About thin arithmetic discrete planes

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Abstract

Arithmetic discrete planes are sets of integer points located within a fixed bounded distance (called thickness) of a Euclidean plane. We focus here on a class of "thin" arithmetic discrete planes, *i.e.*, on a class of arithmetic discrete planes whose thickness is smaller than the usual one, namely the so-called standard one. These thin arithmetic discrete planes have "holes" but we consider a thickness large enough for these holes to be bounded. By applying methods issued from the study of tilings and quasicrystals derived from cut and project schemes, we first consider configurations that occur in thin arithmetic discrete planes. We then discuss substitution rules acting on thin discrete planes, with these geometric rules mapping faces of unit cubes to unions of such faces.

Key words:
digital planes; arithmetic discrete planes; tilings; quasicrystals; cut and project method; substitutions; word combinatorics.

1 Introduction

If arithmetic discrete planes are among the most simple and natural objects in discrete geometry, their study benefits from the various viewpoints under which they can be considered. In particular, discrete planes can be described as codings of simple dynamical systems of an arithmetic flavor (see *e.g.* the survey [Ber10]), or else, they can be seen as simple but nontrivial models of quasicrystals, such as discussed in [Ber09]. The present paper aims at being an illustration of this richness of approaches and methods used in the study of arithmetic discrete planes.

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More precisely, according to [Rev91] in the case of lines, and then to [AAS97] for (hyper)planes, arithmetic discrete hyperplanes are defined as follows. Let $\vec{v} = (v_1, v_2, \ldots, v_d) \in \mathbb{R}^d$, $\mu, \omega \in \mathbb{R}$. The *arithmetic discrete (hyper)plane* $\mathfrak{P}(\vec{v}, \mu, \omega)$ is the set of points $\vec{x} \in \mathbb{Z}^d$ satisfying

$$0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < \omega \ ,$$

where the notation $\langle \, . \, , \, . \, \rangle$ stands for the scalar product. The parameter ω is called the *thickness* of the arithmetic discrete plane, and the interval $[0, \omega)$ is called the *selection window*. For more on their properties, see *e.g.* the survey [BCK07]. Two thicknesses are frequently studied, namely the naive one $\omega = ||\vec{v}||_\infty$ and the standard one $\omega = ||\vec{v}||_1$.

We will focus here on "small" thicknesses. Such a small thickness creates "holes", such as illustrated in Figure 1. If the thickness is to small, these holes can even be unbounded. The aim of this paper is to study thin arithmetic discrete planes under the assumption that these holes are bounded. In particular, we provide a description of local configurations in terms of intervals of the selection window $[0, \omega)$.

An efficient strategy for the study of naive planes consists in exploiting their functionality (see for instance [Rev91,DRR94,AAS97,VC97,VC99]). Indeed, naive planes are well known to be functional, that is, in a one-to-one correspondence with the integer points of one of the coordinate planes by an orthogonal projection map. The notion of functionality for naive arithmetic discrete planes can be extended to a larger family of arithmetic discrete planes, such as described in [BFJP07], by introducing a suitable projection mapping. Functionality allows the reduction of a three-dimensional problem to a two-dimensional one, and thus leads to a better understanding of the combinatorial and geometric properties of arithmetic discrete planes. We propose here an alternative strategy to the functional one developed in [BFJP07] for the study of arithmetic discrete planes that are not necessarily naive or standard. Instead of taking a suitable projection mapping, we continue to work with the standard selection window of size $||\vec{v}||_1$, but we compare our selection window $[0, \omega)$, for $\omega < ||\vec{v}||_1$, with the standard one $[0, ||\vec{v}||_1)$.

This paper is organized as follows. Section 2 discusses arithmetic discrete planes, associated tilings, and generalized faces: this latter notion aims at formalizing the holes that occur in thin arithmetic discrete planes. We then show in Section 3 how to associate with generalized faces intervals of the selection window $[0, \omega)$: this is one of the main tools of the present paper, that we extend to configurations in Section 4. In particular, we will show how to decompose thin arithmetic discrete planes into unions of generalized faces (see Theorem 6). We then will try to "compare" discrete planes having different normal vectors. We thus handle in full details in Section 5 an example of a substitution rule acting on discrete planes, and whose action is described with respect to their normal vector $\vec{v}$.

This paper is an extended version of [Ber09]: it can be seen as an illustration of the way methods discussed in [Ber09] can be applied to the case of a thickness ω that satisfies $v_1+v_3 \leq \omega < ||\vec{v}||_1$, with $\vec{v} = (v_1, v_2, v_3)$, and $0 \leq v_1 \leq v_2 \leq v_3$. This lower bound on ω is a sufficient condition for having bounded holes (see Proposition 5).

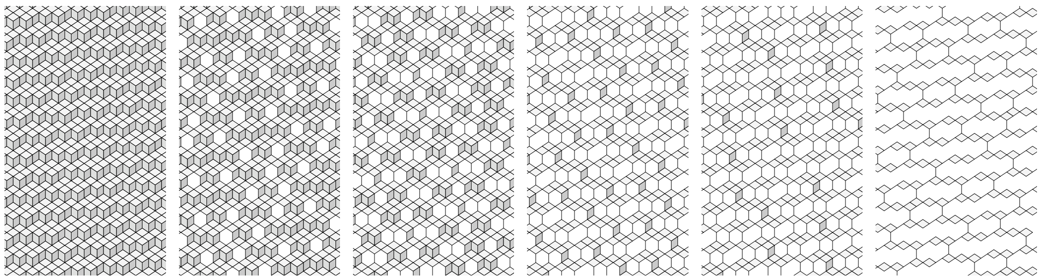


Figure 1. Planes with normal vector $\vec{v} = (v_1, v_2, v_3) = (1, \sqrt{2}, \pi)$ with decreasing thickness ω. Left: standard thickness $\omega = ||\vec{v}||_1$. The two following ones have thickness in $[v_2+v_3, v_1+v_2+v_3)$, the next two ones have thickness in $[v_1+v_3, v_2+v_3)$, the last one has thickness in $[0, v_1+v_3)$. For more explanation on the way edges are chosen to connect points of these arithmetic discrete planes, see Section 2.

2 Faces of arithmetic discrete planes

In this section we introduce basic material on arithmetic discrete planes.

2.1 Arithmetic discrete planes

We first recall the definition of an arithmetic discrete plane.

Definition 1 (Arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$) *Let $\mu, \omega \in \mathbb{R}$ and $\vec{v} = (v_1, v_2, \ldots, v_d) \in \mathbb{R}^d$. The* arithmetic discrete (hyper)plane $\mathfrak{P}(\vec{v}, \mu, \omega)$ *is defined as the set of points $\vec{x} \in \mathbb{Z}^d$ satisfying*

$$0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < \omega \ .$$

Parameter $\vec{v}$ is called normal vector*, ω is called the* thickness*, and μ is called the* translation parameter*.*

The vector $\vec{v}$ is assumed in all that follows to be a nonzero vector with nonnegative coordinates. We work in dimension $d = 3$ but the results and methods of the present paper hold for any larger dimension. We consider here integer as well as irrational parameters $\vec{v}, \mu, \omega$.

There exist two thicknesses ω which play a particular role in the study of arithmetic discrete planes. If $\omega = ||\vec{v}||_\infty = \max(v_1, v_2, v_3)$, then the arithmetic discrete plane is said to be *naive*, whereas if $\omega = ||\vec{v}||_1 = v_1 + v_2 + v_3$, then the arithmetic discrete plane is said to be *standard*. We thus call *naive thickness*, the value $||\vec{v}||_\infty$, and *standard thickness*, the value $||\vec{v}||_1$. As an illustration of the fact that naive and standard thicknesses provide natural objects, note that points of a naive (resp. standard) arithmetic discrete line are connected by horizontal and vertical (resp. horizontal and diagonal) segments. Both notions are strongly related as shown *e.g.* in [SDC04]. More precisely, the correspondence between both types of planes works as follows, by using the formalism and terminology of the topology based on abstract cellular complexes introduced in [Kov89], and recalled in [SDC04]: consider the points in $\mathbb{Z}^3$ of a naive plane with normal vector $\vec{v}$ as voxels; then, the pointels of its surface elements form a standard plane with same normal vector $\vec{v}$. We thus consider in all that follows that the points of $\mathbb{Z}^3$ that make a standard arithmetic discrete plane are pointels of a discrete surface composed of surfels. This leads us to introduce Definition 2 below. But before stating it, we need the following notation.

Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ stand for the canonical basis of $\mathbb{R}^3$. We consider the following

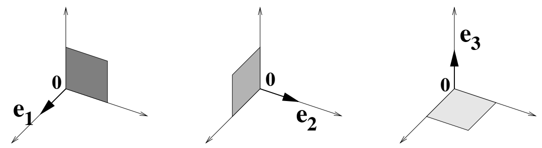


Figure 2. Faces F_1 (left), $F_2 + \vec{e}_1$ (middle), and $F_3 + \vec{e}_2 + \vec{e}_3$ (right).

faces of the unit cube (see Figure 2):

$$F_1 = \{\lambda\vec{e}_2 + \mu\vec{e}_3 \mid 0 \leq \lambda, \mu \leq 1\},$$

$$F_2 = \{-\lambda\vec{e}_1 + \mu\vec{e}_3 \mid 0 \leq \lambda, \mu \leq 1\},$$
$$F_3 = \{-\lambda\vec{e}_1 - \mu\vec{e}_2 \mid 0 \leq \lambda, \mu \leq 1\}.$$

Definition 2 (Stepped plane $\mathcal{P}(\vec{v}, \mu)$) *The stepped plane $\mathcal{P}(\vec{v}, \mu)$ is defined as the union of integer translates of faces of the unit cube whose vertices belong to the standard plane $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$.*

The leftmost image of Figure 1 is an example of a stepped plane.

An arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$ with $\dim_{\mathbb{Q}} \vec{v} = 1$ is called *rational*, otherwise it is called *irrational*, according to [AAS97,BFJP07]. From now on, we shall agree that any representation $\mathfrak{P}(\vec{v}, \mu, \omega)$ of a rational arithmetic discrete plane satisfies:

$$\vec{v} \in \mathbb{Z}^3 \text{ and } \gcd(\vec{v}) = 1, \quad \mu \in \mathbb{Z}, \quad \omega \in \mathbb{N}^\star.$$

We recall that the dimension of the lattice of the period vectors of an arithmetic discrete plane is equal to the dimension of the space minus the dimension of the $\mathbb{Q}$-vector space generated by the coordinates of the normal vector $\vec{v}$.

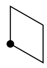
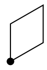
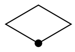
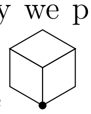
2.2 From discrete planes to tilings

Let $\mathfrak{P}(\vec{v},\mu,\omega)$ be an arithmetic discrete plane, with $\vec{v}=(v_1,v_2,v_3)$ being a nonzero vector with nonnegative coordinates. We first assume in this section that we are in the standard case $\omega=||\vec{v}||_1$.

Recall that the faces of the unit cube are labeled as

$$F_1=\{\lambda\vec{e}_2+\mu\vec{e}_3 \mid 0\le\lambda,\mu\le 1\},$$

$$F_2=\{-\lambda\vec{e}_1+\mu\vec{e}_3 \mid 0\le\lambda,\mu\le 1\},$$
$$F_3=\{-\lambda\vec{e}_1-\mu\vec{e}_2 \mid 0\le\lambda,\mu\le 1\}.$$

In order to point faces, we pick the origin for each face F_i as a particular vertex, and we call it its *distinguished vertex*. Furthermore, for $\vec{x}\in\mathbb{Z}^3$, the distinguished vertex of the integer translate $\vec{x}+F_i$ of the face F_i is defined as $\vec{x}$. This is depicted as follows:  for the face F_1,  for the face F_2 , and lastly  for the face F_3, where the black dot denotes the origin. As an illustration of the way we point faces (we will use it in Section 3.4), the following upper unit cube  is equal to the union $(-\vec{e}_2+F_1)\cup F_2\cup(\vec{e}_3+F_3)$ (the black dot is again located at the origin).

In order to better understand and visualize the stepped plane $\mathcal{P}(\vec{v},\mu)$ we project it orthogonally onto the vectorial plane P_0 with normal vector $(1,1,1)$. We denote by π_0 this projection. This construction will prove its efficiency in Section 3.3 for smaller thicknesses when we will try to formalize the notion of holes created by reducing the thickness ω.

A *tiling* by translation of the plane by a set T of (proto)tiles is a union of translates of elements of T that covers the full space, with any two tiles intersecting either on an empty set, on a vertex, or on an edge. For more details on tilings, see for instance [GS87]. By applying the projection π_0 to $\mathcal{P}(\vec{v},\mu)$ one gets a tiling of the plane P_0 by three kinds of tiles, namely the three regular lozenges being the projections by π_0 of the three faces F_i $(i=1,2,3)$ of the unit cube. We call them $T_i=\pi_0(F_i)$, for $i=1,2,3$. Similarly as for faces, the distinguished vertex of the tile $\vec{y}+T_i$ is defined as $\vec{y}$.

Definition 3 (Tiling $T(\vec{v},\mu)$) *The tiling $T(\vec{v},\mu)$ associated with the stepped* *plane $\mathcal{P}(\vec{v},\mu)$ is the tiling with set of prototiles T_1, T_2, T_3 obtained by applying the projection π_0 to $\mathcal{P}(\vec{v},\mu)$.*

One has a one-to-one correspondence between tiles $\vec{y}+T_i$ of the tiling $T(\vec{v},\mu)$ and faces $\vec{x}+F_i$ in $\mathbb{R}^3$ of the stepped plane $\mathcal{P}(\vec{v},\mu)$: indeed, one easily checks that for any tile $\vec{y}+T_i$ of the tiling $T(\vec{v},\mu)$, there exists a unique $\vec{x}$ such that $\pi_0(\vec{x})=\vec{y}$ and $0 \le \langle \vec{x},\vec{v}\rangle + \mu < ||\vec{v}||_1$.

Remark 1 *More generally, any tiling made of the three lozenge tiles T_i, for $i=1,2,3$, admits a unique lifting as a surface in $\mathbb{R}^3$ up to translation by the vector $(1,1,1)$, with this lifting being equal to $\mathcal{P}(\vec{v},\mu)$ if the tiling equals $T(\vec{v},\mu)$. The idea of the proof is to associate with every vertex of the tiling a height function that is uniquely determined and whose definition is globally consistent. For more details, see [Thu89] and for a proof in this context, see [ABFJ07]. Tilings by the three tiles T_i ($i=1,2,3$) are widely studied in the framework of dimers on the honeycomb graph (see [KO05]).*

Remark 2 *The discrete set of points $\pi_0(\mathfrak{P}(\vec{v},\mu,||\vec{v}||_1))$ of the plane P_0 has a priori no specific algebraic structure (unless $\vec{v}$ has rational entries; in this latter case this set of points is periodic). Nevertheless, it is proved in [BV00] that the set of distinguished vertices of tiles of $T(\vec{v},\mu)$ is a two-dimensional lattice.*

Arithmetic discrete planes and their associated tilings enter the framework of *cut and project constructions*: such constructions consist in projecting a subset that has been selected by slicing a higher dimensional lattice, and are widely used as an efficient method for constructing tilings. Indeed, arithmetic discrete planes are obtained by selecting points of the lattice $\mathbb{Z}^3$ in a slice of width ω of $\mathbb{Z}^3$ along the Euclidean plane with equation $\langle \vec{x},\vec{v}\rangle + \mu = 0$. This is the *cutting* part of the construction. We then obtain a tiling by projecting these points by π_0. We recover via this construction a so-called *quasicrystal*, that is, a discrete structure which displays long-range order without having to be periodic. For more details, see *e.g.* [Sen95,BM2000]. According to this framework, we introduce the following terminology.

Definition 4 *The interval $[0,\omega)$ is called the* selection window*.*

2.3 Nonstandard case and generalized faces

We now consider the case of a thickness ω that is smaller than the standard one. Since $\omega < ||\vec{v}||_1$, one retrieves $\mathfrak{P}(\vec{v},\mu,\omega)$ from the stepped plane $\mathcal{P}(\vec{v},\mu)$ by removing some vertices, edges, and faces. See Figure 1 for an illustration. The question is now to be able to describe $\mathfrak{P}(\vec{v},\mu,\omega)$ similarly as what has been done in the standard case. The key point is to be able to formalize the notion of hole.

A convenient way to do this is to keep in mind the fact that the stepped plane $\mathcal{P}(\vec{v}, \mu)$ is endowed in a natural way with a structure of a two-dimensional discrete manifold as a simplicial complex made of point-cells, edge-cells, surface-cells. We have focused so far either on its surface-cells, namely the faces of unit cubes it is made of, or on its point-cells, *i.e.*, the pointels of $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$. However, when reducing the width ω of an arithmetic discrete plane, some vertices are taken out: it is natural to consider that some edges do not have to be taken into account. This leads us to introduce the following notion of edges of discrete planes, by using the notation

$$E_i := \{\lambda \vec{e}_i \mid 0 \leq \lambda \leq 1\}, \quad \text{for } i = 1, 2, 3$$

for edges of faces of the unit cube.

Definition 5 (Edges) *The* edges *of $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$ are defined as the edges of the faces of unit cubes that are contained in $\mathcal{P}(\vec{v}, \mu)$.*

Let $\omega \leq ||\vec{v}||_1$. The set of edges of $\mathfrak{P}(\vec{v}, \mu, \omega)$ is defined as the subset of edges of $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$ for which both endpoints do belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$.

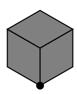
We now have gathered all the required material for being able to define generalized faces.

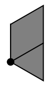
Definition 6 (Generalized face) *A* generalized face *G is defined as an edge-connected union of integer translates of faces F_i ($i = 1, 2, 3$) such that the restriction of the projection π_0 on G is onto.*

The set of edges of a generalized face G is the set of edges of the faces that compose it. The outer edges *of G are the edges whose projection by π_0 belong to the boundary of $\pi_0(G)$. The remaining edges of faces of unit cubes that are included in G are called* inner edges.

A generalized face is said to be finite *if it is made of a finite union of faces.*

Let ω satisfy $0 \leq \omega \leq ||\vec{v}||_1$. A generalized face G is said to be included in $\mathfrak{P}(\vec{v}, \mu, \omega)$ if its outer edges are all edges of $\mathfrak{P}(\vec{v}, \mu, \omega)$, and if either G is reduced to a single face of a unit cube, or if one of its inner edges is not an edge of $\mathfrak{P}(\vec{v}, \mu, \omega)$.

As an example of a generalized face, consider  which is equal to the union $(-\vec{e}_2 + F_1) \cup F_2 \cup (\vec{e}_3 + F_3)$ (the black dot is again located at the origin). The edges E_3, $\vec{e}_3 - E_1$ and $\vec{e}_3 - E_2$ are inner edges. The following union of faces

$F_1 \cup F_2$  is not a generalized face: the restriction of π_0 to this union of faces is not onto. Note that the generalized face $(-\vec{e}_2 + F_1) \cup F_2 \cup (\vec{e}_3 + F_3)$ occurs in the four planes depicted in Figure 1 whose thickness ω belongs to $[v_1 + v_3, v_1 + v_2 + v_3)$.

Definition 7 (Tiling $T(\vec{v}, \mu, \omega)$) *For ω satisfying $0 \leq \omega \leq ||\vec{v}||_1$, we define $T(\vec{v}, \mu, \omega)$ as the* tiling *made of the projections by π_0 of the generalized faces of $\mathfrak{P}(\vec{v}, \mu, \omega)$.*

Note that this terminology is consistent with Definition 3: if $\omega = ||\vec{v}||_1$, then $T(\vec{v}, \mu, ||\vec{v}||_1) = T(\vec{v}, \mu)$. We call *generalized tile* of $T(\vec{v}, \mu, \omega)$ a projection of a generalized face. Furthermore, one easily notices that the projections by π_0 of generalized faces of $\mathfrak{P}(\vec{v}, \mu, \omega)$, *i.e.,* generalized tiles, are connected components of the complement in the plane P_0 (identified with $\mathbb{R}^2$) of the projection by π_0 of the union of edges of $\mathfrak{P}(\vec{v}, \mu, \omega)$. In other words, a generalized tile is a facet of this union of projected edges seen as a planar graph, and can be considered a "projection of a hole" in the arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$.

The generalized tile T is said to *occur* in $T(\vec{v}, \mu, \omega)$ at point $\vec{y} \in P_0$ if there exists $\vec{x} \in \mathbb{Z}^3$ such that $\vec{y} = \pi_0(\vec{x})$, and a generalized face G such that $T = \pi_0(G)$, with the generalized face $\vec{x} + G$ being included in $\mathfrak{P}(\vec{v}, \mu, \omega)$.

If ω is small enough, there might be some infinite generalized faces. In all that follows, we work with the following assumption:

We assume that all generalized faces of $\mathfrak{P}(\vec{v}, \mu, \omega)$ are finite.

A sufficient condition for this property to hold is given in Proposition 5. Note that the tiling $T(\vec{v}, \mu, \omega)$ can have possibly infinitely many tiles. Note also that under the previous assumption, generalized tiles are polygonal tiles. We will give in Section 3.3 a sufficient condition for this assumption to hold.

3 From faces to intervals of the selection window

The aim of this section is to introduce the *localization method* which consists in localizing the values taken by $\langle \vec{x}, \vec{v} \rangle + \mu$ in the selection window $[0, \omega)$ for the distinguished vertices $\vec{x}$ of faces of a given type.

3.1 Faces and intervals

We first come back to the standard case in order to illustrate the method. It is based on Theorem 1 below. Indeed, our convention for the choice of a distinguished vertex of a face implies the following simple classic localization result in the standard case:

Theorem 1 *[BV00] For $i \in \{1, 2, 3\}$, the face $\vec{x} + F_i$ is included in $\mathcal{P}(\vec{v}, \mu)$ if and only if $\langle \vec{x}, \vec{v} \rangle + \mu \in I_{F_i}$, where we have cut the selection window $[0, ||\vec{v}||_1)$ into the three subintervals*

$$I_{F_1} = [0, v_1),\ I_{F_2} = [v_1, v_1 + v_2),\ I_{F_3} = [v_1 + v_2, v_1 + v_2 + v_3).$$

For more details, see [BV00]. The proof is recalled here in order to better understand the nonstandard case in Section 3.3.

Proof. By definition, one has $\vec{x} \in \mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$ if and only if $0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < ||\vec{v}||_1 = v_1 + v_2 + v_3$.

We use the fact that the four vertices of a face belong to $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$ if and only if the corresponding face is included in $\mathcal{P}(\vec{v}, \mu)$.

Assume first that $0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < v_1$. Then $\vec{x} + \vec{e}_2$, $\vec{x} + \vec{e}_3$, $\vec{x} + \vec{e}_2 + \vec{e}_3$ all belong to $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$. We thus deduce that the full face $F_1 + \vec{x}$ is included in $\mathcal{P}(\vec{v}, \mu)$.

Similarly, assume $v_1 \leq \langle \vec{x}, \vec{v} \rangle + \mu < v_1 + v_2$ (resp. $v_1 + v_2 \leq \langle \vec{x}, \vec{v} \rangle + \mu < v_1 + v_2 + v_3$). Then $\vec{x} - \vec{e}_1$, $\vec{x} + \vec{e}_3$, $\vec{x} - \vec{e}_1 + \vec{e}_3$ (resp. $\vec{x} - \vec{e}_1$, $\vec{x} - \vec{e}_2$, $\vec{x} - \vec{e}_1 - \vec{e}_2$) all belong to $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$. We thus deduce that the full face $F_2 + \vec{x}$ (resp. $F_3 + \vec{x}$) is included in $\mathcal{P}(\vec{v}, \mu)$.

We thus have proved for $\vec{x} \in \mathbb{Z}^3$ and for $i = 1, 2, 3$ that if

$$\sum_{k<i} v_k \leq \langle \vec{x}, \vec{v} \rangle + \mu < \sum_{k \leq i} v_k \text{ then } \vec{x} + F_i \subset \mathcal{P}(\vec{v}, \mu).$$

The converse is established following the same lines. ■

3.2 Frequencies

More can be deduced from this simple localization result. We first need a preliminary definition.

The *frequency of occurrence* of a generalized face G in $\mathfrak{P}(\vec{v}, \mu, \omega)$ is defined as the limit, if it exists, of the number of occurrences of $T = \pi_0(G)$ in central patterns of the tiling $T(\vec{v}, \mu, \omega)$:

$$\lim_{n \to \infty} \frac{\mathrm{Card}\{\vec{y} \in [\![-n, n]\!]^2,\ T \text{ occurs at } \vec{y} \text{ in } T(\vec{v}, \mu, \omega)\}}{(2n+1)^2}.$$

Let us recall a simple statement that will be used in Section 3.3 and 4.1 when studying frequencies of generalized faces:

Lemma 2 *If $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$ is rational, then $\{\langle \vec{x}, \vec{v} \rangle + \mu \mid \vec{x} \in \mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)\} = \{0, \cdots, ||\vec{v}||_1 - 1\}$, and otherwise, the set $\{\langle \vec{x}, \vec{v} \rangle + \mu \mid \vec{x} \in \mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)\}$ is dense, and even equidistributed, in the selection interval $[0, ||\vec{v}||_1)$.*

Proof. The first statement is a direct consequence of Bezout's lemma together with the fact that the coordinates of $\vec{v}$ are assumed to be coprime. The second statement is a direct consequence of the fact the sequence $(\{n\alpha\})_n$ is dense, and even equidistributed in $(0, 1)$, as soon as α is irrational. ■

We thus can already deduce the following corollary as a simple consequence of Theorem 1 and Lemma 2 in the standard case.

Corollary 3 *Let $i \in \{1, 2, 3\}$. The frequency of occurrence of the face F_i in the standard arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$ is equal to v_i.*

Proof. If the arithmetic discrete plane is rational, we use Bezout's lemma together with the fact that the coordinates of $\vec{v}$ are assumed to be coprime. Otherwise, we use the equidistribution properties of the sequence $(\{n\alpha\})_n$, where α is an irrational number (see Lemma 2). ■

We also recall the classic following statement that will be used in the next section. For more details, see *e.g.* [Sla67].

Theorem 4 *Let α be an irrational number in $(0, 1)$ and let I be an interval of $[0, 1)$. The sequence $(n\alpha)_{n \in \mathbb{N}}$ enters the interval I with bounded gaps, that is, there exists $N \in \mathbb{N}$ such that any sequence of N successive values of the sequence $(n\alpha)_{n \in \mathbb{N}}$ contains a value in I.*

3.3 Back to the nonstandard case

Our aim is to be able to obtain a statement analogous to Theorem 1 for

$\omega < ||\vec{v}||_1$, that is, to cut the selection window into a finite number of intervals, and to associate with each of these intervals at least a finite set of edges, or even a generalized face, such as defined in Section 2.3.

We will not handle in full generality the case $\omega < ||\vec{v}||_1$. Indeed, connectivity issues which are not trivial introduce a further level of complexity in the problem. We will restrict ourselves to parameters ω and $\vec{v} = (v_1, v_2, v_3)$ satisfying

$$0 \leq v_1 \leq v_2 \leq v_3, \quad \text{and} \quad v_1 + v_3 \leq \omega \leq v_1 + v_2 + v_3.$$

We will see (below with Proposition 5) that this condition is a sufficient condition for the generalized faces of $\mathfrak{P}(\vec{v}, \mu, \omega)$ to be finite. Our motivation is mainly to illustrate the power of the localization method in the flavor of Theorem 1. These restrictions on ω will become clearer with Theorem 6 and the following proposition.

Proposition 5 *Let ω and $\vec{v} = (v_1, v_2, v_3)$ satisfying*

$$0 \leq v_1 \leq v_2 \leq v_3, \quad \textit{and} \quad v_1 + v_3 \leq \omega \leq v_1 + v_2 + v_3.$$

The generalized faces of $\mathfrak{P}(\vec{v}, \mu, \omega)$ are finite.

PROOF. We assume $v_1 + v_3 \leq \omega \leq v_1 + v_2 + v_3$. Let $\vec{x} \in \mathbb{Z}^3$ such that $\vec{x} \notin \mathfrak{P}(\vec{v}, \mu, \omega)$ and $\vec{x} \in \mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$, *i.e.*, $\omega \leq \langle \vec{x}, \vec{v} \rangle + \mu < ||\vec{v}||_1$. We first note that if $\vec{x} \pm \vec{e_i}$ also belongs to $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1) \setminus \mathfrak{P}(\vec{v}, \mu, \omega)$, then $i = 1$. Indeed, one has $\vec{x} - \vec{e}_3 \in \mathfrak{P}(\vec{v}, \mu, \omega)$, since $\langle \vec{x} - \vec{e}_3, \vec{v} \rangle + \mu = \langle \vec{x}, \vec{v} \rangle + \mu - v_3 \in [\omega - v_3, v_1 + v_2)$. One has similarly $\vec{x} - \vec{e}_2 \in \mathfrak{P}(\vec{v}, \mu, \omega)$. Furthermore, $\vec{x} + \vec{e}_3, \vec{x} + \vec{e}_2 \notin \mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$, since $\omega \geq v_1 + v_3$. We thus have proved that if $\vec{x} \pm \vec{e_i}$ also belongs to $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1) \setminus \mathfrak{P}(\vec{v}, \mu, \omega)$, then $i = 1$. This implies that the generalized faces of $\mathfrak{P}(\vec{v}, \mu, \omega)$ are all finite. Otherwise, there would exist an infinite sequence of points $(\vec{x}_n)_{n \in \mathbb{N}}$ with values in $\mathbb{Z}^3$ such that, for all n, $\omega \leq \langle \vec{x}_n, \vec{v} \rangle + \mu < ||\vec{v}||_1$ and $\vec{x}_{n+1} - \vec{x}_n \in \{\pm \vec{e_i} \mid i = 1, 2, 3\}$. From what precedes, one deduces that $\vec{x}_{n+1} - \vec{x}_n = \pm \vec{e}_1$ for all n. We then get a contradiction by applying Theorem 4 to the subinterval $[0, \omega)$ of $[0, ||\vec{v}||_1)$ in the irrational case. In the rational case, we conclude by noticing that $[0, \omega)$ is large enough for not being avoided. ■

Before proving a general statement (see Theorem 6 below), let us revisit what has been done in the proof of Theorem 1. We want to be able to localize with respect to the value $\langle \vec{x}, \vec{v} \rangle$ in the selection window $[0, \omega)$ vertices of edges of a given type that belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$. We distinguish two cases with respect to ω, namely $v_2 + v_3 \leq \omega < v_1 + v_2 + v_3$, and $v_1 + v_3 \leq \omega < v_2 + v_3$.

Case $v_2 + v_3 \leq \omega$

Assume first that $0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < v_1$. According to Theorem 1, we know that the four edges of $\vec{x} \ + F_1$ belong to $\mathfrak{P}(\vec{v}, ||\vec{v}||_1)$. We would like to know which edges of the face $\vec{x} \ + F_1$ still belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$. One has $\vec{x} + \vec{e}_2 \in \mathfrak{P}(\vec{v}, \mu, \omega)$ since $\omega \geq v_1 + v_2$. Hence, the edge $\vec{x} + E_2$ belongs to $\mathfrak{P}(\vec{v}, \mu, \omega)$. Moreover if $0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < \omega - (v_2 + v_3)$, then $\vec{x} + \vec{e}_2$, $\vec{x} + \vec{e}_3$, $\vec{x} + \vec{e}_2 + \vec{e}_3$ all belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$. If $\langle \vec{x}, \vec{v} \rangle + \mu \geq \omega - (v_2 + v_3)$, then $\vec{x} + \vec{e}_2$, $\vec{x} + \vec{e}_3$ belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$. Hence we divide $[0, v_1)$ into two intervals

$$[0, \omega - (v_2 + v_3)), \ [\omega - (v_2 + v_3), v_1)$$

in the following way: if $\langle \vec{x}, \vec{v} \rangle + \mu$ belongs to the first interval, then the four edges of the face $\vec{x} + F_1$ belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$, otherwise we only can say that the edges $\vec{x} + E_2$ and $\vec{x} + E_3$ belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$.

Assume now $v_1 \leq \langle \vec{x}, \vec{v} \rangle + \mu < v_1 + v_2$. Then $\vec{x} - \vec{e}_1$ belongs to $\mathfrak{P}(\vec{v}, \mu, \omega)$. Hence, the edge $\vec{x} - E_1$ belongs to $\mathfrak{P}(\vec{v}, \mu, \omega)$. Furthermore, $\vec{x} + \vec{e}_3 \in \mathfrak{P}(\vec{v}, \mu, \omega)$ if and only if $\langle \vec{x}, \vec{v} \rangle + \mu \in [v_2, \omega - v_3)$. One has $\omega - v_3 \leq v_1 + v_3$. Hence we divide $[v_1, v_1 + v_2)$ into two intervals

$$[v_1, \omega - v_3), \ [\omega - v_3, v_1 + v_2)$$

which correspond respectively to the four edges of the face $\vec{x} + F_2$, and to the edges $\vec{x} - E_1$, $\vec{x} + E_3$, and $\vec{x} - E_1$, $\vec{x} - E_1 + E_3$.

Assume $v_1 + v_2 \leq \langle \vec{x}, \vec{v} \rangle + \mu < \omega < v_1 + v_2 + v_3$. Then $\vec{x} - \vec{e}_1$, $\vec{x} - \vec{e}_2$, $\vec{x} - \vec{e}_1 - \vec{e}_2$ all belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$. We thus deduce that the four edges of $F_3 + \vec{x}$ belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$. We have only one interval

$$[v_1 + v_2, \omega).$$

Case $v_1 + v_3 \leq \omega < v_2 + v_3$

One similarly checks that one never finds the four edges of a translate of a face F_1, but that the interval $[0, v_1)$ corresponds to the edges $\vec{x} + E_2$ and $\vec{x} + E_3$.

We divide $[v_1, v_1 + v_2)$ into three intervals

$$[v_1, \omega - v_3), \ [\omega - v_3, \omega - v_3 + v_1), \ [\omega - v_3 + v_1, v_1 + v_2)$$

which correspond respectively to the four edges of the face $\vec{x} + F_2$, to the edges $\vec{x} - E_1$ and $\vec{x} - E_1 + E_3$, and to $\vec{x} - E_1$.

Lastly, the interval $[v_1 + v_2, \omega)$ corresponds to the four edges of $F_3 + \vec{x}$.

We are now ready to give a general statement generalizing Theorem 1 and Corollary 3: this is the object of Theorem 6 below. Let us first note that this theorem can be considered as a generalization of the results of [Lam98] and [GMP03] on discrete lines: it is proved in [GMP03] that there are finitely (and even three) possible distances between adjacent points after projection on the underlying Euclidean line of the vertices of a discrete line. This is a consequence of the so-called three-gap theorem (see [Sla67]). This implies that it is possible to code any discrete line as an infinite word over a three-letter or a two-letter alphabet, according to the thickness ω: these words are either Sturmian words [Lot02,PF02] (if there are only two lengths), or three-interval exchange words. The interest of such a formulation is that one can deduce easily properties concerning their configurations (number of configurations of a given size, frequencies *etc.*). For the range of values ω we are considering, we show in this section that an arithmetic discrete plane can be decomposed into at most four finite generalized faces.

Before stating Theorem 6, we need to introduce the following class of generalized faces.

Definition 8 *Let $k \in \mathbb{N}$. The face H_k is defined as*

$$H_k := ((k-1)\vec{e}_1 - \vec{e}_2 + F_1) \bigcup \bigcup_{0 \le i \le k-1} \Big((i\vec{e}_1 + F_2) \bigcup (\vec{e}_3 + i\vec{e}_1 + F_3)\Big).$$

The distinguished vertex of the generalized face $\vec{x} + H_k$ is defined as $\vec{x}$.

For an illustration of Definition 8, see Figure 3 below.

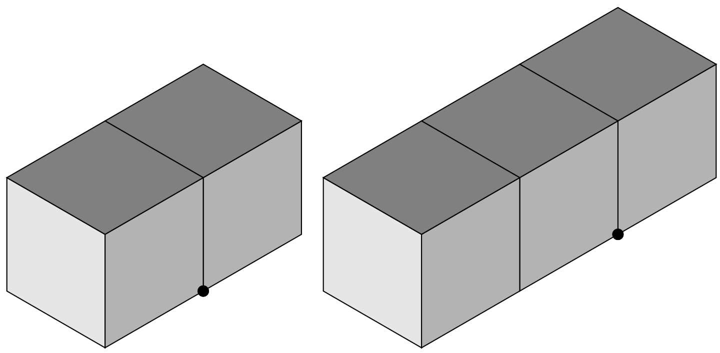


Figure 3. The generalized face H_2 (left), and the generalized face H_3 (right) with their distinguished vertex (the black dot) being located at the origin.

Theorem 6 *Let $\vec{v} = (v_1, v_2, v_3)$ be a nonzero vector in $\mathbb{R}^3$ and let $\omega \in \mathbb{R}^+$ that satisfy*

$$0 \le v_1 \le v_2 \le v_3, \;\; v_1 + v_3 \le \omega < v_1 + v_2 + v_3.$$

Then, the arithmetic discrete plane $\mathfrak{P}(\vec{v},\mu,\omega)$ admits exactly 4 types of generalized faces.

Let k be the smallest nonnegative integer such that $\omega + kv_1 \geq v_2 + v_3$. If $k = 0$, i.e., *$\omega \geq v_2 + v_3$, these generalized faces are F_1, F_2, F_3, H_0. If $k \geq 1$,* i.e., *$v_1 + v_3 \leq \omega < v_2 + v_3$, these generalized faces are equal F_2, F_3, H_{k+1}, H_k.*

Furthermore, the generalized face G (with $G \in \{F_1, F_2, F_3, H_k, H_{k+1}\}$) occurs at vector $\vec{x}$ in $\mathfrak{P}(\vec{v},\mu,\omega)$ if and only if $\langle \vec{x}, \vec{v} \rangle + \mu$ belongs to I_G, with:

- $I_{F_1} = [0, \omega - (v_2 + v_3))$, *if* $\omega \geq v_2 + v_3$, $I_{F_1} = \emptyset$ *otherwise,*
- $I_{F_2} = [v_1, \omega - v_3)$,
- $I_{H_{k+1}} = [\omega - v_3, v_2 - (k-1)v_1)$,
- $I_{H_k} = [v_2 - (k-1)v_1, \omega - v_3 + v_1)$, *if* $k \geq 1$, $I_{H_0} = \emptyset$ *otherwise,*
- $I_{F_3} = [v_1 + v_2, \omega)$.

The frequency of occurrence of each of these generalized faces is equal to the length (resp. to the cardinality) of the corresponding interval if the arithmetic discrete plane is irrational (resp. rational).

Proof. The proof is similar to the proof of Theorem 1. Assume first $\omega \geq v_2 + v_3$. We have seen in Section 3.3 that the four edges of the face $\vec{x} + F_1$ all belong to $\mathfrak{P}(\vec{v},\mu,\omega)$ if and only if $\langle \vec{x}, \vec{v} \rangle + \mu \in I_{F_1}$. Similarly, the four edges of the face $\vec{x} + F_2$ all belong to $\mathfrak{P}(\vec{v},\mu,\omega)$ if and only if $\langle \vec{x}, \vec{v} \rangle + \mu \in I_{F_2}$. Now, assume that $\langle \vec{x}, \vec{v} \rangle + \mu \in I_{H_1}$. By using the description made in Section 3.3, one checks that the outer edges of H_1 all belong to $\mathfrak{P}(\vec{v},\mu,||\vec{v}||_1)$:

$\vec{x}-E_1$, $\vec{x}-\vec{e}_1+E_3$, $\vec{x}-\vec{e}_1+\vec{e}_3-E_2$, $\vec{x}-\vec{e}_1+\vec{e}_3-\vec{e}_2+E_1$, $\vec{x}+\vec{e}_3-\vec{e}_2-E_3$, $\vec{x}-\vec{e}_2+E_2$;

since $\vec{e}_3$ does not belong to $\mathfrak{P}(\vec{v},\mu,\omega)$, none of its inner edges does belong to $\mathfrak{P}(\vec{v},\mu,\omega)$. This implies that $\vec{x} + H_1$ is a generalized face of $\mathfrak{P}(\vec{v},\mu,\omega)$. We similarly prove that this condition is also necessary. Lastly, we also have seen that the four edges of the face $\vec{x} + F_3$ all belong to $\mathfrak{P}(\vec{v},\mu,\omega)$ if and only if $\langle \vec{x}, \vec{v} \rangle + \mu \in I_{F_3}$.

The proof works in the same way for the case $v_1 + v_3 \leq \omega < v_2 + v_3$.

Finally, the statement concerning the frequencies is obtained similarly as for Corollary 3. ■

Remark 3 *Let us note that the union of intervals associated with generalized faces is not equal to $[0,\omega)$, whereas $\bigcup_{i=1,\ldots,4} I_{F_i} = [0, ||\vec{v}||_1)$ in Theorem 1. Indeed, this is not crucial to have a partition of the selection windows into intervals. We could have chosen to make a partition into intervals and to associate with each interval sets of edges as done in Section 3.3. Nevertheless, the decomposition that we have made in Theorem 6 (and which does not cor-*

respond to a partition), allows us a more convenient description in terms of generalized faces.

Remark 4 *If $\omega \leq v_1 + v_3$, then there might exist an infinite sequence $(\vec{x}_n)_{n\in\mathbb{N}}$ with values in $\mathbb{Z}^3$ such that, for all n, $\omega \leq \langle \vec{x_n}, \vec{v}\rangle + \mu < ||v||_1$ and $\vec{x}_{n+1} - \vec{x}_n \in \{\pm\vec{e}_1, \pm\vec{e}_2\}$, which could prevent the generalized faces of $\mathfrak{P}(\vec{v}, \mu, \omega)$ to be finite. Note that there exists an important difference with the case of a discrete line, where such a situation cannot happen, according to Theorem 4.*

Remark 5 *In the case $\omega \geq ||\vec{v}||_1$, a similar study can be performed, by setting $\omega' := \omega - \lfloor \omega/||\vec{v}||_1 \rfloor ||\vec{v}||_1$. Indeed some generalized faces will occur with multiplicity $\lfloor \omega/||\vec{v}||_1 \rfloor ||\vec{v}||_1$, and other generalized faces will occur with multiplicity $\lfloor \omega/||\vec{v}||_1 \rfloor ||\vec{v}||_1 - 1$, where the notation $\lfloor\ \rfloor$ stands as usual for the integer part.*

4 From generalized faces to configurations

We have been so far able to associate with generalized faces intervals of the selection window $[0, \omega)$: this was the object of Theorem 6. Our aim is to extend this result to more general configurations, that is, not only to generalized faces but also to finite unions of generalized faces. In all that follows we assume that we are under the assumptions of Theorem 6.

4.1 Configurations and intervals

We define a *configuration* of the tiling $T(\vec{v}, \mu, \omega)$ as an edge-connected finite union of generalized tiles contained in the tiling. We assume that $\vec{0}$ is always a distinguished vertex of one of the generalized faces of a configuration. We consider occurrences of configurations up to translation. Note that preimages by π_0 in $\mathfrak{P}(\vec{v}, \mu, \omega)$ of configurations correspond to usual local configurations of arithmetic discrete planes. By abuse of terminology, we also call them *configurations* of $\mathfrak{P}(\vec{v}, \mu, \omega)$. The configuration C is said to occur at $\vec{y}$ in the tiling $T(\vec{v}, \mu, \omega)$ if $C + \vec{y}$ is included in it. In particular, we have seen in Theorem 6 (we use here its notation) that a generalized tile $T = \pi_0(G)$ occurs at vector $\vec{y} = \pi_0(\vec{x})$ in the tiling $T(\vec{v}, \mu, \omega)$ if and only if $\langle \vec{x}, \vec{v}\rangle + \mu \in I_G$ (here $G \in \{F_1, F_2, F_3, H_k, H_{k+1}\}$).

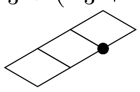
Let $C = \bigcup_n \vec{y_n} + \pi_0(L_n)$ be a configuration, where for all n, $\vec{y_n} = \pi_0(\vec{x}_n)$, $\vec{x}_n \in \mathfrak{P}(\vec{v}, \mu, \omega)$, $L_n \in \{F_1, F_2, F_3, H_k, H_{k+1}\}$, with k being defined in Theorem 6, and $\vec{y}_0 = \vec{0}$. One sets

$$J_C := \bigcap_n (-\langle \vec{x}_n, \vec{v}\rangle + I_{L_n}),$$

where in this intersection, intervals are considered as intervals of the one-dimensional torus $\mathbb{R}/(||\vec{v}||_1\mathbb{Z})$.

The notion of frequency for faces extends in a natural way to configurations. This yields the following result.

Theorem 7 *Assume that* $0 \leq v_1 \leq v_2 \leq v_2$ *and* $v_1 + v_3 \leq \omega \leq ||\vec{v}||_1$. *Let* C *be a an edge-connected finite union of generalized tiles of* $T(\vec{v}, \mu, \omega)$. *One has* $J_C \neq \emptyset$ *if and only if* C *is a configuration of* $T(\vec{v}, \mu, \omega)$. *The set* J_C *is an interval of the selection window of the arithmetic discrete plane if it is irrational, otherwise it is a connected set of integers. The frequency of occurrence of the configuration* C *is equal to the cardinality of* J_C *if it is rational, and to its length if it is irrational.*

Before proving Theorem 7, let us illustrate it on one example in the standard case. Consider the configuration $C = T_3 \cup (T_3 + \vec{e}_1) \cup (T_3 + 2\vec{e}_1)$ of a standard arithmetic discrete plane, depicted as . Configuration C occurs at $\vec{y} = \pi_0(\vec{x})$, with $\vec{x} \in \mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$, if and only if $\langle \vec{x}, \vec{v} \rangle + \mu \in I_{F_3}$, $\langle \vec{x} + \vec{e}_1, \vec{v} \rangle + \mu = \langle \vec{x}, \vec{v} \rangle + v_1 + \mu \in I_{F_3}$ and $\langle \vec{x} + 2\vec{e}_1, \vec{v} \rangle + \mu = \langle \vec{x}, \vec{v} \rangle + 2v_1 + \mu \in I_{F_3}$, that is, $\langle \vec{x}, \vec{v} \rangle + \mu \in I_{F_3} \cap (-v_1 + I_{F_3}) \cap (-2v_1 + I_{F_3})$. Hence $J_C \neq \emptyset$ if and only if $v_3 > 2v_1$, and $J_C = [v_1 + v_2, v_3 + v_2 - v_1)$.

Proof. The proof is classic and follows the same lines as similar proofs in [BV00] (see Lemma 2, Lemma 3). One first checks that if C occurs at $\pi_0(\vec{x})$, then $\langle \vec{x}, \vec{v} \rangle + \mu \in J_C$, which implies that $J_C \neq \emptyset$. Conversely, if $J_C \neq \emptyset$, then it contains an element of the form $\langle \vec{x}, \vec{v} \rangle + \mu$, by Lemma 2.

For proof of the fact that J_C is an interval in the irrational case and a set of consecutive integers in the rational case, see [BV00], Lemma 3. It uses the fact that J_C is described as an intersection of intervals whose lengths prevent disconnectedness. Indeed, they are preimages of intervals of length smaller than the v_i under the action of the translations $x \mapsto x + v_j$ modulo $||\vec{v}||_1$, for $j \in \{1, 2, 3\}$. ∎

4.2 Applications

We thus have been able to associate with a configuration C an interval J_C of the selection window $[0, \omega)$ thanks to the localization method. Let us discuss several properties that can be deduced from this correspondence.

First, Theorem 7 provides us a simple and effective way to check whether a configuration occurs or not: a configuration C occurs if and only if J_C is nonempty. Let us come back to the example of the previous section. The preimage $F_3 \cup (F_3 + \vec{e}_1) \cup (F_3 + 2\vec{e}_1)$ of the configuration C occurs up to translation in $\mathcal{P}(\vec{v}, \mu)$ if and only if $J_C = [v_1 + v_2, v_2 + v_3 - v_1)$ is nonempty, that is, $v_3 > 2v_1$. Note that this does not depend on the parameter μ.

More generally, one gets the following result.

Corollary 8 *Let $\vec{v} = (v_1, v_2, v_3)$ be a nonzero vector in $\mathbb{R}^3$ and $\omega \in \mathbb{R}^+$ be such that*

$$0 \le v_1 \le v_2 \le v_3, \; v_1 + v_3 \le \omega < v_1 + v_2 + v_3.$$

Two arithmetic discrete planes with the same normal vector $\vec{v}$ and the same width ω have the same set of configurations.

PROOF. This a direct consequence of Theorem 7 since intervals J_C do not depend on μ, but only on $\vec{v}$. ■

Note that this was already the case for generalized faces in Theorem 6: the set of generalized faces of an arithmetic discrete plane does not depend on the parameter μ.

These methods are classic in word combinatorics, symbolic dynamics, or tiling theory. For instance, we can count the number of configurations of a given size and shape: indeed we have to determine the bounds of the intervals J_C and then count them. For more details, see *e.g.* [BV00,BFJP07,Ber10].

Consider now repetitivity results. The *radius of a configuration* is defined as the minimal radius of a disk containing this configuration. Two configurations in the plane P_0 are said to be identical if they only differ by a translation vector. A tiling is said to be *repetitive* if for every configuration C of radius r there exists a positive number R such that every configuration of radius R contains C. This is a counterpart of the notion of uniform recurrence in word combinatorics and symbolic dynamics. In other words, configurations appear "with bounded gaps". Repetitive tilings can be considered as ordered structures.

Theorem 9 *Let $\vec{v} = (v_1, v_2, v_3)$ be a nonzero vector in $\mathbb{R}^3$ and $\omega \in \mathbb{R}^+$ be such that*

$$0 \le v_1 \le v_2 \le v_3, \; v_1 + v_3 \le \omega < v_1 + v_2 + v_3.$$

The tiling $T(\vec{v}, \mu, \omega)$ associated with the arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$ is repetitive.

PROOF. Let C be a given configuration with associated interval J_C. Repetitivity is a direct consequence of Theorem 7 together with Theorem 4 in the irrational case, and of the periodicity in the rational case. ■

5 Substitutions

The localization method has proved its efficiency in the previous section for the study of configurations, by working with the image by the mapping $\vec{x} \mapsto \langle \vec{x}, \vec{v} \rangle + \mu$ in the selection window $[0, \omega)$ of points of $\mathfrak{P}_{\vec{v},\mu,\omega}$. Recall that the value $\langle \vec{x}, \vec{v} \rangle + \mu$, and more precisely the interval of the selection window it belongs to, indicates that the point $\vec{x}$ is the distinguished vertex of a certain type of generalized face or configuration.

We have worked so far with a fixed normal vector $\vec{v}$ and a fixed selection window $[0, \omega)$. We consider now a different type of mechanism that is also very useful in the study of arithmetic discrete planes and that can also be described in terms of the selection window. Such a mechanism consists in letting both the normal vector $\vec{v}$ and the thickness ω vary under the action of a unimodular linear transformation. We fix a matrix $M \in SL(3, \mathbb{N})$ (*i.e.*, a square matrix of size 3 with determinant ± 1 with entries in $\mathbb{N}$) and we will construct an algorithmic way to go from $\mathcal{P}_{M\vec{v},\mu,||M\vec{v}||_1}$ to $\mathcal{P}_{\vec{v},\mu,||\vec{v}||_1}$, and even from $\mathcal{P}_{M\vec{v},\mu,\omega}$ to $\mathcal{P}_{\vec{v},\mu,\omega'}$ for some ω'. This algorithmic process is defined as substitution rule that replaces generalized faces by finite unions of generalized faces. Recall that a substitution is a classic object in word combinatorics. It replaces letters by finite words in a morphic way with respect to the concatenation rule: a substitution is a morphism of the free monoid. For more details, see *e.g.* [Que87,PF02]. We discuss here similar objects acting on unions of generalized faces. The key idea is to use the fact that

$$\langle \vec{x}, M\,\vec{v} \rangle = \langle {}^t M\,\vec{x}, \vec{v} \rangle \text{ for any } \vec{x} \in \mathbb{Z}^3. \tag{1}$$

We illustrate our approach with an example worked in full details in Section 5.1. This example, which is produced thanks to the formalism of [AI02], is an illustration of the general method discussed in [Ber09]. The novelty of such an example relies mainly in the fact that it works for more general thicknesses ω than the standard one, *i.e.*, $\omega = ||\vec{v}||_1$. Indeed, only the standard thickness has been considered in the seminal paper [AI02], or in references using generalized substitutions in this context of discrete geometry (see *e.g.*, [ABI02,Fer06,ABFJ07,Fer09,Ber09,BF11].

5.1 An example of application of a substitution

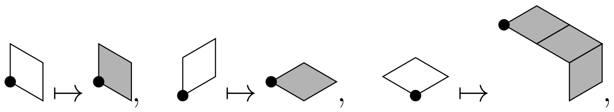
Let a be a positive integer. We consider the mapping Σ_a^* that acts on the set of faces and generalized faces introduced in Section 2.3, with the following "morphic type" rule: if G, H are unions of generalized faces, then

$$\Sigma_a^*(G \cup H) = \Sigma_a^*(G) \cup \Sigma_a^*(H). \tag{2}$$

Let $M_a := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & a \end{bmatrix}$ in $SL(3, \mathbb{N})$. We first define Σ_a^* on translates of faces F_i $(i = 1, 2, 3)$:

$$\begin{cases} \Sigma_a^*(\vec{x} + F_1) = M_a^{-1}\vec{x} + F_1, \\ \Sigma_a^*(\vec{x} + F_2) = M_a^{-1}\vec{x} + \vec{e}_2 + F_3, \\ \Sigma_a^*(\vec{x} + F_3) = (M_a^{-1}\vec{x} + a\vec{e}_2 - \vec{e}_3 + F_2) \cup \\ \qquad \cup (M_a^{-1}\vec{x} + a\vec{e}_2 + F_3) \cup \cdots \cup (M_a^{-1}\vec{x} + \vec{e}_2 + F_3). \end{cases} \tag{3}$$

This can be depicted for $a = 2$ for the faces F_1, F_2, F_3 as follows:

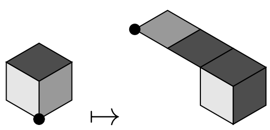
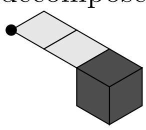


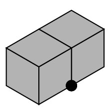
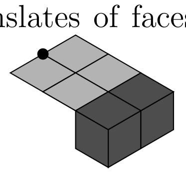
with the black dot indicating the origin. We stress the fact that the image of the translate of a face by the vector $\vec{x}$ is equal to the image of this face translated by $M_a^{-1}\vec{x}$.

Note that M_a can be seen as an *incidence matrix* for Σ_a^*: it counts the number of faces of each type in the images of the faces.

One checks by applying (3) that the image of the generalized face H_1 satisfies

$$\Sigma_a^*(\vec{x} + H_1) = \bigcup_{1 \le i \le a} (M_a^{-1}\vec{x} + i\vec{e}_2 + F_3) \bigcup (M_a^{-1}\vec{x} + (a+1)\vec{e}_2 - \vec{e}_3 + H_1),$$

which can be depicted as:  . This image can be decomposed as a union of a translate of H_1 and of a translates of faces F_3:  .

Similarly, the image of H_2  is equal to  .

More generally, one checks that the image of a generalized face H_k ($k \geq 1$) can be decomposed as the union of a translate of a face H_k and of $k \times a$ translates of faces F_3, with the interiors of these faces having no intersection. Hence, we can extend the definition of Σ_a^* to generalized faces. The map Σ_a^* being now defined on generalized faces, it is also defined for any union of generalized faces thanks to (2).

We can now state the main result of this section, by defining the distinguished vertices of the image of a generalized face as the distinguished vertices of the faces it is made of.

Theorem 10 *Let $\vec{v} = (v_1, v_2, v_3)$ be a nonzero vector in $\mathbb{R}^3$ and $\omega \in \mathbb{R}^+$ that satisfy*

$$0 \leq v_1 \leq v_2 \leq v_3, \;\; v_1 + v_3 \leq \omega < v_1 + v_2 + v_3.$$

We set $\delta := ||\vec{v}||_1 - \omega = v_1 + v_2 + v_3 - \omega$. Let a be a positive number.

The generalized faces contained in the image by Σ_a^* of the generalized faces of $\mathfrak{P}_{\vec{v},\mu,||\vec{v}||_1-\delta}$ are in a one-to-one correspondence with the generalized faces of $\mathfrak{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta}$.

Remark 6 *Note that the matrix M_a being symmetric, we could avoid the use of ${}^tM_a$. We chose to keep it here since we will make a frequent use of Equation (1).*

Proof. The proof is done here under the assumption $\omega \geq v_2 + v_3$, *i.e.*, $0 < \delta \leq v_1$. The proof of the remaining case $v_1 + v_3 \leq \omega \leq v_2 + v_3$ follows the same lines.

Note that ${}^tM_a\vec{v}$ has coordinates $(v_1, v_3, v_2 + av_3)$ and that its parameters obey the assumptions of Theorem 6, namely,

$$0 \leq v_1 \leq v_3 \leq v_2 + av_3$$

and since $0 < \delta \leq v_1$,

$$v_2 + (a+1)v_3 \leq ||{}^tM_a\vec{v}||_1 - \delta = v_1 + v_2 + (a+1)v_3 - \delta < v_1 + v_2 + (a+1)v_3.$$

We work here with the two discrete planes $\mathfrak{P}_{\vec{v},\mu,||\vec{v}||_1-\delta}$ and $\mathfrak{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta}$. There is a first division of the larger selection window $[0, ||{}^tM_a\vec{v}||_1-\delta)$ provided by Theorem 6:

$$[0, v_1 - \delta), \;\; [v_1, v_1 + v_3 - \delta), \;\; [v_1 + v_3 - \delta, v_1 + v_3), [v_1 + v_3, v_1 + v_2 + (a+1)v_3 - \delta).$$

These four intervals correspond to the four types of generalized faces that occur in $\mathcal{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta}$.

The idea of the proof is to embed points issued from the initial selection windows $[0,\omega) = [0, ||\vec{v}||_1 - \delta)$ corresponding to $\mathfrak{P}_{\vec{v},\mu,||\vec{v}||_1-\delta}$ into the larger one $[0, ||{}^tM_a\vec{v}||_1 - \delta)$, corresponding to $\mathfrak{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta}$. We thus refine the previous division of $[0, ||{}^tM_a\vec{v}||_1 - \delta)$ into intervals of respective lengths $v_1, v_2, v_3, v_3 - \delta$. We thus consider the division:

$$[0, v_1), [v_1, v_1 + v_3 - \delta),\ [v_1 + v_3 - \delta, v_1 + v_3),\ [v_1 + v_3, v_1 + v_3 + v_2),$$

$$[v_1 + v_3 + v_2, v_1 + v_3 + v_2 + v_3), \cdots, [v_1 + v_3 + v_2 + av_3, v_1 + v_3 + v_2 + av_3 - \delta).$$

In order to work in the selection window $[0, ||{}^tM_a\vec{v}||_1 - \delta)$, we consider the values taken by the distinguished vertices of the images of faces under the mapping $\varphi\colon \mathbb{Z}^3 \to \mathbb{R},\ \vec{x} \mapsto \langle \vec{x}, {}^tM_a\vec{v}\rangle + \mu$, together with Equation (1).

To prove Theorem 10, it is sufficient to show that the distinguished vertices of the images by Σ_a^* of the generalized faces of the four types of $\mathfrak{P}(\vec{v},\mu,\omega)$ are mapped by φ *onto* the intersection of $\varphi(\mathfrak{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta})$ with the interval $[0, v_1 - \delta)$ for the distinguished vertices of translates of faces F_1 in $\mathfrak{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta}$, $[v_1, v_1 + v_3 - \delta)$ for translates of faces F_2, $[v_1 + v_3 - \delta, v_1 + v_3)$ for translates of faces H_1, and $[v_1 + v_3, v_1 + v_2 + (a+1)v_3 - \delta)$, for translates of faces 3. Note that the main point is to prove that that the distinguished vertices of the images by Σ_a^* of the generalized faces are mapped by φ *into* the respective intersections of $\varphi(\mathfrak{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta})$ with the corresponding intervals. The fact that this mapping is indeed *onto* comes from the fact that M_a is invertible as a matrix with entries in $\mathbb{Z}$.

We first prove that the set of values taken by the distinguished vertices of the images of translates of faces F_1 of $\mathfrak{P}(\vec{v},\mu,\omega)$ are mapped by φ onto $[0, v_1 - \delta) \cap \varphi(\mathfrak{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta})$. By (3), these distinguished vertices are distinguished vertices of translates of faces F_1 in $\mathfrak{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta}$. According to Theorem 6, the distinguished vertices of the translates of faces F_1 that belong to $\mathfrak{P}(\vec{v},\mu,\omega)$ are the points $\vec{x} \in \mathbb{Z}^3$ that satisfy

$$0 \le \langle \vec{x}, \vec{v}\rangle + \mu < \omega - (v_2 + v_3) = v_1 - \delta.$$

By definition of Σ_a^*, the distinguished vertices of their images are of the form $M_a^{-1}\vec{x}$. According to (1), their images by φ satisfy

$$\langle M_a^{-1}\vec{x}, {}^tM_a\vec{v}\rangle + \mu = \langle \vec{x}, \vec{v}\rangle + \mu \in [0, v_1 - \delta).$$

We thus have proved that the set of values taken by the distinguished vertices of the images of translates of faces F_1 that belong to $\mathfrak{P}(\vec{v},\mu,\omega)$ is included in $[0, v_1) \cap \varphi(\mathfrak{P}_{{}^tM_a\vec{v},\mu,||{}^tM_a\vec{v}||_1-\delta})$. The converse inclusion follows from the fact that M_a is invertible as a matrix with entries in $\mathbb{Z}$.

We now prove that the set of values taken by the distinguished vertices of the images of translates of faces F_2 of $\mathfrak{P}(\vec{v},\mu,\omega)$ are mapped by φ onto $[v_1 +$

$v_3, v_1 + v_3 + v_2 - \delta)$. They are distinguished vertices of translates of faces F_2 in $\mathfrak{P}_{{}^t M_a \vec{v}, \mu, ||{}^t M_a \vec{v}||_1 - \delta}$, hence they satisfy

$$v_1 \le \langle \vec{x}, \vec{v} \rangle + \mu < \omega - v_3 = v_1 + v_2 - \delta.$$

The distinguished vertices of their images are of the form $M_a^{-1}\vec{x} + \vec{e}_2$. Their images by φ satisfy

$$\langle M_a^{-1}\vec{x} + \vec{e}_2, {}^t M_a \vec{v} \rangle + \mu = \langle \vec{x}, \vec{v} \rangle + v_3 + \mu \in [v_1 + v_3, v_1 + v_2 + v_3 - \delta).$$

For the reverse inclusion, it follows again from the invertibility of M_a, which ends the treatment of translates of faces F_2.

We consider now images of translates of faces F_3. The distinguished vertices of the translates of faces F_3 of $\mathfrak{P}(\vec{v}, \mu, \omega)$ are the points $\vec{x} \in \mathbb{Z}^3$ that satisfy

$$v_1 + v_2 \le \langle \vec{x}, \vec{v} \rangle + \mu < \omega = v_1 + v_2 + v_3 - \delta.$$

Their images contain translates of faces F_2 with distinguished vertices of the form $M_a^{-1}\vec{x} + a\vec{e}_2 - \vec{e}_3$, and translates of faces F_3 with distinguished vertices of the form $M_a^{-1}\vec{x} + k\vec{e}_2$, for $1 \le k \le a$. The images by φ of the distinguished vertices of translates of faces F_2 satisfy

$$\langle M_a^{-1}\vec{x} + a\vec{e}_2 - \vec{e}_3, {}^t M_a \vec{v} \rangle + \mu = \langle \vec{x}, \vec{v} \rangle + av_3 - (v_2 + av_3) + \mu \in [v_1, v_1 + v_3 - \delta).$$

The images by φ of the distinguished vertices of translates of faces F_3 satisfy for $1 \le k \le a$

$$\langle M_a^{-1}\vec{x} + k\vec{e}_2, {}^t M_a \vec{v} \rangle + \mu = \langle \vec{x}, \vec{v} \rangle + kv_3 + \mu \in [v_1 + v_2 + kv_3, v_1 + v_3 + (k+1)v_3 - \delta).$$

Hence, the distinguished vertices of images of translates of faces F_3 that belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$ are mapped by φ on $[v_1, v_1 + v_3)$ for the translates of faces F_2, and on $[v_1 + v_3, v_1 + v_3 + v_2)$ for the a translates of faces F_3. For the reverse inclusion, it follows again from the invertibility of M_a.

Note that we have covered so far the intervals

$$[0, v_1 - \delta), [v_1, v_1 + v_3 - \delta), [v_1 + v_3, v_1 + v_3 + v_2 - \delta),$$

$$[v_1 + v_3 + v_2, v_1 + v_3 + v_2 + v_3 - \delta), \cdots, [v_1 + v_3 + v_2 + av_3, v_1 + v_3 + v_2 + (a+1)v_3 - \delta).$$

Let us see how to cover the still uncovered right subintervals of length δ of intervals except the last one, by involving now the generalized face H_1. By covered, we mean that the distinguished vertices of the images by Σ_a^* of the generalized faces are mapped by φ onto the respective intersections of $\varphi(\mathfrak{P}_{{}^t M_a \vec{v}, \mu, ||{}^t M_a \vec{v}||_1 - \delta})$ with the corresponding intervals, with surjectivity coming from the invertibility of M_a.

The set of values taken by the distinguished vertices of the images of the translates of faces H_1 in $\mathfrak{P}(\vec{v}, \mu, \omega)$ are the points $\vec{x} \in \mathbb{Z}^3$ that satisfy

$$\omega - v_3 = v_1 + v_2 - \delta \leq \langle \vec{x}, \vec{v} \rangle + \mu < v_1 + v_2.$$

Their images contain a number of a translates of faces F_3 with distinguished vertices of the form $M_a^{-1}\vec{x} + k\vec{e}_2$, for $1 \leq k \leq a$, and one translate of F_1 with distinguished vertex $M_a^{-1}\vec{x} + (a+1)\vec{e}_2 - \vec{e}_3$. Their images by φ satisfy

$$\langle M_a^{-1}\vec{x} + k\vec{e}_2, {}^tM_a\vec{v} \rangle + \mu = \langle \vec{x}, \vec{v} \rangle + kv_3 + \mu \in [v_1 + v_2 + kv_3 - \delta, v_1 + v_2 + kv_3)$$

with $1 \leq k \leq a$ for the translates of F_3, and

$$\langle M_a^{-1}\vec{x} + (a+1)\vec{e}_2 + \vec{e}_3, {}^tM_a\vec{v} \rangle + \mu = \langle \vec{x}, \vec{v} \rangle - v_2 + v_3 + \mu \in [v_1 + v_3 - \delta, v_1 + v_3)$$

for the generalized faces H_1, which ends the proof. ■

Remark 7 *When* $\delta = 0$, i.e.,, $\omega = ||\vec{v}||_1$, *Theorem 10 is a consequence of the results of [AI02,Fer06], see also [BF11]. The main interest of Theorem 10 relies in the fact that the case* $\omega < ||\vec{v}||_1$ *has not yet been handled in the literature.*

5.2 General case

We have produced in Section 5.1 an example of a generalized substitution acting on a class of arithmetic discrete planes that are thiner than standard arithmetic discrete planes. In the standard case, *i.e.*, $\omega = ||\vec{v}||_1$, such examples of generalized substitutions are well known. The idea underlying them is a suitable decomposition of the interval $[0, ||{}^tM\vec{v}||_1)$ into subintervals of respective lengths v_i, for $i = 1, 2, 3$: one has to choose a way of tiling the larger interval by these smaller intervals. For more details, see [Ber09].

The strategy developed in [AI02] for such a choice of a tiling is to use a unimodular substitution σ on words, *i.e.*, a substitution such that its incidence matrix has determinant ± 1. By a duality process introduced in [AI02], one can associate with any unimodular substitution σ a generalized substitution acting on faces, denoted by $E_1^*(\sigma)$, and called *generalized substitution*. Such a formalism allows one to relate two discrete planes with different normal vectors $\vec{v}$ and $\vec{v}'$ in the standard case when $\vec{v} = {}^tM\vec{v}'$, where $M \in SL(3, \mathbb{N})$:

Theorem 11 *[AI02,Fer06] Let* σ *be a unimodular substitution. Let* $\vec{v} \in \mathbb{R}_+^d$ *be a positive vector. The generalized substitution* $E_1^*(\sigma)$ *satisfies*

$$E_1^*(\sigma)(\mathcal{P}_{\vec{v},\mu,||\vec{v}||_1}) = \mathcal{P}_{{}^tM_\sigma\vec{v},\mu,||{}^tM_\sigma\vec{v}||_1}.$$

The generalized substitution Σ_a^* has been obtained thanks to this formalism with the substitution $\sigma_a \colon 1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 3^a 2$. Theorem 10 is a generalization of Theorem 11 for this particular class of substitutions.

Remark 8 A priori, *not every mapping $E_1^*(\sigma)$ associated with a unimodular substitution σ can be applied to "thin" arithmetic discrete planes. In particular, we have used the fact that the image of the generalized faces H_k can be decomposed into a union of the generalized faces F_i and H_j.*

One motivation for Theorem 10 and Theorem 11 is that unimodular transformations are the basic steps when expanding vectors under the action of a unimodular multidimensional continued fraction algorithm, such as Jacobi-Perron or Brun algorithms (here we expand the normal vector $\vec{v}$ of a plane). For more on multidimensional continued fraction algorithms, see [Bre81,Sch00]. Note that arithmetic discrete lines and their codings as Sturmian words are perfectly well described by Euclid's algorithm and by the continued fraction expansion of their slope. For more details, see *e.g.* [Lot02,PF02]. Generalized substitutions aim at generalizing this approach to the higher-dimensional case. Here, the generalized substitution Σ_a^* comes from the application of Brun algorithm to (v_1, v_2, v_3), by chosing a as the largest positive number such that $v_3 - a v_2 \geq 0$ (for more details, see [BF11]). Generalized substitutions associated with multidimensional continued fraction algorithms are used for the generation and the recognition of standard arithmetic discrete planes. See in particular for the Jacobi-Perron algorithm [ABI02,BLPP11], and [Fer09,BF11] for Brun algorithm.

6 Concluding remarks

We have focused here on the information provided by the selection window $[0, \omega)$ of a thin arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$ by using methods issued from tiling theory and word combinatorics. Note that the connections between word combinatorics and discrete geometry have recently proved their efficiency, in particular through the notion of boundary word. Let us mention in particular the nice characterization of digitally convex polyominoes in terms of the Lyndon decomposition of the word coding their boundary given in [BLPR08]. See also [BKP09,BFP09,BBGL09] for related results.

Theorem 11 can be fruitfully applied in discrete geometry, in particular for the generation of discrete planes (see *e.g.* [Fer09,BLPP11]). Let us stress that we are not only able to substitute, *i.e.*, to replace faces by unions of faces, but also to de-substitute, *i.e.*, to perform the converse operation, by using the algebraic property $E_1^*(\sigma)^{-1} = E_1^*(\sigma^{-1})$ (when σ is considered as a morphism of the free group is an automorphism). We aim at extending the approach developed in [Fer09,BF11] to thin arithmetic discrete planes for the digital plane recognition and the digital plane generation problems.

Let us quote a further classical question in the study of discrete planes that can be handled under the formalism of generalized substitutions $E_1^*(\sigma)$. The question is to find the smallest width ω for which the plane $\mathfrak{P}_{\vec{v},\mu,\omega}$ is connected (either edge-connected or vertex-connected) such as first discussed in [BB04]. The case of rational parameters has been solved in [JT09a]. For the case of irrational parameters, see [DJT09]. The method used in both papers relies on the use of a particular unimodular multidimensional continued fraction algorithm (the so-called fully subtractive algorithm, see [Sch00]) and on the use of Equation (1).

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