On continued fraction expansions in positive characteristic: equivalence relations and some metric properties

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Abstract The aim of this paper is to survey some properties of analogues of continued fraction expansions for formal power series with coefficients in a finite field. We discuss in particular connections between equivalence relations for continued fractions and the action of $SL(2, \mathbb{F}_q[X])$.

1 Introduction

Let p be a prime, q be a power of p, and let $\mathbb{F}_q$ be the finite field with q elements. By analogy with the real case, one can classically extend arithmetical results concerning the ring $\mathbb{Z}$ of integers to the ring $\mathbb{F}_q[x]$ of polynomials with coefficients in $\mathbb{F}_q$ (see for instance [23]). The purpose of this paper is to discuss properties concerning some analogues of classical continued fraction transformations and to compare them with the real case. These transformations are defined on the set of Laurent formal power series (with coefficients in $\mathbb{F}_q(X)$) of positive $1/X$-valuation: they describe continued fraction expansions, with polynomials playing the rôle of the "digits". More precisely, we investigate three different types of continued fraction expansions: the first one, which we call "+"-expansion, corresponds to the classical continued fraction (it was introduced by Artin in [5]); the second one, which we call "−"-expansion, corresponds to the real semiregular expansions, i.e., the expansion with "−" sign; and the third one corresponds to the Lüroth series expansion, which can be considered as an analogue of the decimal expansion. We discuss here some classical properties (modular equivalence, metric results) of these expansions. Most of these results are known in the "+" case. We survey the existing proofs and offer new ones.

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In the real case, we have the following situation. Two real numbers x and y have ultimately the same "+"-continued fraction expansion (respectively the same "−"-continued fraction expansion) if and only if x and y are equivalent under the action of $GL(2, \mathbb{Z})$ (respectively $SL(2, \mathbb{Z})$) (see for instance [84]). In this sense, "−" expansion seems to be natural. On the other hand, the transformation associated with this continued fraction does not have an absolutely continuous finite invariant measure (but an infinite one). The situation in positive characteristic is different. The equivalence relation by the semiregular expansion is a subrelation of the $SL(2, \mathbb{F}_q[X])$ equivalence relation and is not identical to it. The following question is thus natural. How can we characterize the $SL(2, \mathbb{F}_q[X])$ relation (or the $GL(2, \mathbb{F}_q[X])$ relation) by using "−" and "+" expansions? We answer in detail this question in Section 4. Note that the case of the $GL(2, \mathbb{F}_q[X])$ equivalence for "+" expansions was studied for instance by de Mathan in [48], see also [71].

Furthermore, by introducing a realization of the natural extension of both transformations associated to "+"-extensions and "−"-extensions, we give a proof, in Section 5, of the analogue of Galois' Theorem (originally proved by Artin [5]): let f be a quadratic power series of negative degree and let $\overline{f}$ be its quadratic conjugate; then the continued fraction expansion of f is purely periodic if and only if $\overline{f}$ has positive degree.

In our situation, both transformations T_+ and T_- (associated to "+" and "−" continued fraction expansions, respectively) preserve the Haar measure and are isomorphic to the transformation L associated with the Lüroth series expansion. Actually this Haar measure preserving property for the transformation T_+ is first proved in [17, 19] as a special case of the Jacobi-Perron algorithm, as proved in [31, 36] for the transformation L. It is easy to see that these transformations T_+, T_- and L are isomorphic to each other in the sense of ergodic transformations, though they are certainly not in the classical case, that is, in the case of real numbers. In Section 6 we discuss the difference between the "+" and "−" continued fractions, and Lüroth series from a metric theoretic point of view. We end this paper by giving in Section 7 an overview of the abundant literature devoted to continued fraction expansions for formal power series with coefficients in a finite field.

2 Notations

By analogy with the real case, consider the following sets. The set which plays here the rôle of $\mathbb{Z}$ is the ring $\mathbb{F}_q[X]$ of polynomials with coefficients in $\mathbb{F}_q$ and the analogue of $\mathbb{Q}$ is the fraction field of $\mathbb{F}_q[X]$, i.e., the field $\mathbb{F}_q(X)$ of fractions with coefficients in $\mathbb{F}_q$. Let $\mathbb{F}_q((1/X))$ be the field of Laurent formal power series:

$$\mathbb{F}_q((1/X)) = \{f = \sum_{n \geq n_0} f_n X^{-n}, \text{ where } f_n \in \mathbb{F}_q \text{ and } n_0 \in \mathbb{Z}\}.$$

This field is a complete metric space with respect to the valuation

$$v(f) = -deg(f) = \inf\{n \in \mathbb{Z},\ f_n \neq 0\}.$$

It is thus considered as the analogue of $\mathbb{R}$. We define a non-Archimedean absolute values in $\mathbb{F}_q((1/X))$ normalized by

$$|f| := q^{\deg(f)}, \text{ for any } f \in \mathbb{F}_q((1/X)).$$

Let $\mathbf{C}$ be the completion of an algebraic closure of $\mathbf{R}$; $\mathbf{C}$ is algebraically closed. This field will be considered as an analogue of $\mathbb{C}$.

Let $\mathcal{X}$ denote the valuation ideal of $\mathbb{F}_q((1/X))$:

$$\mathcal{X} = \{f,\ v(f) \geq 1\} = \{f = \sum_{n \geq 1} f_n X^{-n}\}.$$

The field $\mathbb{F}_q((1/X))$ is locally compact. Let μ denote the Haar measure on $\mathbb{F}_q((1/X))$ normalized to 1 on $\mathcal{X}$.

Every $f \in \mathbb{F}_q((1/X))$ has a unique (Artin) decomposition as

$$f = [f] + \{f\},$$

where the integral part $[f]$ of f belongs to $\mathbb{F}_q[X]$ and the fractional part $\{f\}$ of f belongs to $\mathcal{X}$. For any $a \in \mathbb{F}_q((1/X))$ and any $r \in \mathbb{Z}$, let

$$\mathcal{B}(a, r) = \{f \in \mathbb{F}_q((1/X)),\ v(f - a) > r\} = \{f \in \mathbb{F}_q((1/X)),\ \deg(f - a) \leq -(r+1)\}$$

be the closed disc (or cylinder) of center a and radius r We have

$$\mu(\mathcal{B}(a, r)) = q^{-r}.$$

The set $\mathcal{X}$ is isomorphic to $\prod_{n \geq 1} \mathbb{F}_q$. The measure μ can be seen as the product of the equirepartition measure on $\mathbb{F}_q$.

In all that follows, $\mathbb{F}_q^*$ *denotes the set of non-zero elements of* $\mathbb{F}_q$. *The set* $\mathbb{N}$ *denotes the set of natural integers greater than or equal to* 0.

3 Some continued fraction expansions

Consider by analogy with the real case, the following transformations from $\mathcal{X}$ onto itself defined by:

$$\begin{array}{ll} T_+ : & f \mapsto \{\frac{1}{f}\}, \text{ if } f \neq 0 \text{ and } T_+(0) = 0, \\ T_- : & f \mapsto -\{\frac{1}{f}\}, \text{ if } f \neq 0 \text{ and } T_-(0) = 0, \\ L : & f \mapsto (c-1)(cf-1), \text{ where } c(f) = [\frac{1}{f}] \text{ if } f \neq 0, \text{ and } L(0) = 0. \end{array}$$

The map T_+ describes the regular continued fraction and has been introduced by Artin in [5]. For a brief sketch of the continued fraction theory in this framework, see for instance [5, 7]. The map $T_- = (-T_+)$ corresponds to the semiregular continued fraction. The map L describes the Lüroth type expansion and has been introduced in [31, 36] for formal power series.

3.1 Continued fraction expansion

Every $f \in \mathcal{X}$ has the following continued fraction expansion:

$$f \;=\; \cfrac{1}{a_1(X) + \cfrac{1}{a_2(X) + \ddots}} := [0; a_1(X), a_2(X), \cdots],$$

where the digits $a_i(X)$ are polynomials of strictly positive degree and are defined by

$$\forall n \geq 1,\; a_n(X) = [\frac{1}{(T_+)^{n-1}(f)}].$$

For $f \in \mathbb{F}_q((1/X))$, we put $a_0(X) = [f]$ and have

$$f = a_0(X) + \cfrac{1}{a_1(X) + \cfrac{1}{a_2(X) + \ddots}} := [a_0(X); a_1(X), a_2(X), \cdots].$$

Such an expansion is unique (provided $\forall n,\; \deg(a_n) \geq 1$), even if f does not belong to $\mathbb{F}_q(X)$. Furthermore, the expansion terminates after a finite number of terms if and only if f belongs to $\mathbb{F}_q(X)$.

As in the real case, we do not have for such an expansion any admissibility condition, i.e., given any sequence of polynomials of positive degree $(a_n)_{n\in\mathbb{N}}$, then the series

$$[a_0(X); a_1(X), a_2(X), a_3(X), \cdots] := a_0(X) + \cfrac{1}{a_1(X) + \cfrac{1}{a_2(X) + \ddots}},$$

is easily seen to converge in $\mathcal{X}$ to a formal power series f, say, which satisfies: $\forall n \geq 1$, $a_n(X) = [\frac{1}{(T_+)^{n-1}(f)}]$.

Let $(\frac{p_n(X)}{q_n(X)})_{n\in\mathbb{N}}$ be the sequence of convergents in the expansion of f, i.e.,

$$\frac{p_n(X)}{q_n(X)} = a_0(X) + \cfrac{1}{a_1(X) + \cfrac{1}{a_2(X) + \ddots + \frac{1}{a_n(X)}}} = [0; a_1(x), \cdots, a_n(X)].$$

It is easily seen as in the real case that, for every $n \geq 1$

$$p_n(X)q_{n-1}(X) - p_{n-1}(X)q_n(X) = (-1)^{n-1},$$

i.e.,

$$\frac{p_n(X)}{q_n(X)} = \frac{p_{n-1}(X)}{q_{n-1}(X)} + \frac{(-1)^{n-1}}{q_{n-1}(X)q_n(X)},$$

which implies that

$$f = \sum_{k=0}^{+\infty} \frac{(-1)^k}{q_k(X)q_{k+1}(X)},$$

and

$$\frac{p_n(X)}{q_n(X)} = \sum_{k=0}^{n-1} \frac{(-1)^k}{q_k(X)q_{k+1}(X)}.$$

Since the absolute value is non-archimedean, we have for any $n \geq 1$

$$|q_n(X)f - p_n(X)| = \frac{1}{|q_{n+1}(X)|} = \frac{1}{|a_{n+1}(X)||q_n(X)|}. \tag{1}$$

In the real case, we have a strict inequality: $|q_n(X)f - p_n(X)| < \frac{1}{|a_{n+1}(X)||q_n(X)|}$ (see for instance [26]).

As another difference, we have the following characterization of convergents (see for example [7]): let $(p(X), q(X)) \in (\mathbb{F}_q[X])^2$;

$$\text{if } |q(X)f - p(X)| < \frac{1}{|q(X)|}, \text{ then } \frac{p(X)}{q(X)} \text{ is a convergent of } f.$$

The corresponding property in the real case is (see for instance [26]):

$$\text{if } |qf - p| < \frac{1}{2|q|}, \text{ then } \frac{p}{q} \text{ is a convergent of } f.$$

3.2 Semiregular continued fraction expansion

Consider now the semiregular continued fraction expansion corresponding to the map T_-. Every $f \in \mathcal{X}$ has the following expansion:

$$f = \cfrac{1}{b_1(X) - \cfrac{1}{b_2(X) - \ddots}} := [0; b_1(X), b_2(X), \cdots]_-$$

where the digits are polynomials with strictly positive degree and are defined by

$$\forall n \geq 1, \; b_n(X) = [\frac{1}{T_-^{n-1}(f)}].$$

We also call this expansion *"$-$"expansion* or *semiregular continued fraction*.

The expansion terminates after a finite number of terms if and only if f belongs to $\mathbb{F}_q(X)$. For $f \in \mathbb{F}_q((1/X))$, we put $b_0(X) = [f]$ and have

$$f = b_0(X) + \cfrac{1}{b_1(X) - \cfrac{1}{b_2(X) - \ddots}} := [b_0(X); b_1(X), b_2(X), \cdots]_-.$$

In the real case, such an expansion is unique provided there are infinitely many digits i such that $b_i \geq 3$ (we have $1 = [0; 2, 2, 2, \cdots]_-$). Such a phenomenon does not occur here: this expansion is unique in all cases.

Conversely, given any sequence $(b_n)_{n\in\mathbb{N}}$ of polynomials of positive degrees, then the series

$$[b_0(X); b_1(X), \cdots, b_n(X), \cdots]_-$$

converges in $\mathcal{X}$ to a formal power series f, say, which satisfies: $\forall n \geq 1$, $b_n(X) = [\frac{1}{T_-^{n-1}(f)}]$.

Note that the convergents $\frac{p_n}{q_n}$ are exactly the same as in the classical expansion, but p_n and q_n are multiplied by (-1) for even n.

3.3 Lüroth continued fraction expansion

Consider the map (introduced in [31, 36]) defined for any $f \in \mathcal{X}$ by

$$L : f \mapsto (c-1)(cf-1),$$

where $c := [\frac{1}{f}]$, if $f \neq 0$ and $L(0) = 0$. It is easily seen that $L(\mathcal{X}) \subset \mathcal{X}$. Indeed $\frac{1}{f} - c \in \mathcal{X}$, hence

$$\deg(cf-1) < \deg(f), \text{i.e., } \deg[(c-1)(cf-1)] < 0.$$

Let $f \in \mathcal{X}$. Define for $n \geq 1$ the sequence

$$c_n(X) = [\frac{1}{L^{n-1}(f)}].$$

If $L^{n-1}(f) = 0$, put $c_k = 0$, for $k \geq n$. We thus have the following "Lüroth expansion"

$$f = \frac{1}{c_1} + \frac{L(f)}{c_1(c_1-1)} = \frac{1}{c_1} + \frac{1}{c_1(c_1-1)c_2} + \frac{L^2(f)}{c_1(c_1-1)c_2(c_2-1)}$$
$$= \frac{1}{c_1} + \frac{1}{c_1(c_1-1)c_2} + \frac{1}{c_1(c_1-1)c_2(c_2-1)c_3} + \dots$$

Such an expansion is unique if f does not belong to $\mathbb{F}_q(X)$. For $f \in \mathbb{F}_q((1/X))$, we put $c_0(X) = [f]$ and have

$$f = c_0(X) + \frac{1}{c_1} + \frac{1}{c_1(c_1-1)c_2} + \frac{1}{c_1(c_1-1)c_2(c_2-1)c_3} + \dots.$$

Furthermore, the expansion terminates after a finite number of terms, or is periodic, if and only if f belongs to $\mathbb{F}_q(X)$.

Given any sequence of polynomials (c_n) with positive degree, the series

$$f = \frac{1}{c_1} + \frac{1}{c_1(c_1-1)c_2} + \frac{1}{c_1(c_1-1)c_2(c_2-1)c_3} + \dots$$

is easily seen to be convergent. Hence, as in the real case, we do not have for such an expansion any admissibility condition.

We put for $n \geq 1$

$$\begin{cases} s_{n+1} = & (c_n-1)c_{n+1}s_n + t_n, \\ t_{n+1} = & (c_n-1)c_{n+1}t_n, \end{cases}$$

with $s_1 = 1$ and $t_1 = c_1$. These equalities define the n-th (Lüroth series) convergents, i.e.,

$$\frac{s_n}{t_n} = \frac{1}{c_1} + \frac{1}{c_1(c_1-1)c_2} + \dots + \frac{1}{c_1(c_1-1)\dots c_{n-1}(c_{n-1}-1)c_n};$$

however, we have to notice that $\frac{s_n}{t_n}$ is in general not reduced. Furthermore it is easy to see that

$$|t_n||f - \frac{s_n}{t_n}| = \frac{1}{|c_n||c_{n+1}|} = q^{-(\deg(c_{n+1})+\deg(c_n))}. \tag{2}$$

For other continued-fraction-like expansions, see [33, 34].

4 Equivalence relations of continued fraction expansions

The aim of this section is to give a characterization of equivalence relations for "+" and "−" continued fraction expansions. The $GL(2, \mathbb{F}_q[X])$ equivalence for "+" continued fraction expansions has been considered in [48], see also [71], and of course [5]. Both approaches [48, 71] are based on the following property: if $\alpha = \frac{A\beta+B}{C\beta+D}$, where $\alpha, \beta \notin \mathbb{F}_q(X)$, $\deg(\beta) \geq 0$, $A, B, C, D \in \mathbb{F}_q[X]$, $|D| < |C|$, and $AD - BC$ is invertible, then $\frac{A}{C}$ and $\frac{B}{D}$ are successive convergents of α. Note that this result holds more generally for any function field $k(X)$. We present here a different (yet classical in the real case) approach.

4.1 Statement of the results

Let $GL(2, \mathbb{F}_q[X])$ (respectively $SL(2, \mathbb{F}_q(X))$) be the set of 2×2 matrices of invertible determinant (respectively of determinant 1) with $\mathbb{F}_q[X]$ entries. Let Q be a matrix in $GL(2, \mathbb{F}_q[X])$; in all that follows the notation $g = Q.f$ stands for

$$g = \frac{a(X)f + b(X)}{c(X)f + d(X)},$$

where $Q = \begin{bmatrix} a(X) & b(X) \\ c(X) & d(X) \end{bmatrix}$.

Consider the two equivalence relations, say $\sim_S$ and $\sim_G$, associated with the actions of $GL(2, \mathbb{F}_q[X])$ and $SL(2, \mathbb{F}_q(X))$. This means

$$f \sim_S g \quad \text{(respectively } f \sim_G g\text{)}$$

if and only if there exists $Q \in SL(2, \mathbb{F}_q[X])$ (respectively $GL(2, \mathbb{F}_q[X])$) such that $g = Q.f$.

On the other hand, we also define two equivalence relations, say $\sim$ and $\approx$, associated with semiregular continued fractions:

$$f \sim g, \text{respectively } f \approx g$$

if and only if there exist positive integers m, n and $\nu \in \mathbb{F}_q^*$ such that

$$\forall k \geq 0,\ d_{m+k}(X) = (\nu^2)^{(-1)^k} e_{n+k}(X),$$

or

$$\forall k \geq 0,\ d_{m+k}(X) = \nu^{(-1)^k} e_{n+k}(X),\ \text{respectively,}$$

where $f = [d_0(X); d_1(X), d_2(X), \cdots]_-$ and $g = [e_0(X); e_1(X), e_2(X), \cdots]_-$.

Theorem 1 *Let $f, g \in \mathbb{F}_q((1/X))$. The following holds:*

- *$f \sim g$ if and only if $f \sim_S g$,*
- *$f \approx g$ if and only if $f \sim_G g$.*

A similar result holds with the classical expansion corresponding to the map T_+. The equivalence relations become:

$$f \sim_+ g \text{ or } f \approx_+ g$$

if and only if there exist positive integers m, n and $\nu \in \mathbb{F}_q^*$ such that

$$\forall k \geq 0,\ d_{m+k}(X) = (-\nu^2)^{(-1)^k} e_{n+k}(X),$$

or

$$\forall k \geq 0,\ d_{m+k}(X) = \nu^{(-1)^k} e_{n+k}(X),\ \text{respectively,}$$

where $f = [d_0(X); d_1(X), d_2(X), \cdots]$ and $g = [e_0(X); e_1(X), e_2(X), \cdots]$.

Theorem 2 *Let $f, g \in \mathbb{F}_q((1/X))$. The following holds:*

- *$f \sim_+ g$ if and only if $f \sim_S g$,*
- *$f \approx_+ g$ if and only if $f \sim_G g$.*

We will give detailed proofs for the "$-$" case in the next section. The "+" case can be handled in the same way.

4.2 Proof of Theorem 1

The sketch of the proof is the following. We will first produce a set of generators for the group $SL(2, \mathbb{F}_q(X))$ (Lemma 1), then we will see how these generators act on the continued fraction expansion.

Lemma 1 *For any $Q \in SL(2, \mathbb{F}_q[X])$, there exists a matrix B of the form $\begin{bmatrix} \eta & \gamma(X) \\ 0 & \eta^{-1} \end{bmatrix}$ or $\begin{bmatrix} 0 & \eta^{-1} \\ -\eta & \gamma(X) \end{bmatrix}$, with $\gamma(X) \in \mathbb{F}_q[X]$ and $\eta \in \mathbb{F}_q^*$ such that*

$$Q = A_1 A_2 \ldots A_n B,$$

with A_i, for $1 \leq i \leq n$, either of the form

$$\begin{bmatrix} 1 & 0 \\ \alpha(X) & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & \beta(X) \\ 0 & 1 \end{bmatrix}, \tag{3}$$

where $\alpha(X), \beta(X) \in \mathbb{F}_q[X]$ and $\deg(\alpha(X)) \geq 1,\ \deg(\beta(X)) \geq 1$.

Proof of Lemma 1 The proof works exactly as in the classical case $SL(2,\mathbb{Z})$. Let $Q = \begin{bmatrix} a(X) & b(X) \\ c(X) & d(X) \end{bmatrix} \in SL(2, \mathbb{F}_q[X])$ and

$$r(Q) = \min(|a(X)|, |c(X)|).$$

Suppose $r(Q) \geq 1$. Then, using the Euclidean division in $\mathbb{F}_q[X]$, we can transform the matrix Q into a matrix Q' with $r(Q') < r(Q)$, by multiplying Q by the inverse of a matrix of type (3), which is also of type (3).

More precisely, suppose $\deg(a(X)) < \deg(c(X))$, then there exists $\alpha(X) \in \mathbb{F}_q[X]$ (with $\deg(\alpha(X)) \geq 1$) such that

$$\deg(\alpha(X)a(X) + c(X)) < \deg(a(X)).$$

If $\deg(a(X)) \geq \deg(c(X))$, then there exists $\beta(X) \in \mathbb{F}_q[X]$ such that

$$\deg(a(X) + \beta(X)c(X)) < \deg(c(X)).$$

Hence, by iterating this process, there exist $A_1 \dots A_n$ of type (3) such that $Q = A_1 \dots A_n B$, with $B = \begin{bmatrix} a'(X) & b'(X) \\ c'(X) & d'(X) \end{bmatrix}$ and $r(B) = 0$. We have furthermore $a'(X), b'(X), c'(X), d'(X) \in \mathbb{F}_q[X]$ and $a'(X)d'(X) - b'(X)c'(X) = 1$. Suppose $a'(X) = 0$. Then $b'(X) \in \mathbb{F}_q^*$ and $c'(X) = -b'(X)^{-1}$. Suppose $c'(X) = 0$. Then $a'(X) \in \mathbb{F}_q^*$ and $d'(X) = a'(X)^{-1}$. ■

Lemma 2 *Let $c \in \mathbb{F}_q^*$ and $f = [d_0(X); d_1(X), d_2(X), \cdots]_-$. We have*

1. $cf = [cd_0(X); c^{-1}d_1(X), cd_2(X), c^{-1}d_3(X), \cdots]_-$;
2. $-1/f = [0; -d_0(X), d_1(X), d_2(X), d_3(X), \cdots]_-$;
3. *if furthermore $d_0(X) = 0$, then*

$$\frac{1}{c-f} = [1/c; -c + c^2 d_1(X), c^{-2}d_2(X), c^2 d_3(X), c^{-2}d_4(X), \cdots]_-.$$

Proof of Lemma 2 We have $[cf] = c[f]$, which proves the first assertion by induction. The proof of the second assertion is immediate. The third assertion comes from

$$\frac{1}{c-f} - \frac{1}{c} = \frac{f}{c(c-f)} = \frac{1}{c^2/f - c}.$$

We thus apply the first assertion by noticing that $\deg(d_1(X)) \geq 1$ and hence, $-c + c^2 d_1(X) \neq 0$. ■

Proof of Theorem 1 Let us first prove that the two equivalence relations $\sim$ and $\sim_S$ are equal. It is easily seen that if $f \sim g$, then $f \sim_S g$: suppose that there exist positive integers m, n and $\eta \in \mathbb{F}_q^*$ such that

$$\forall k \geq 0,\ d_{m+k}(X) = (\eta^2)^{(-1)^k} e_{n+k}(X),$$

with $f = [d_0(X); d_1(X), d_2(X), \cdots]_-$ and $g = [e_0(X); e_1(X), e_2(X), \cdots]_-$. Consider the matrix

$$Q = \begin{bmatrix} \eta^{-1} & 1 \\ 0 & \eta \end{bmatrix}$$

and apply Lemma 2 (1) to $T_-^{m-1}(f - d_0(X))$ and $T_-^{n-1}(g - e_0(X))$. We have

$$T_-^{m-1}(f - d_0(X)) = Q.T_-^{n-1}(g - e_0(X))$$

and thus $f \sim_S g$.

Let us prove now the converse assertion. Let

$$f = [d_0(X); d_1(X), d_2(X), d_3(X), \cdots]_-.$$

Consider the action of the generators of $SL(2, \mathbb{F}_q[X])$ introduced in Lemma 1 on the expansion of f. Let us check that the image g of f under the action of any generator satisfies $f \sim g$.

- Let $\beta \in \mathbb{F}_q[X]$. Let $g = f + \beta(X)$. These series have the same fractional part, i.e., $g = [d_0(X) + \beta(X); d_1(X), d_2(X), d_3(X), \cdots]_-$. This equality corresponds to the action of the matrix $\begin{bmatrix} 1 & \beta(X) \\ 0 & 1 \end{bmatrix}$.

- Let $\alpha \in \mathbb{F}_q[X]$ with $\deg(\alpha(X)) \geq 1$. Let

  $$g = \frac{f}{\alpha(X)f + 1} = \frac{1}{\alpha(X) + 1/f},$$

  which corresponds to the action of $\begin{bmatrix} 1 & 0 \\ \alpha(X) & 1 \end{bmatrix}$.

  Let us distinguish three cases according to $d_0(X)$.

  - Suppose $\deg(d_0(X)) \geq 1$. Recall that $\deg(\alpha) \geq 1$. Then we see

    $$g = [0; \alpha(X), -d_0(X), d_1(X), d_2(X), d_3(X), \ldots]_-.$$

  - Suppose $d_0(X) \in \mathbb{F}_q^*$. Put $c = d_0$. In this case we have

    $$g = \cfrac{1}{\alpha(X) - \cfrac{1}{-c - \cfrac{1}{d_1(X) \ddots}}},$$

    which implies, with assertion 3 of Lemma 2,

    $$g = [0; \alpha(X) + 1/c, c + c^2 d_1(X), c^{-2} d_2(X), c^2 d_3(X), c^{-2} d_4(X), \ldots]_-.$$

  - Suppose $d_0(X) = 0$. If $\deg(\alpha(X) + d_1(X)) \geq 1$, then

    $$g = [0; \alpha(X) + d_1(X), d_2(X), d_3(X), \ldots]_-.$$

    If $\alpha(X) + d_1(X) = 0$, then

    $$g = [-d_2(X); d_3(X), d_4(X), d_5(X), \ldots]_-.$$

    If $\alpha(X) + d_1(X) = c \in \mathbb{F}_q^*$. By Lemma 2 (3), we have

    $$g = [1/c; -c + c^2 d_2(X), c^{-2} d_3(X), c^2 d_4(X), \ldots]_-.$$

- Let $\gamma \in \mathbb{F}_q[X]$ and $\eta \in \mathbb{F}_q^*$. Let

$$g = \frac{\eta f + \gamma(X)}{\eta^{-1}}$$

which corresponds to the action of $\begin{bmatrix} \eta & \gamma(X) \\ 0 & \eta^{-1} \end{bmatrix}$. Then Lemma 2 (1) implies

$$g = [\eta^2 d_0(X) + \eta\gamma(X); \eta^{-2} d_1(X), \eta^2 d_2(X), \eta^{-2} d_3(X), \ldots]_-.$$

- Finally, let $\gamma(X) \in \mathbb{F}_q[X]$ and $\eta \in \mathbb{F}_q^*$. Let

$$g = \frac{\eta^{-1}}{-\eta f + \gamma} = \frac{1}{-\eta^2 f + \eta\gamma(X)}$$

which corresponds to $\begin{bmatrix} 0 & \eta^{-1} \\ -\eta & \gamma(X) \end{bmatrix}$.

Let us distinguish two cases according to $d_0(X)$.

  - Suppose $d_0(X) = 0$. If $\deg(\gamma(X)) \geq 1$, then

$$g = [0; \eta\gamma(X), \eta^{-2} d_1(X), \eta^2 d_2(X), \ldots]_-.$$

  If $\gamma(X) = 0$, then

$$g = [-\eta^{-2} d_1(X); \eta^2 d_2(X), \eta^{-2} d_3(X), \ldots]_-.$$

  If $\gamma(X) = c \in \mathbb{F}_q^*$, then by Lemma 2 (3)

$$g = \frac{1}{\eta c - \eta^2 [0; d_1(X), d_2(X), \ldots]_-} = [\frac{1}{\eta c}; -\eta c + c^2 d_1(X), c^{-2} d_2(X), \ldots]_-.$$

  - Suppose $d_0(X) \neq 0$. We have

$$g = \frac{1}{\eta\gamma(X) - \eta^2 d_0(X) - \eta^2 [0; d_1(X), d_2(X), d_3(X), \ldots]_-}.$$

  If $\deg(\eta\gamma(X) - \eta^2 d_0(X)) \geq 1$, then

$$g = [0; \eta\gamma(X) - \eta^2 d_0(X), \eta^{-2} d_1(X), \eta^2 d_2(X), \ldots]_-.$$

  If $\deg(\eta\gamma(X) - \eta^2 d_0(X)) = 0$, then

$$g = [-\eta^{-2} d_1(X); \eta^2 d_2(X), \eta^{-2} d_3(X)]_-.$$

We thus have proved that if f and g are equivalent under the action of $SL(2, \mathbb{F}_q[X])$, then $f \sim g$.

Suppose now that f and g are equivalent under the action of $GL(2, \mathbb{F}_q[X])$. Then there exists a matrix $Q \in GL(2, \mathbb{F}_q[X])$ such that $g = Q.f$. Consider the matrix $Q' = \begin{bmatrix} \eta^{-1} & 0 \\ 0 & 1 \end{bmatrix}$. Then $Q'Q \in SL(2, \mathbb{F}_q[X])$, whenever $\det Q = \eta$. Let $g' = Q'Q.f$. We thus have from above $g' \sim f$. Then

$$g = (Q'^{-1}).g' = \eta g = [\eta e_0(X); \eta^{-1} e_1(X), \eta e_2(X), \ldots]_-,$$

where

$$g = [e_0(X); e_1(X), e_2(X), \ldots]_-.$$

Then $g \approx f$. Conversely, it is easily seen that if $f \approx g$, then $f \sim g$. Consider indeed a matrix of the form

$$Q = \begin{bmatrix} \eta^{-1} & 1 \\ 0 & 1 \end{bmatrix},$$

where $\eta \in \mathbb{F}_q^*$, and apply Lemma 2 (1). ■

4.3 Remarks

If $q = 2, 5$, then two formal power series are equivalent under the action of $SL(2, \mathbb{F}_q[X])$ (respectively $GL(2, \mathbb{F}_q[X])$) if and only if their "$-$" (or "$+$") expansions are ultimately equal up to a non-zero square (respectively up to a non-zero multiplicative constant).

Furthermore, (-1) is a non-zero square in $\mathbb{F}_q$ if and only if $q \equiv 1$ modulo 4. In this case, the relations $\sim$ and $\sim_+$ are equal.

Let us introduce a continued fraction type expansion which describes the equivalence classes of the action of $GL(2, \mathbb{F}_q(X))$ as the formal power series which have ultimately the same expansion.

Let us write

$$f = [d_0(X); d_1(X), d_2(X), \cdots]_-.$$

For any $k \geq 0$, let μ_k be the leading coefficient of $d_k(X)$ (i.e., the coefficient of its term of highest degree) and let $e_k(X)$ satisfy $d_k(X) = \mu_k(X) e_k(X)$. We thus have

$$f = \mu_0 e_0(X) + \cfrac{1}{\mu_1(X) e_1(X) - \cfrac{1}{\mu_2(X) e_2(X) - \ddots}},$$

which yields

$$f = \eta_0 e_0(X) + \cfrac{\eta_1}{e_1(X) - \cfrac{\eta_2}{e_2(X) - \ddots}},$$

with $\eta_0 = \mu_0$, $\eta_1 = \mu_1^{-1}$ and $\eta_{k+1} = (\mu_k \mu_{k+1})^{-1}$, for every $k \geq 1$. Hence every $f \in \mathbb{F}_q((1/X))$ can be uniquely expanded as

$$f = \eta_0 e_0(X) + \cfrac{\eta_1}{e_1(X) - \cfrac{\eta_2}{e_2(X) - \ddots}},$$

where the coefficients η_i belong to $\mathbb{F}_q^*$ and the digits $e_i(X)$ are polynomials of positive degree with leading coefficient equal to 1.

Hence two formal power series f and f' are equivalent under the action of $GL(2, \mathbb{F}_q[X])$ if and only if the two corresponding sequences $(\eta_k, e_k(X))$ and $(\eta'_k, e'_k(X))$ are ultimately equal, i.e., if there exist m, n such that

$$\forall k \geq 0, \ \eta_{k+m}(X) = \eta'_{k+n}(X) \text{ and } e_{k+m}(X) = e'_{k+n}(X). \tag{4}$$

One can similarly describe the action of $SL(2, \mathbb{F}_q[X])$. Two formal power series f and f' are equivalent under the action of $SL(2, \mathbb{F}_q[X])$ if and only if the two corresponding sequences $(\eta_k, e_k(X))$ and $(\eta'_k, e'_k(X))$ satisfy Equation (4), and there exists $l, l' \geq m, n$ such that

$$\frac{\prod_{1 \leq k \leq l} \eta_{2k} \prod_{0 \leq k \leq l'-1} \eta'_{2k+1}}{\prod_{1 \leq k \leq l} \eta'_{2k} \prod_{0 \leq k \leq l'-1} \eta_{2k+1}} \text{ is a non-zero square.}$$

The study of the action of a homography on the continued fraction expansion often appears in the literature and has many applications, in particular in Diophantine approximation. We will review some connected works in the last section of this paper.

As a direct consequence of Theorem 1, we obtain the following: an irrational series is a fixed point of a non-trivial invertible homography if and only if its continued fraction expansion is ultimately periodic. We will study in detail the quadratic formal power series in the next section. For the corresponding study for formal power series with coefficients in a field of positive characteristic (not necessarily finite), see for example [71]. Periodicity is then replaced by the notion of pseudoperiodicity or quasi-periodicity [9, 67, 27] (i.e., periodicity up to a multiplicative constant). We will come back to the notion of quasi-periodicity in the last section.

5 Natural extension transformation and quadratic formal power series

The aim of this section is to prove an analogue of Galois' Theorem. For this goal, we will introduce a realization of the natural extension for the transformation T_-. But we first need to prove the invariance of the Haar measure.

5.1 Invariant measure

Theorem 3 *The Haar measure μ is invariant for the transformations T_+, T_- and L. These transformations are ergodic with respect to the Haar measure μ. For each of these transformations, the dynamical system $(\mathcal{X}, T, \mu)$ ($T = T_+, T_-, L$) is conjugated to the symbolic dynamical system*

$$(\prod_{n \geq 1} \mathbb{F}_q^0[X], \sigma, m),$$

where $\mathbb{F}_q^0[X] = \{h(X) \in \mathbb{F}_q[X],\ \deg(h) \geq 1\}$, σ *denotes the unilateral shift and* m *is the product of the measure* m_0 *defined on* $\mathbb{F}_q^0[X]$ *by*

$$m_0\{h(X)\} = \left(\frac{1}{q}\right)^{2\deg(h(X))}.$$

This conjugation is one-to-one except for a set of measure zero.

This Haar measure preserving property is a classical one. It has been first proved in [17, 18, 19] for the transformation associated with "+"-expansion, as a special case of the Jacobi-Perron algorithm. See also [28] for the ergodicity and other metric properties. The case of the transformation L is studied in detail in [31, 36].

Proof The sketch of the proof is the same for the three transformations. One shows that the random variables corresponding to the digit functions are independent and identically distributed random variables relatively to the Haar measure: the image measure is the product of the measure m_0 defined on the set of polynomials of positive degree $\mathbb{F}_q^0[X]$.

Consider for instance the case of the transformation T_+. For any non-negative integer k, consider the map $t_k : \mathbb{F}_q^* \times \mathbb{F}_q^k \to \mathbb{F}_q^* \times \mathbb{F}_q^k$, $(\alpha_0, \ldots, \alpha_k) \mapsto (\beta_0, \ldots, \beta_k)$, where the quantities $\beta_0, \ldots, \beta_k$ are defined inductively by $\beta_0 = 1/\alpha_0$ and $\sum_{i+j=n} \alpha_i \beta_j = 0$, for $1 \leq n \leq k$. The map t_k is easily seen to be one-to-one and onto.

We have $f = \frac{\alpha_0}{X^n} + \ldots + \frac{\alpha_n}{X^n}$, with $\alpha_0 \neq 0$, if and only if $[\frac{1}{f}] = \beta_0 X^n + \ldots + \beta_n$, with $(\beta_0, \ldots, \beta_n) = t_n(\alpha_0, \ldots, \alpha_n)$.

Hence, for any fixed polynomial $h(X)$ of non-zero degree, we have

$$\mu\{f|\ a_1(f) = h(X)\} = (1/q)^{2(\deg(h(X))},$$

and for any fixed polynomials $h_1(X), \ldots, h_n(X)$ of non-zero degree, we prove in the same way

$$\mu(\{f|\ a_1(f) = h_1(X); \ldots;\ a_n(f) = h_n(X)\}) =$$

$$\prod_{i=1}^{n} \mu(\{f|\ a_i(f) = h_i(X)\}) = (1/q)^{2(\deg(h_1)+\ldots+\deg(h_n))}.$$

Hence the maps $f \mapsto h_i(X)$ are independent and identically distributed random variables. ■

Remark Note that the transformation associated with this "−" continued fraction expansion does not have an absolutely continuous finite invariant measure (but an infinite one) in the real case; recall that the regular continued fraction transformation and the transformation associated with the Lüroth expansion have a finite invariant measure [29].

5.2 A realization of the natural extension

The dynamical system $(\mathcal{X}, T_-, \mu)$ is conjugated to the symbolic dynamical system $(\prod_{n\geq 1} \mathbb{F}_q^0[X], \sigma, m)$, where $\mathbb{F}_q^0[X] = \{h(X) \in \mathbb{F}_q[X],\ \deg(h) \geq 1\}$. Furthermore we denote by $\mathbb{F}_q^0((1/X))$ the set of those $f \in \mathbb{F}_q((1/X))$ with $\deg(f) > 0$. We have a probability measure m_0 on $\mathbb{F}_q^0[X]$ defined by $m_0\{h(X)\} = q^{-2\deg(h(X))}$. Next we define a probability measure ν on $\mathbb{F}_q^0((1/X)) :\equiv \mathbb{F}_q^0[X] \times \mathcal{X}$ by $\nu = m_0 \times \mu$. Finally we define $\overline{\mu}$ on $\overline{\mathcal{X}} :\equiv \mathcal{X} \times \mathbb{F}_q^0((1/X))$ by $\overline{\mu} = \mu \times \nu$.

We define a map $\overline{T} : \overline{\mathcal{X}} \to \overline{\mathcal{X}}$ by

$$\overline{T}(f, g) = (T_-(f), [1/f] - 1/g) = ([1/f] - 1/f, [1/f] - 1/g).$$

Clearly $\overline{T}$ is 1-1 and onto on $\overline{\mathcal{X}} \backslash [(\mathcal{X} \cap \mathbb{F}_q(X) \times \mathbb{F}_q^0((1/X))]$, where the exceptional set is a null set.

Theorem 4 *The system $(\overline{X}, \overline{\mu}, \overline{T})$ is a realization of the natural extension of (X, μ, T_-).*

One can also consider the realization $(\overline{X}, \overline{\mu}, \overline{T}_+)$ of the natural extension of $(\mathcal{X}, \mu, T)$ defined by

$$\overline{T}_+(f, g) = (T_+(f), 1/g - [1/f]).$$

Proof Let $(f, g) \in \overline{\mathcal{X}} = \mathcal{X} \times \mathbb{F}_q^0((1/X))$.

We suppose

$$f = [0; d_1(X), d_2(X), \cdots]_-$$

and

$$g = e_0(X) - [0; e_1(X), e_2(X), e_3(X), \cdots]_-,$$
$$\text{i.e., } 1/g = [0; e_0(X), e_1(X), e_2(X), \cdots]_-.$$

Then we see

$$\overline{T}(f, g) = (f^1, g^1),$$

with

$$\begin{cases} f^1 & = [0; d_2(X), d_3(X), \cdots]_- \\ g^1 & = d_1(X) - [0; e_0(X), e_1(X), e_2(X), \cdots]_- \end{cases}$$

Let $\Omega = \prod_{-\infty}^{+\infty} \mathbb{F}_q^0[X]$ be endowed with the probability measure $P = \prod_{-\infty}^{+\infty} m_0$ and σ be the shift operator on Ω. Then it is easy to see that the map

$$\theta \ : \ (f, g) \to (\cdots e_1(X), e_0(X), d_1(X), d_2(X), d_3(X), \cdots)$$

induces a conjugacy between $(\overline{X}, \overline{\mu}, \overline{T})$ and (Ω, P, σ). Namely, we see that

$$\theta\overline{T}(f, g) = \sigma\theta(f, g) \text{ and } \overline{\mu} = \theta^{-1}P,$$

which implies the assertion of the theorem. ∎

Remark Put $(f^n, g^n) = \overline{T}^n(f, g)$. For (f, g_1) and $(f, g_2) \in \overline{X}$, we see that $|g_1^n - g_2^n|$ tends to 0 exponentially fast as n goes to infinity. The same contracting property holds in the real case. We will use it in the next section to prove an analogue of Galois' Theorem.

5.3 An analogue of Galois' Theorem

Let us deduce a characterization of the formal power series whose expansion is purely periodic. Recall that an expansion $(d_n)_{n\geq 0}$ is said *purely periodic* (respectively *eventually periodic*) if there exists $T>0$ such that: $\forall n\geq 1,\ d_n = d_{n+T}$ (respectively there exist $T>0, n_0\geq 1$ such that: $\forall n\geq n_0,\ d_n = d_{n+T}$).

The following analogue of Lagrange's Theorem is classical.

Proposition 1 *Let f be an algebraic element over $\mathbb{F}_q(X)$. The series f is quadratic if and only if the continued fraction expansion of f is ultimately periodic.*

One can prove this result by following the proof in the real case as in [37], see also [48, 71]. Note that Lagrange's Theorem does not hold for any function field $k(X)$ [71].

Theorem 5 *Suppose that $f\in\mathcal{X}\setminus\mathbb{F}_q(X)$ is a quadratic power series and let $\overline{f}$ denote its algebraic conjugate. Then the continued fraction expansion of f is purely periodic if and only if $(f,\overline{f})\in\overline{\mathcal{X}}$, i.e.,* $\deg(f)<0$ *and* $\deg(\overline{f})>0$.

Proof Let $f\in\mathcal{X}$ be a quadratic power series and let $\overline{f}$ be its algebraic conjugate. We have $f\neq\overline{f}$ and $f\notin\mathbb{F}_q(X)$. We can extend the definition of the map $\overline{T}$ to a map on $\mathcal{X}\times\mathbb{F}_q((1/X))$:

$$(f,g)\mapsto([1/f]-1/f,[1/f]-1/g).$$

As $f\notin\mathbb{F}_q(X)$, then it is easy to see that $h=T(f)$ is quadratic (of algebraic conjugate $\overline{h}$, say) and that

$$\overline{T}(f,\overline{f})=(h,\overline{h}).$$

We will use the following notation:

$$\forall n\geq 0,\overline{T}^n(f,\overline{f}):\equiv(f^{(n)},\overline{f}^{(n)}).$$

The algebraic conjugate $\overline{f}^{(n)}$ of f^n equals $\overline{f}^{(n)}$.

- Suppose the continued fraction expansion of f is strictly periodic. Suppose that $\overline{f}\notin\mathbb{F}_q^0((1/X))$, i.e., $\deg(\overline{f})\leq 0$. We write

  $$f=[0;a_1(X),a_2(X),\cdots]_-$$

  and

  $$\overline{f}=c-[0;b_1(X),b_2(X),b_3(X),\cdots]_-,\ \text{where } c\in\mathbb{F}_q.$$

  Let us prove that there exists n such that $\overline{T}^{n+1}(f,\overline{f})\in\mathcal{X}$. Suppose $c=0$. Then we have

  $$(\overline{f})^1=(a_1(X)-b_1(X))-[0;b_2(X),b_3(X),\cdots]_-.$$

  If furthermore $a_1(X)-b_1(X)=0$, we see that

  $$(\overline{f})^2=(a_2(X)-b_2(X))-[0;b_3(X),b_4(X),\cdots]_-.$$

We thus consider the smallest integer n such that $a_n(X) \neq b_n(X)$. (Such an integer exists since $f \neq \overline{f}$.) If $c \neq 0$, we put $n = 0$. We have

$$(\overline{f})^n = (a_n(X) - b_n(X)) - [0; b_{n+1}(X), b_{n+2}(X), \cdots]_-.$$

Hence $\deg(\overline{f})^n \geq 0$ and $(\overline{f})^n \neq 0$.

Suppose $\deg(\overline{f}^n) = 0$. Then we put $c = a_n(X) - b_n(X)$: we have

$$(\overline{f})^n = c - [0; b_{n+1}(X), b_{n+2}(X), \cdots]_-, \ c \neq 0.$$

We deduce from Lemma 2 (3)

$$\begin{aligned}(\overline{f})^{n+1} &= a_{n+1}(X) - 1/(\overline{f})^n \\ &= a_{n+1}(X) - 1/c - [0; -c + c^2 b_{n+1}(X), c^{-2} b_{n+2}(X), \cdots]_-.\end{aligned}$$

Hence we have

$$\overline{T}^{n+1}(f, \overline{f}) \in \overline{\mathcal{X}}.$$

Suppose that f is strictly periodic, then there exists $m > n$ such that

$$\overline{T}^m(f, \overline{f}) = (f, \overline{f}) \not\in \overline{\mathcal{X}}.$$

On the other hand,

$$\overline{T}^m(f, \overline{f}) \in \overline{\mathcal{X}},$$

whenever $m > n$, which is impossible.

Consequently, it turns out that if f is purely periodic, then $(f, \overline{f})$ belongs to $\overline{\mathcal{X}}$.

- Now suppose the expansion of f is not purely periodic, say,

$$f = [0; a_1(X), a_2(X), \ldots, a_{k-1}(X), a_k(X), \underbrace{a_{k+1}(X), \ldots, a_{k+n}(X)}, \underbrace{a_{k+1}(X), \ldots, a_{k+n}(X)}, \ldots]_-$$

and

$$[0; a_{k-u}(X); a_{k-u+1}(X), \cdots, a_{k-1}(X), a_k(X), a_{k+1}(X), \ldots, a_{k+n}(X), a_{k+1}(X), \ldots, a_{k+n}(X), \ldots]_-$$

is not purely periodic for any $0 \leq u < k$. We can define a strictly periodic g by

$$g = [0; a_{k+1}(X), \ldots, a_{k+n}(X), a_{k+1}(X), \ldots, a_{k+n}(X), \ldots]_-.$$

We have from above

$$(g, \overline{g}) \in \overline{\mathcal{X}}.$$

Since $T^k g = T^k f$, we see that $\overline{T}^k(g, \overline{g}) = \overline{T}^k(f, \overline{f})$. On the other hand, $\overline{T}^k$ is 1-1 on $\overline{\mathcal{X}} \setminus (\mathbb{F}_q(X) \times \mathbb{F}_q^0((1/X)))$. Hence $(f, \overline{f})$ cannot belong to $\overline{\mathcal{X}}$, which ends the proof. ∎

Corollary 1 *If* $(f, \overline{f}) \in \overline{\mathcal{X}}$ *and*

$$f = [0; a_1(X), a_2(X), \cdots, a_n(X), a_1(X), a_2(X), \cdots, a_n(X), \cdots]_-,$$

then

$$\overline{f} = a_n(X) - [0; a_{n-1}(X), \cdots, a_2(X), a_1(X), a_n(X), a_{n-1}(X), \cdots, a_2(X), a_1(X), a_n(X), \cdots]_-.$$

Proof The proof comes from

$$\overline{T}^n(f,\overline{f}) = (f,\overline{f}).$$

■

Remark Note the following characterization of strictly periodic quadratic power series [56, 57]: a quadratic formal power series f of non zero integral part is purely periodic if and only if f satisfies an equation

$$Af^2 + Bf + C = 0,$$

with $A, B, C \in \mathbb{F}_q[X]$, $\deg(B) > \deg(A)$ and $\deg(B) > \deg(C)$.

6 Metric results

The purpose of this section is first to deduce, from the ergodicity of the transformations T_+, T_-, L, some classical distribution properties on the digits in the expansions, and then to compare these transformations from a metric theoretical point of view.

6.1 Ergodicity

As an application of the ergodicity, we obtain easily the following results. For what concerns the map T_+, see [28]. The case of the Lüroth series expansions is considered in [31]. These results can be sharpened as in [36]. For the corresponding results in the real case, see for instance [10].

Theorem 6 *The following results hold for all $f \in \mathcal{X}$ outside a set of Haar measure 0. Let $T \in \{T_-, T_+, L\}$. Let $a_n(f) = [\frac{1}{T^{n-1}(f)}]$, for $n \geq 1$.*

1. *Let $h(X) \in \mathbb{F}_q[X]$ and $\deg(h) \geq 1$. The digit value $h(X)$ has asymptotic frequency $q^{-2\deg(h(X))}$, i.e.,*

$$\lim_{N \to +\infty} \frac{card\{1 \leq n \leq N;\ a_n(f) = h(X)\}}{N} = q^{-2\deg(h(X))} \quad a.e.;$$

2. *there exists a Khintchine-type constant, namely $\frac{q}{q-1}$:*

$$\lim_{n \to +\infty} \frac{1}{n}(\deg(a_1(f)) + \cdots + \deg(a_n(f))) = \frac{q}{q-1} \quad a.e.,$$

i.e.,

$$\lim_{n \to +\infty} |a_1(f) \dots a_n(f)|^{1/n} = q^{\frac{q}{q-1}} \quad a.e.;$$

3. *the three transformations have entropy $\frac{2q}{q-1}$;*

4. *suppose $T = T_+$ or T_- and let $\frac{p_n}{q_n}$ denote the n-th convergent of f; we have*

$$\lim_{n\to+\infty} \frac{1}{n}\log_q |q_n| = \frac{q}{q-1} \quad a.e.,$$

$$\lim_{n\to+\infty} \frac{1}{n}(\log_q |f - \frac{p_n}{q_n}|) = -\frac{2q}{q-1} \quad a.e.;$$

suppose $T = L$ and let $\frac{s_n}{t_n}$ denote the n-th Lüroth convergent of f; we have

$$\lim_{n\to+\infty} \frac{1}{n}\log_q |q_n| = \frac{2q}{q-1} \quad a.e.,$$

$$\lim_{n\to+\infty} \frac{1}{n}(\log_q |f - \frac{s_n}{t_n}|) = -\frac{2q}{q-1} \quad a.e.$$

Proof To prove the first assertion, we apply the ergodic theorem to the characteristic function of the set $a_1^{-1}\{h(X)\}$. For assertion 2, we apply the ergodic theorem to the map $f \mapsto \deg(a_1(f))$.

Assertion 3 is a direct consequence of the Shannon-McMillan theorem [10].

Let us prove assertion 4. Note that

$$\deg(q_n) = \deg(a_n(f)) + \deg(q_{n-1}) = \sum_{1\leq i\leq n} \deg(a_i(f)).$$

Hence

$$\frac{1}{n}\log_q |q_n(X)| = \frac{1}{n}\sum_{1\leq i\leq n} \deg(a_i(f)).$$

We have furthermore from Equation (1)

$$|f - \frac{p_n(f)}{q_n(f)}| = \frac{1}{|q_n(f)|\;|q_{n+1}(f)|}.$$

We thus have

$$\begin{aligned} \log_q |f - \tfrac{p_n(f)}{q_n(f)}| &= -\log_q(q_n(f)) - \log_q(q_{n+1}(f)) \\ &= -2\textstyle\sum_{1\leq i\leq n} \deg(a_i(f)) - \deg(a_{n+1}(f)), \end{aligned}$$

hence the result.

Similarly,

$$\deg(t_n) = 2\sum_{1\leq i\leq n-1} \deg(a_i(f)) + \deg(a_n(f));$$

we get furthermore from Equation (2)

$$|f - \frac{s_n(f)}{t_n(f)}| = \frac{1}{|a_n(f)||a_{n+1}(f)||t_n(f)|},$$

i.e.,

$$\begin{aligned} \log_q |f - \tfrac{s_n(f)}{t_n(f)}| &= -\log_q(t_n(f)) - \log_q(a_n(f)) - \log_q(a_{n+1}(f)) \\ &= -2\textstyle\sum_{1\leq i\leq n} \deg(a_i(f)) - \deg(a_{n+1}(f)). \end{aligned}$$

∎

6.2 Remarks

- We have normalized the absolute value as follows:
$$|f| = q^{\deg(f)}.$$
Consider another absolute value equivalent to the previous one on $\mathbb{F}_q((1/X))$, and defined by
$$||f||_C = C^{\deg(f)},$$
where $C > 1$. Let us compute the expectation of the random variable $||a_1(f)||_C$. We have
$$\mu\{f;\ ||a_1(f)||_C = C^n\} = \frac{q-1}{q^n},$$
$$E(||c_1(f)||_C) = \sum_{n=1}^{+\infty} \frac{q-1}{q^n} C^n = \frac{(q-1)C}{q} \sum_{n\geq 0} (\frac{C}{q})^n.$$
Hence the expectation equals $+\infty$ if and only if $C \geq q$. This difference of behaviour of the expectation with respect to the normalization of the absolute value does not appear in the real case.

- We have seen (Theorem 6) that
$$\lim_{n\to+\infty} \frac{1}{n} \log_q |t_n| = 2 \lim_{n\to+\infty} \log_q |q_n|, \text{ a.e.},$$
and
$$\log|f - \frac{p_n}{q_n}| \sim \log|f - \frac{s_n}{t_n}|.$$
In this sense, the metrical constants appeared in Theorem 6 of continued fractions (either "+" or "−" expansions) and the corresponding constants for Lüroth series are the same. On the other hand, the next propositions show that the error term distributions are not the same to each other.

Proposition 2 *For a.e. $f \in \mathcal{X}$, we have*
$$\lim_{N\to+\infty} \frac{card\{n;\ 1 \leq n \leq N,\ |q_n|^2 |f - \frac{p_n}{q_n}| = \frac{1}{|q|^k}\}}{N} = \frac{q-1}{q^k},$$
for any $k \geq 1$. The same holds for "−" expansions.

Proof Because
$$|q_n|^2 |f - \frac{p_n}{q_n}| = |a_{n+1}|^{-1}$$
and the sequence (a_n) is an i.i.d. sequence with distribution $q^{-2\deg(h)}$ for any polynomial h (with $\deg(h) \geq 1$), the assertion is an easy consequence of the strong law of large numbers. ∎

Proposition 3 *For a.e. $f \in \mathcal{X}$, we have*
$$\lim_{N\to+\infty} \frac{card\{n;\ 1 \leq n \leq N,\ |t_n||f - \frac{s_n}{t_n}| = \frac{1}{|q|^k}\}}{N} = (q-1)^2 \frac{(k-1)}{q^k},$$
for any $k \geq 1$.

Proof This follows from the equality

$$|t_n||f - \frac{s_n}{t_n}| = |a_n|^{-1}|a_{n+1}|^{-1}$$

and the i.i.d. property of (a_n). ∎

Remark For the corresponding results in the real case, see [12] for the regular continued fractions and [29, 6, 16] for the Lüroth series in the case of real numbers. Because the absolutely continuous measure for T (in the real case) is not a finite measure, we can not get such a property for "−" continued fraction expansion of real numbers.

6.3 Full Schweiger systems and f-expansions

The study of f-expansions was essentially introduced by A. Renyi [70] in 1957. In his paper, the most important examples of f-expansions were continued fraction expansions, r-adic expansions, and β-expansions. Recently, Schweiger summarized in his book [72] a number of results obtained by many authors until the first half of 1990th, by introducing *fibred systems.* We can extend his unifying treatment to non-Archimedean fields f-expansions. Following Schweiger, we can define a full Schweiger system as follows. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and $\{\Omega_a;\ a \in A\}$ be a countable partition of Ω with the following properties:

- $\forall a \in A,\ \Omega_a \in \mathcal{B}$;
- $\mu(\Omega_a) > 0$.

We define a transformation T of Ω onto itself by the following

$$T\omega = T_a\omega,\ \text{if}\ \omega \in \Omega_a,\ a \in A,$$

where $T_a : \Omega_a \to \Omega$ is bijective, bi-nonsingular. We assume that the following holds

$$\text{for any } a \in A,\ \frac{\mathrm{d}T_a\mu}{\mathrm{d}\mu} = \mu(\Omega_a) \text{ a.e.}$$

We call such a system a *full Schweiger system.* For each $\omega \in \Omega$, we define

$$a_i(\omega) = a,\ \text{if}\ T^{i-1}(\omega) \in \Omega_a, \quad \text{for } i \geq 1.$$

Then it is not difficult to see that if $(\Omega, \mathcal{B}, \mu)$ is a full Schweiger system, then T is μ-preserving. Moreover, the sequence of digits $(a_i(.))$ is an i.i.d. sequence.

7 A brief incursion into the literature

Let us end this paper by surveying some of the many works devoted to continued fraction expansions for formal power series with coefficients in a finite field. If these works are mostly devoted to generalizations of classical results on continued fractions (most of the basic facts in the real case have analogies for formal power series), numerous results illustrate the fundamental differences between positive and zero characteristic, as for instance in Diophantine approximation with the failure of Roth's Theorem. The list of references given here although long, is far from being exhaustive.

Let us begin by some explicit computations of continued fraction expansions. In the real case very few explicit examples of continued fraction expansions are known; one can mostly expand power series or roots of some particular equations. Furthermore, it is a still open problem to determine if the set of partial quotients in the expansion of an algebraic number of degree greater than 2 is bounded. Only a few examples of such expansions are known (for more references see the survey [73]). The situation is drastically different here. Indeed Baum and Sweet [7] have produced a cubic series with partial quotients of bounded degree. We review in the next paragraph (devoted to results connected with Diophantine approximation) some more examples of algebraic series with bounded partial quotients issued from the Baum and Sweet series. Note also the explicit expansions in [4, 53, 68], and Thakur's results on the expansion of the Carlitz analogue of the exponential [76, 77, 78].

Various unique expansions of Laurent formal power series over a field F are introduced in [33, 34], as the sum of reciprocals of polynomials, involving digits in $F[X]$. Such expansions include Lüroth expansions dealt in detail from a metric point of view in [31, 36]. See also [35] for expansions as a product. For a generalization of Jacobi-Perron algorithm, see [17, 18, 19, 20].

7.1 Continued fractions and Diophantine approximation

It is well known that Liouville's Theorem holds in the context of Laurent series with coefficients in a finite field, but that Roth's Theorem fails, as proved by Mahler's counterexample [47] (see also [66]), which is of the form:

$$f = \frac{a(X)f^{p^k} + b(X)}{c(X)f^{p^k} + d(X)}, \tag{5}$$

where $k \in \mathbb{N}$, $a(X), b(X), c(X), d(X) \in \mathbb{F}_q[X]$ and $a(X)d(X) - b(X)c(X) \neq 0$. In particular there are numerous works devoted to the study of such examples of irrational algebraic series [7, 8, 38, 39, 40, 42, 48, 50, 59, 60, 71, 74, 75, 80, 81]. For a nice survey of Diophantine approximation in fields of power series, see [41].

Recall in particular the remarkable example of Baum and Sweet [7], i.e., a cubic series the continued fraction expansion of which has partial quotients with bounded degree; see also [8]. Mills and Robbins have extended Baum and Sweet's approach to produce in [60] explicit expansions of algebraic elements in characteristic $p > 2$ for which the degrees of the partial quotients are bounded: they construct an efficient algorithm which produces the continued fraction expansion for a series f satisfying an equation of the form (5). For a study of rational approximations of such series and connected results see [13, 38, 40, 42, 43, 50, 80, 81]. A description of the approximation spectrum of those algebraic series which satisfy (5) is given in [71]. See also [79] for explicit continued fraction expansions for algebraic series with prescribed rational approximation exponent (including a nice review of Diophantine approximation for Laurent formal power series). Note the criterion in [39] for α's to satisfy Roth's Theorem, especially for $q = 2$.

Simultaneous approximation properties are considered in [49, 74, 75] for pairs of algebraic series, and also in [44] for a non-Archimedean analogue of the classical Kronecker sequences. See also [65] for connections with low-discrepancy sequences.

7.2 Continued fractions with bounded partial quotients

As in the real case (see for instance the survey [73]), there is an abundant literature devoted to continued fraction expansions with partial quotients of small degree, bearing an extremely close analogy to the real case.

Some classical problems (see [52] for a list of those problems in the real case) concerning limited continued fraction expansions (i.e., the rational and the periodic case) have been explored. Mkaouar gives an upper-bound for the period of a quadratic formal power series [56, 58]. Grisel uses in [24] an algorithm due to Mendès France [51] (which multiplies the continued fraction expansion of a formal power series by a rational fraction $A/B \in \mathbb{F}_q(X)$) in order to investigate the length of the continued fraction expansion of α^n as n tends to infinity, for a rational fraction α. The distribution of the length of the continued fraction expansion of a rational series and asymptotic estimates for the average degree of the partial quotients are given in [30]. Note also [32] in connection with the length of the Euclidean algorithm. For other results connected to the degree (and to its distribution) of the partial quotients of a rational series, see [22], see also [54, 62].

As an application, there are some remarkable relationships with pseudorandom numbers generated by the digital multistep method [61], and with low-discrepancy sequences [65]. Niederreiter gives a description in [65] of a general principle for the construction of (t, m, s)-nets and (t, s)-sequences (i.e., high-dimensional point sets or sequences extremly well distributed with respect to special classes of subintervals), with explicit connections with continued fraction expansions (note the problems asked at the end of [61]). Indeed the function which associates to a rational series the value of the maximum of the degrees of its partial quotients plays a fundamental rôle in the construction of (t, m, s)-nets. Probabilistic theorems on the distribution of sequences constructed by the digital method are also given in [44].

Several papers (for instance [11, 45] and also [13, 62, 45]) investigate the notion of orthogonal multiplicity (the orthogonal multiplicity of a monic polynomial g is the number of polynomials f coprime to g and of degree less than that of g, such that all the partial quotients of f/g have degree one). The polynomials which occur as the denominators of rational series whose partial quotients have all degree one (i.e., the polynomials of positive orthogonal multiplicity) occur in several fields, as reviewed in [11, 45] and [46] (where more generally families of continued fractions expansions are considered with partial quotients all lying in a given set). In particular polynomials of positive orthogonal multiplicity appear in stream cipher theory as minimal polynomials of initial segments of sequences with perfect linear complexity profiles [63].

The linear complexity profile of a sequence $(s_n)_{n\in\mathbb{N}}$ with values in $\mathbb{F}_q$ is defined as the sequence $(L_N)_{N\in\mathbb{N}}$, where L_N denotes the length of the shortest linear recurrence satisfied by $(s_n)_{n\leq N}$. The jumps profile, i.e., the sequence $(L_N - L_{N-1})_{N\in\mathbb{N}}$, (with $L_{-1} = 0$), is intimately connected with the degrees of the partial quotients of the series $\sum_n s_n X^{-n}$ [63, 82]. Furthermore a linear complexity profile is said perfect if the jumps have size 1 or equivalently if the partial quotients have degree 1. See [63, 82] for a method of construction of sequences with prescribed complexity profile and [63] for a criterion for a sequence to have a perfect linear complexity profile.

The algebraic series with bounded partial quotients produced respectively by Baum and Sweet [7], and by Mills and Robbins [60], have raised many interrogations, in particular concerning the automaticity of their coefficients. Indeed recall Christol's criterion of algebraicity based on automata theory ([14] and also [15]): a formal power series f with coefficients in a finite field $\mathbb{F}_q(X)$ is algebraic if and only if the sequence of its coefficients is q-automatic (i.e., generated by a finite automaton). In the case where the partial quotients in the continued fraction expansion of f take finitely many values, Mendès France asked whether this sequence is itself p-automatic. A positive answer to this question has been given in [2, 3] in the case of Mills and Robbins examples [60] in characteristic > 2. But Mkaouar [55] (see also [83]) showed that the Baum and Sweet series [7] provides a negative answer to the question asked by Mendès France. Note the connection in [45] with folded continued fractions [69].

7.3 Connections with arithmetic

The theory of Laurent formal power series contains many analogues of classical objects deeply connected with continued fractions, as for instance analogues of the Poincaré upper half-plane, modular forms, cusp forms [23]... Explicit formulas for the analytic continuation of the Selberg zeta function for the "modular" group $PGL(2, \mathbb{F}_q[X])$ are given in [37] (see also [1]).

Continued fraction expansions play a natural rôle in the study of quadratic function fields. They have been introduced for that purpose by Artin [5]. In particular continued fractions are used to give explicit formulas for class numbers of quadratic function fields [5, 21, 25, 27]. A characterization of quasi-periodicity (i.e., periodicity up to a non-zero multiplicative constant) for an algebraic function in a hyperelliptic function field is given in [9, 67, 27], in connection with the Pell equation. See also [71] for an algebraic characterization of quasiperiodicity using $SL(2, \mathbb{F}_q[X])$ equivalence.

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