Effective S-adic symbolic dynamical systems

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Abstract. We focus in this survey on effectiveness issues for S-adic subshifts and tilings. An S-adic subshift or tiling space is a dynamical system obtained by iterating an infinite composition of substitutions, where a substitution is a rule that replaces a letter by a word (that might be multi-dimensional), or a tile by a finite union of tiles. Several notions of effectiveness exist concerning S-adic subshifts and tiling spaces, such as the computability of the sequence of iterated substitutions, or the effectiveness of the language. We compare these notions and discuss effectiveness issues concerning classical properties of the associated subshifts and tiling spaces, such as the computability of shift-invariant measures and the existence of local rules (soficity). We also focus on planar tilings.

Keywords: Symbolic dynamics; adic map; substitution; S-adic system; planar tiling; local rules; sofic subshift; subshift of finite type; computable invariant measure; effective language.

1 Introduction

Decidability in symbolic dynamics and ergodic theory has already a long history. Let us quote as an illustration the undecidability of the emptiness problem (the *domino problem*) for multi-dimensional subshifts of finite type (SFT) [8, 40], or else the connections between effective ergodic theory, computable analysis and effective randomness (see for instance [14, 33, 44]). Computability is a notion that has also appeared as a major understanding tool in the study of multi-dimensional subshifts of finite type with the breakthrough characterization by M. Hochman and T. Meyerovitch of the entropies of multi-dimensional subshifts of finite type as the non-negative right recursively enumerable numbers [32] (see also [28] in the one-dimensional case). Let us mention also the realization of effective subshifts (with factor and projective subaction operation) from higher-dimensional subshifts of finite type [3, 19, 31]. It is now clear that sofic and effective subshifts are closely related, in particular for substitutive subshifts and tilings. Indeed, contrarily to the one-dimensional case, substitution subshifts are known to have (colored) local rules (they are SFT or sofic) in the higher-dimensional framework [26, 29, 36].

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We focus here on effectiveness issues for S-adic subshifts (and tilings). An S-adic expansion is a way to represent (or to generate) words (one-dimensional and multi-dimensional ones), or tilings, by composing infinitely many substitutions. A (word) substitution is a morphism of the free monoid: it replaces letters by words. Substitutions can also be defined in the higher-dimensional framework: they replace letters by multi-dimensional patterns, and act on multi-dimensional words (configurations) in $\mathbb{Z}^d$; more generally, substitutions can also generate and act on tilings, by replacing a tile by a finite union of tiles. An infinite word u (or a d-dimensional configuration, or a tiling) admits an S-adic expansion if

$$u = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_{n-1}(a_n),$$

where $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of substitutions, and $(a_n)_{n\in\mathbb{N}}$ a sequence of letters. For more on substitutions, see e.g. [38], and for more on S-adic words and tilings, see [9, 39]. There is a deep parallelism between subshifts associated with such expansions (under natural assumptions like primitivity) and Bratteli–Vershik systems endowed with adic transformations, hence the terminology 'adic', with the letter S referring to 'substitution'. This connection between adic models and substitutions has been widely investigated; see e.g. [24], or [10] and the references therein. Recall also that any Cantor minimal system admits a Bratteli–Vershik representation [30], which illustrates the representation power of this notion.
Without any further assumption on the S-adic representation, every infinite word admits an S-adic expansion (according to Cassaigne's construction, see e.g. [9, Remark 3]). One thus needs to introduce suitable assumptions on these S-adic representations in order to find a good balance between the expressive power of such representations and the information provided by their existence. Let us illustrate this with [2] where it is proved that multi-dimensional S-adic subshifts, obtained by applying an effective sequence of substitutions chosen among a finite set of substitutions, are sofic subshifts.

Basic notions and definitions on substitutions and S-adic subshifts and tilings are recalled in Section 2. We discuss some decidability results in the one-dimensional setting for substitutive words in Section 3. Section 4 focuses on effectiveness for S-adic subshifts. Lastly, multi-dimensional Sturmian words and planar tilings are considered in Section 5.

2 Definitions

2.1 Subshifts

Let $\mathcal{A}$ be finite *alphabet* and $d \geq 1$. A *configuration* u is an element of $\mathcal{A}^{\mathbb{Z}^d}$. A pattern p is an element of $\mathcal{A}^D$, where $D \subset \mathbb{Z}^d$ is a finite set, called its *support*. Denote $\mathcal{A}^*$ the set of patterns. A *translate* of the pattern p by $\mathbf{m} \in \mathbb{Z}^d$ is denoted $p+\mathbf{m}$ and has $D+\mathbf{m}$ for support. A pattern $p \in \mathcal{A}^D$ is a *factor* of a configuration $u = (u_\mathbf{n})_{\mathbf{n}\in\mathbb{Z}^d}$ if there exists $\mathbf{m} \in \mathbb{Z}^d$ such that the restriction of u to $D + \mathbf{m}$ coincides with $p + \mathbf{m}$. The set of factors (up to translation) of u is called its *language*.

The set $\mathcal{A}^{\mathbb{Z}^d}$ endowed with the product topology is a compact metric space. A d-dimensional *subshift* $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a closed and shift-invariant set of configurations in $\mathcal{A}^{\mathbb{Z}^d}$, where the *shifts* $\sigma_{\mathbf{m}}$ with $\mathbf{m} \in \mathbb{Z}^d$ are defined as $\sigma_{\mathbf{m}} \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$, $(u_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^d} \mapsto (u_{\mathbf{n}+\mathbf{m}})_{n\in\mathbb{Z}^d}$. The shifts provide a natural action of $\mathbb{Z}^d$.
A subshift X can be defined by providing its *language*, that is, the set of patterns (up to translation) that occur in configurations in X. It can be defined equivalently by providing the set of forbidden patterns. Subshifts of *finite type* (also called SFT) are the subshifts such that the set of their forbidden patterns is finite. *Sofic subshifts* are images of SFT under a factor map, where a factor map $\pi : X \to Y$ between two subshifts X and Y is a continuous, surjective map such that $\pi \circ \sigma_{\mathbf{m}} = \sigma_{\mathbf{m}} \circ \pi$, for all $\mathbf{m} \in \mathbb{Z}^d$.

Definition 1 (Computable subshift). *A subshift is said to be*

- Π_1**-computable** *or* **effective** *if its language is co-recursively enumerable;*
- Σ_1**-computable** *if its language is recursively enumerable;*
- Δ_1**-computable** *or* **decidable** *if its language is recursive.*

A subshift is said to *be linearly recurrent* if there exists $C > 0$ such that every pattern whose support is a translate of $[-Cn, Cn]^d$ contains every factor whose support is a translate of $[-n, n]^d$. The *frequency* $f(p)$ of a pattern in a d-dimensional configuration u is defined as $\limsup_n |x_n|_p/(2n+1)^d$, where x_n is the restriction of u to $[-n, n]^d$, and $|x_n|_p$ stands for the number of occurrences of p in x_n. If the lim sup is in fact a limit, then the frequency is said to exist. A subshift is said to be *uniquely ergodic* if it admits a unique shift-invariant measure; in this case, pattern frequencies do exist. A subshift is said to be *minimal* if every non-empty closed shift-invariant subset is equal to the whole set. A minimal and uniquely ergodic subshift is said *strictly ergodic*. Any pattern which appears in a strictly ergodic subshift has a positive frequency. For more on multidimensional subshifts, see e.g. [11, Chapter 8,9].

2.2 Substitutions and S-adic Subshifts

A *substitution* s over the alphabet $\mathcal{A}$ is a map $s : \mathcal{A} \longrightarrow \mathcal{A}^*$. Let $\mathscr{S}$ be a finite set of substitutions; we want to define how a pattern of substitutions $\mathbf{s} \in \mathscr{S}^D$ acts on a pattern $p \in \mathcal{A}^D$, with $D \subset \mathbb{Z}^d$ finite. This general definition allows us to apply simultaneously different substitutions; we are in the non-deterministic case of [36]. We thus introduce *concatenation rules* which specify how the respective images of two adjacent tiles must be glued. A pattern of substitutions $\mathbf{s} \in \mathscr{S}^D$ is said to be *compatible* with a pattern $p \in \mathcal{A}^D$ (made of cells) if it is *consistent* (the image of a cell does not depend on the sequence of concatenation rules that are used, patterns have a unique image) and *non-overlapping* (the images of two cells do not overlap).
When $\mathbf{s}$ and p are compatible, the (unique) image of p by $\mathbf{s}$ is denoted as $\mathbf{s}(p)$. If all letters of $\mathbf{s}$ are equal to the same substitution $s \in \mathscr{S}$, the $\mathscr{S}$-pattern is said to be *s-constant*. This corresponds to the classical case of the action

of one substitution (the deterministic case in [36]). An *$\mathscr{S}$-super-tile of order n* corresponds to n iterations of (compatible) patterns of substitutions applied to a letter. We define the $\mathscr{S}$-adic subshift $X_{\mathscr{S}}$ as the set of the configurations for which every pattern appears in an $\mathscr{S}$-super-tile. Here we can compose all substitutions in $\mathscr{S}$. This notion plays a role for the description of planar tilings introduced in Section 5.

We now introduce the usual S-adic setting by applying only constant patterns of substitutions. We take a sequence of substitutions $S = (s_n)_{n\in\mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$; the shift acting on S is denoted as σ ($\sigma(S) = (s_{n+1})_{n\in\mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$). The *$S$-super-tile of order 0 and type $a \in \mathcal{A}$* is defined as the letter a, whereas the S-super-tile of order $n+1$ and type a is the image of the $\sigma(S)$-super-tile of order n and type a by a s_0-constant $\mathscr{S}$-pattern. A super-tile of order n can thus be defined by a word of size n in $\mathscr{S}^*$ together with a letter. The sequence S is said to be a *directive sequence*. We then define the S-adic subshift X_S as the set of the configurations for which every pattern appears in an S-super-tile. For a closed subset $\mathbf{S} \subset \mathscr{S}^{\mathbb{N}}$, we also define the $\mathbf{S}$-adic subshift $X_{\mathbf{S}} = \bigcup_{S\in\mathbf{S}} X_S$.

For $d = 1$, a natural way to define concatenation rules is to consider that the image of two consecutive letters is obtained as the concatenation of the two image words. Thus a substitution can be viewed as a non-erasing endomorphism of the free monoid $\mathcal{A}^*$. For example the Fibonacci substitution on the alphabet $\{a, b\}$ is defined by $\sigma\colon a \mapsto ab,\ b \mapsto a$. For $d = 2$, if all the supports of the images by an element of $\mathscr{S}$ are rectangular, concatenation rules of two adjacent letters consist in the concatenation of the two image patterns as long as the two glued edges have the same size. Rectangular substitutions are considered e.g. in [36] (see also [16] for the notion of shape-symmetric rectangular substitutions).

It is possible to extend the notion of substitution to geometric tilings. A *tiling* of $\mathbb{R}^d$ is a collection of compact sets which cover topologically $\mathbb{R}^d$, that is, with the interiors of the tiles being pairwise disjoint. In general, a tile-substitution in $\mathbb{R}^d$ is given by a set of prototiles $T_1, \dots, T_m \subset \mathbb{R}^d$, an expanding map and a rule how to dissect each expanded prototile into translated copies of some prototiles T_i. These *geometric tiling substitutions* are considered e.g. in [29]. It is also possible to define S-adic tilings in this context (see [39]).

There are further strategies for defining substitutions such as described in [38]. For instance, one can also use a global information; see the formalism introduced in [1] that allows the generation of multi-dimensional Sturmian words considered in Section 5; this formalism also provides concatenation rules [25].

3 Some Decisions Problems for Substitutions

In the one-dimensional case, numerous decidability results exist for fixed points of substitutions (D0L words), and their images by general morphisms (HD0L words). More precisely, let $\mathcal{A}$, $\mathcal{B}$, be finite alphabets. We consider two morphisms $\sigma\colon \mathcal{A}^* \to \mathcal{A}^*$, $\phi\colon \mathcal{A}^* \to \mathcal{B}^*$; an infinite word of the form $\lim_n \sigma^n(u)$ (respectively $\phi(\lim_n \sigma^n(u))$) is a D0L word (respectively an HD0L or morphic word), for u finite word.

We focus here on some decision problems that can be solved using the notion of return word and derived sequence (see e.g. [20]). Let σ be a primitive substitution. It generates a minimal subshift X_σ. A return word to a word u of its language is a word w of the language such that uw admits exactly two occurrences of u, with the second occurrence of u being a suffix of uw. One can recode sequences of the subshift via return words, obtaining *derived sequence* (see e.g. [20]). Note that even if analogous notions exist in the higher-dimensional case and for tilings [37], this is not sufficient to yield a direct generalization of the results described below.

The HD0L ω-equivalence problem (which has been open for more than 30 years) is solved in [21] for primitive morphisms: it is decidable to know whether two HD0L words are equal (see the references in [21] for the D0L case). The decidability of the ultimate periodicity of HD0L infinite sequences has also been a long-standing problem: it is decidable to know whether an HD0L word is ultimately periodic. See [21] for the primitive case, and [22] for the general case. See also the references in [22] for the D0L case. This problem is closely related to the decidability of the ultimate periodicity of recognizable sets of integers in some abstract numeration systems [7]. It is also proved in [23] that the uniform recurrence of morphic sequences is decidable.

The particular case of constant-length substitutions (automatic sequences) has also been widely studied; see e.g. [17, 42] where decision procedures are produced based on the connections between first-order logic and automata such as developed in [15] where the equivalence between being p-recognizable and p-definability is developed. For more references, see also the book [41].

4 Effectiveness for S-adic subshifts and Local Rules

We discuss here several effectivity notions for S-adic subshifts concerning their directive sequences, pattern frequencies, or else their language. We also focus on the existence of local rules. We only consider here iterations by constant $\mathscr{S}$-patterns. We recall that $\mathscr{S}$ is finite.

A closed subset $\mathbf{S} \subset \mathscr{S}^{\mathbb{N}}$ is *effectively closed* if the set of (finite) words which do not appear as prefixes of elements of $\mathbf{S}$ is recursively enumerable (one enumerates forbidden prefixes). An effectively closed set is not necessarily a subshift.

A set of substitutions $\mathscr{S}$ has *a good growing property* if there are finitely many ways of gluing super-tiles, and if the size of the super-tiles of order n grows with n: there exists a finite set of patterns $\mathcal{P} \subset \mathcal{A}^*$ such if a pattern formed by a super-tile of order n surrounded by super-tiles of order n is in the language of $X_{\mathscr{S}^{\mathbb{N}}}$, then it appears as the n-iteration by a constant $\mathscr{S}$-pattern of a pattern of $\mathcal{P}$, and, moreover, if for every ball of radius R, there exist $n \in \mathbb{N}$ such a translate of this ball is contained in all the supports of super-tiles of order n. Clearly non-trivial rectangular substitutions or geometrical substitutions (such as defined in [29]) verify this property.

Proposition 1. *Let $\mathbf{S} \subset \mathscr{S}^{\mathbb{N}}$ be a closed subset. If $X_{\mathbf{S}}$ is effective, then there exists an effective closed subset $\mathbf{S}' \subset \mathscr{S}^{\mathbb{N}}$ such that $X_{\mathbf{S}} = X_{\mathbf{S}'}$. The reciprocal is true if $\mathscr{S}$ has the good growing property.*

Proof. Assume that $X_{\mathbf{S}}$ is effective. The complement of its language is recursively enumerable. Let $\mathbf{S}'$ be the effective closed set such that a word in $\mathscr{S}^*$ is a forbidden prefix if the associated super-tiles are in the complement of the language of $X_{\mathbf{S}}$. Clearly $X_{\mathbf{S}'} = X_{\mathbf{S}}$.
Conversely, consider the $\mathbf{S}$-adic subshift $X_{\mathbf{S}}$ where $\mathbf{S}$ is effectively closed and let $\mathcal{P}$ be the set of patterns given by the good growing property. Let $\mathcal{P}'$ be the set of patterns in $\mathcal{P}$ that occur in $X_{\sigma^n(\mathbf{S})}$ for infinitely many n. A pattern p is in the language of $X_{\mathbf{S}}$ if it appears in the image by an n-iteration of a pattern of $\mathcal{P}'$, where n is the first order where the support of p is included in all super-tiles of order n. Since $\mathcal{P}'$ is finite and the prefixes of $\mathbf{S}$ are co-recursively enumerable, the same holds for the language of $X_{\mathbf{S}}$.

Definition 2 (Computable frequencies and measure). *Let X be a subshift. X is said to have* computable frequencies *if the frequencies of patterns exist and are uniformly computable. A shift-invariant measure is said to be* computable *if the measure of any cylinder is uniformly computable.*

Remark 1. Computability of letter frequencies does not say much on the algorithmic complexity of a subshift: take a subshift $X \subset \{0,1\}^{\mathbb{Z}}$ and consider the subshift Y obtained by applying to each configuration of X the substitution $0 \mapsto 01, 1 \mapsto 10$. The subshift Y admits letter frequencies (they are both equal to $1/2$), and it has the same algorithmic complexity as X.

Proposition 2. *Let X be a subshift. If X is effective and uniquely ergodic, then its invariant measure is computable and X is decidable. If X is minimal and its frequencies are computable, then its language is recursively enumerable. If X is minimal and effective, then it is decidable.*

Proof. Let X be a d-dimensional subshift. We assume X effective and uniquely ergodic. Let us prove that the frequency of any pattern is computable. We use the following algorithm that takes as an argument the parameter e that stands for the precision. We consider a finite pattern p. At step n, one produces all 'square' patterns of size n with support being a translate of $[-n,n]^d$ that do not contain the n first forbidden patterns (they do not need to belong to the language of X). For each of these square patterns of size n, one computes the number of occurrences of p in it, divided by $(2n+1)^d$. We continue until these quantities belong to an interval of length e. This algorithm then stops, and taking an element of the interval provides an approximation of the frequency of p up to precision e. Indeed, the square patterns of size n contain the square patterns of size n of X. It remains to prove that the algorithm stops. Suppose it does not, then, for all n, one can find two patterns of size n, x_n and x_n', that do not contain the n first forbidden patterns and such that $||x_n|_p/(2n{+}1)^d - |x_n'|_p/(2n{+}1)^d| > e$. By compactness, we can extract two configurations x and x' that do not contain

forbidden patterns (they thus belong to the subshift X) such that the frequency of p in x is distinct from the frequency of p in x'. This contradicts the unique ergodicity of X.
We now assume X minimal with computable pattern frequencies (frequencies are positive). One can decide whether the frequency of a pattern is larger than a given value. This thus implies that the language is recursively enumerable.
Assume that X is minimal and effective. Consider the square patterns of size n that do not contain the n first forbidden patterns. If a pattern belongs to all these patterns for some n, then it belongs to the language. Otherwise, consider size $n+1$. The algorithm stops if a pattern is in the language by minimality.

Corollary 1. *Let X_S be a strictly ergodic S-adic subshift defined with respect to a directive sequence $S \in \mathscr{S}^{\mathbb{N}}$ such that $\mathscr{S}$ satisfies the good growing property. The following conditions are equivalent:*

1. *there exists a computable sequence S' such that $X_S = X_{S'}$;*
2. *the unique invariant measure of X_S is computable;*
3. *the subshift X_S is decidable.*

Proof. We first assume (1). By Proposition 1, X_S is effective and Proposition 2 yields that its unique measure is computable and that X_S is decidable.
We now assume (2). Let d stand for the cardinality of the alphabet of the substitutions in $\mathscr{S}$. The letter frequency vector is in the cone defined by the product of the incidence matrices of the directive sequence. The incidence matrix of a substitution s is a square matrix whose entry of index (i, j) counts the number occurrences of the letter i in $s(j)$. Let M_n stand for the incidence matrix of the substitution s_n. The letter frequency belongs to the cone $\bigcap_n M_1 \cdots M_n \mathbb{R}_+^d$, which is one-dimensional by unique ergodicity. Given a precision e, one can compute n such that the columns of $M_1 \cdots M_n$ are expected to be at a distance less than e from the letter frequency vector. We fix a cylinder around the direction provided by the letter frequency vector with precision e. Now we test finite products of n substitutions in $\mathscr{S}$. We consider the cone obtained by taking the product of the incidence matrices, and check whether it intersects the cylinder. If it does not intersect the cylinder, one gets a forbidden product of substitutions, which proves that $\{S\}$ is effectively closed.
It remains to prove that (3) implies (1). There exists a closed effective set such that $X_S = X_{\mathbf{S}}$, by Proposition 1. For every $S' \in \mathbf{S}$, one has $X_{S'} = X_S$, by minimality. We then exhibit a computable S' in $\mathbf{S}$ as follows: for any n, take the first prefix for the lexicographic order among the prefixes of elements in $\mathbf{S}$ such that $s_0 s_1 \cdots s_n(a)$ is in the language of X_S.

Existence of local rules. A natural question in tiling theory is to find local rules which only produce aperiodic tilings. The first examples of aperiodic subshifts of finite type were based on hierarchical structures [8, 40]: substitutive structures are known to be able to force aperiodicity. Note that a non-trivial substitutive subshift cannot be sofic in dimension 1: it has zero topological entropy

whereas non-trivial sofic subshifts have positive entropy. In dimension $d \geq 2$, under natural assumptions, it is known for different types of substitutions that substitutive tilings can be enforced with (colored) local rules. The ideas is always to force a hierarchical structure, as in Robison's tiling, where each change of level is marked by the type of the super-tile of this level, and the rule used is transmitted for super-tiles of lower order. For rectangular substitutions, the result is proved in [36] (with the result being more general since the substitutions are non-deterministic). The case of geometrical substitutions is handled in [29] and the result is also true in a more combinatorial way [26].
In the case of rectangular substitutions it is shown in [2] that the $\mathbf{S}$-adic subshift $X_{\mathbf{S}}$ is sofic if and only if it can be defined by a set of directive sequences $\mathbf{S}$ which is effectively closed. A similar result for more general substitutions is expected; the difficulty relies in the ability to exhibit a rectangular grid to use the simulation (see [3, 19]) of a one-dimensional effective subshift by a two-dimensional sofic subshift. One can also ask whether linearly recurrent effective subshifts are sofic. Note that such a statement cannot hold for computability reasons (there are uncountably many linearly recurrent subshifts) without any effectivity assumption. Note also that in the one-dimensional case case, linearly recurrent subshifts are primitive S-adic [20].

5 An application: Planar Tilings

As an example of S-adic configurations, we consider *multi-dimensional Sturmian words*. The associated tilings belong to the more general class of *planar tilings*. A (canonical) planar tiling is an approximation of an affine d-plane E in $\mathbb{R}^n$, via the cut-and-project method (see e.g. [4]). Such a tiling can be lifted into the tube $E + [0,t]^n$: the space E is called the *slope* and the smallest possible t the *thickness*. Planar tilings are closely related to discrete planes in discrete geometry and provide models of quasicrystals. The case $t = 1$ and $d = n - 1$ corresponds to the multi-dimensional Sturmian case. In terms of configurations, a multidimensional Sturmian word is defined as the coding of a $\mathbb{Z}^d$-action by d rotations $R_{\alpha_i} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto x + \alpha_i$ $(1 \leq i \leq d)$, where the α_i are positive real numbers. We assume $1, \alpha_1, \cdots, \alpha_d$ rationally independent and $\sum_i \alpha_i < 1$. A *multidimensional Sturmian word* $u \in \{1, 2, \cdots, d+1\}^{\mathbb{Z}^d}$ is defined as follows: there exists ρ, a partition of $\mathbb{R}/\mathbb{Z}$ into $d+1$ semi-open intervals, d of lengths α_i, and one of length $1-\sum \alpha_i$, such that $u_{\mathbf{n}} = i$ if and only if $n_1\alpha_1+\cdots+n_d\alpha_d+\rho \in I_i$ [12]. The cut-and-project framework is larger than the S-adic framework but multidimensional Sturmian words are S-adic [13] (via the formalism of [1]).
The study of the connections between the existence of local rules for a planar tiling and the parameters of its slope started with [18, 34, 35, 43]. In particular, it was proven in [34] that a slope enforced by undecorated local rules is necessarily algebraic (this is however not sufficient, see e.g. [5, 6]). However, computability comes into play when the tiles can be decorated. Decorations indeed allow the transfer of information through the tiling, and this was used in [27] to prove that a slope can be enforced by such rules if and only if it is computable.

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