétiques et dynamiques du fractal de Rauzy*, J. Théor. Nombres Bordeaux, 10 (1998), pp. 135–162.

[102] ——, *Frontière du fractal de Rauzy et système de numération complexe*, Acta Arith., 95 (2000), pp. 195–224.

[103] M. Morse and G. A. Hedlund, *Symbolic dynamics II. Sturmian trajectories*, Amer. J. Math., 62 (1940), pp. 1–42.

[104] P. Q. Nguyen and B. Vallée, *(ed.) The LLL algorithm. Survey and applications*, Information Security and Cryptography. Dordrecht: Springer, 2010.

[105] A. Nogueira, *The three-dimensional Poincaré continued fraction algorithm*, Israel J. Math., 90 (1995), pp. 373–401.

[106] A. Ostrowsky, *Bemerkungen zur Theorie der Diophantischen Approximationen I, II*, Abh. Math. Sem. Hamburg, 1 (1922), pp. 77–98 and 250–251.

[107] R. Paysant-Le Roux and E. Dubois, *Une application des nombres de Pisot l'algorithme de Jacobi-Perron*, Monatshefte für Mathematik, 98 (1984), pp. 145–155.

[108] O. Perron, *Über diophantische approximationen*, Math. Annalen, 83 (1921), pp. 77–84.

[109] E. V. Podsypanin, *A generalization of the continued fraction algorithm that is related to the Viggo Brun algorithm*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 67 (1977), pp. 184–194, 227. Studies in number theory (LOMI), 4.

[110] N. Priebe-Franck, *A primer on substitution tilings of euclidean space*, Expositiones Mathematicae, 26 (2008), pp. 295–326.

[111] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, vol. 1794 of Lecture Notes in Mathematics, Springer Verlag, 2002. Ed. by V. Berthé and S. Ferenczi and C. Mauduit and A. Siegel.

[112] M. Queffélec, *Substitution dynamical systems—spectral analysis*, vol. 1294 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, second ed., 2010.

[113] G. Rauzy, *Une généralisation du développement en fraction continue*, in Séminaire Delange-Pisot-Poitou, 1976/77, Théorie des nombres, Fasc. 1, Paris, 1977, pp. Exp. No. 15, 16.

[114] G. Rauzy, *Nombres algébriques et substitutions*, Bull. Soc. Math. France, 110 (1982), pp. 147–178.

[115] ——, *Ensembles à restes bornés*, in Seminar on number theory, 1983–1984 (Talence, 1983/1984), Univ. Bordeaux I, Talence, 1984, pp. Exp. No. 24, 12.

[116] ——, *Sequences defined by iterated morphisms*, in Sequences (Naples/Positano, 1988), Springer, New York, 1990, pp. 275–286.

[117] J.-P. Reveillès, *Géométrie discrète, calcul en nombres entiers et algorithmique*, Thèse de Doctorat, Université Louis Pasteur, Strasbourg, 1991.

[118] E. A. Robinson, Jr., *Symbolic dynamics and tilings of* $\mathbb{R}^d$, in Symbolic dynamics and its applications, vol. 60 of Proc. Sympos. Appl. Math., Amer. Math. Soc. Providence, RI, 2004, pp. 81–119.

[119] K. Schmidt, *Algebraic coding of expansive group automorphisms and two-sided beta-shifts*, Monatsh. Math., 129 (2000), pp. 37–61.

[120] W. S. Schmidt, *Diophantine Approximation*, vol. 785 of Lecture Notes in Mathematics, Springer Verlag, 1996.

[121] B. Schratzberger, *On the singularization of the two-dimensional Jacobi-Perron algorithm.*, Exp. Math., 16 (2007), pp. 441–454.

[122] B. R. Schratzberger, *The quality of approximation of brun?s algorithm in three dimensions*, Monatshefte fr Mathematik, 134 (2001), pp. 143–157.

[123] ——, *A conversion algorithm based on the technique of singularization.*, Theor. Comput. Sci., 391 (2008), pp. 138–149.

[124] F. Schweiger, *On the invariant measure for Jacobi-Perron algorithm*, Math. Pannon., 1 (1990), pp. 91–106.

[125] ——, *Ergodic theory of fibred systems and metric number theory*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1995.

[126] ——, *The exponent of convergence for the 2-dimensional Jacobi-Perron algorithm*, in Proceedings of the Conference on Analytic and Elementary Number Theory (Vienna), W. G. Nowak and J. Schoissengeier, eds., 1996, pp. 207–213.

[127] ——, *Multidimensional continued fractions*, Oxford Science Publications, Oxford University Press, Oxford, 2000.

[128] F. Schweiger, *A note on Lyapunov theory for Brun algorithm*, in Diophantine Approximation, H. P. Schlickewei, K. Schmidt, and R. F. Tichy, eds., vol. 16 of Developments in Mathematics, Springer Vienna, 2008, pp. 371–379.

[129] N. Sidorov, *Arithmetic dynamics*, in Topics in dynamics and ergodic theory, S. B. et al., ed., vol. 310 of Lond. Math. Soc. Lect. Note Ser., Cambridge University Press, 2003, pp. 145–189.

[130] A. Siegel, *Représentation des systèmes dynamiques substitutifs non unimodulaires*, Ergodic Theory Dynam. Systems, 23 (2003), pp. 1247–1273.

[131] ——, *Pure discrete spectrum dynamical system and periodic tiling associated with a substitution*, Ann. Inst. Fourier, 54 (2004), pp. 288–299.

[132] R. J. Simpson and R. Tijdeman, *Multi-dimensional versions of a theorem of Fine and Wilf and a formula of Sylvester*, Proc. Amer. Math. Soc., 131 (2003), pp. 1661–1671 (electronic).

[133] V. F. Sirvent and Y. Wang, *Self-affine tiling via substitution dynamical systems and Rauzy fractals*, Pacific J. Math., 206 (2002), pp. 465–485.

[134] B. Solomyak, *Dynamics of self-similar tilings*, Ergodic Theory Dynam. Systems, 17 (1997), pp. 695–738.

[135] C. Steineder, *Subword complexity and projection bodies*, Advances in Mathematics, 217 (2008), pp. 2377–2400.

[136] M. Stewart, *Irregularities of uniform distribution*, Acta Math. Acad. Scient. Hung., 37 (1981), pp. 185–221.

[137] G. Szekeres, *Multidimensional continued fractions*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., 13 (1970), pp. 113–140 (1971).

[138] W. Thurston, *Groups, tilings and finite state automata.* AMS Colloquium Lecture Notes, 1989.

[139] R. Tijdeman and L. Q. Zamboni, *Characterizations of words with many periods*, Integers, 9 (2009), pp. A27, 333–342.

[140] R. Tijdeman and L. Q. Zamboni, *Fine and Wilf words for any periods. II*, Theoret. Comput. Sci., 410 (2009), pp. 3027–3034.

[141] B. Vallée, *Euclidean dynamics*, Discrete Contin. Dyn. Syst., 15 (2006), pp. 281–352.

[142] A. Vershik and N. Sidorov, *Arithmetic expansions associated with the rotation of a circle and continued fractions*, St. Petersburg Math. J., 5 (1994), pp. 1121–1136.

[143] P. Walters, *An introduction to ergodic theory*, Springer-Verlag, New York, 1982.

[144] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liège, 41 (1972), pp. 179–182.