Brun expansions, substitutions and discrete geometry

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Abstract. The aim of this lecture is to present a strategy for the problem of discrete plane recognition based on multidimensional continued fractions and S-adic systems. The problem of the discrete plane recognition consists in deciding whether a given set of points with integer coordinates can be described as a plane discretization. The role played respectively by words, substitutions, and classical continued fractions will be played here respectively by stepped surfaces [1], generalized substitutions [2, 4], and Brun's algorithm. We thus give a geometric interpretation of Brun's continued fraction algorithm in terms of the so-called generalized substitutions introduced by Arnoux and Ito.

1 Plane recognition

The discrete plane recognition problem can be stated as follows: given a set of points in $\mathbb{Z}^d$, does there exist a (standard) arithmetic discrete plane that contains them?

Let us first recall the definition of a standard arithmetic discrete plane, according to [7]. Let $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d)$ stand for the canonical basis of $\mathbb{R}^d$. For any $\boldsymbol{x} \in \mathbb{Z}^d$ and $i \in \{1, \ldots, d\}$, we denote by $(\boldsymbol{x}, i^*)$ the following translate of a face of the unit hypercube:

$$(\boldsymbol{x}, i^*) = \{\boldsymbol{e}_i + \sum_{j \neq i} \lambda_j \boldsymbol{e}_j \mid 0 \leq \lambda_j \leq 1\}.$$

Then, for any non-negative non-zero vector $\boldsymbol{\alpha} \in \mathbb{R}^d_+ \backslash \{\boldsymbol{0}\}$ and $\rho \in \mathbb{R}$, we define

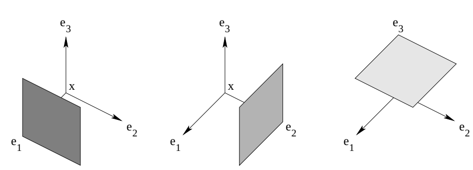

Fig. 1. The faces $(\boldsymbol{x}, 1^*)$, $(\boldsymbol{x}, 2^*)$ and $(\boldsymbol{x}, 3^*)$ (from left to right), in the $d = 3$ case.

the following set of faces:

$$\mathcal{P}_{\boldsymbol{\alpha},\rho} = \{(\boldsymbol{x}, i^*) \mid 0 \leq \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle - \rho < \langle \boldsymbol{e}_i|\boldsymbol{\alpha}\rangle\},$$

where $\langle.|.\rangle$ stands for the canonical inner product. One checks that $\boldsymbol{x} \in \mathbb{Z}^d$ is a *vertex* of $\mathcal{P}_{\boldsymbol{\alpha},\rho}$ (that is, $\boldsymbol{x}$ belongs to a face of $\mathcal{P}_{\boldsymbol{\alpha},\rho}$) if and only if it satisfies:

$$0 \leq \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle - \rho < \sum_{i=1}^{d} \langle \boldsymbol{e}_i|\boldsymbol{\alpha}\rangle,$$

i.e., the set of vertices of $\mathcal{P}_{\boldsymbol{\alpha},\rho}$ is a so-called *standard arithmetic discrete hyperplane.*

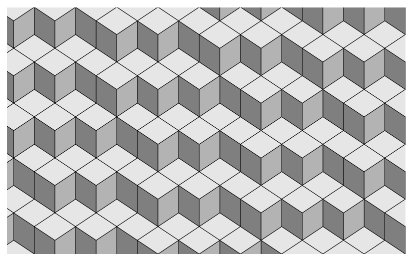

Fig. 2. A piece of a standard arithmetic discrete plane.

There exist various strategies in discrete geometry for the recognition of arithmetic discrete planes, such as described for instance in the survey [3]. The aim of this lecture is to present a strategy based on multidimensional continued fractions and S-adic systems, inspired by the one-dimensional Sturmian case. We will see that the role played respectively by words, substitutions, and classical continued fractions will be played here by stepped surfaces [1], generalized substitutions [2, 4], and Brun's algorithm.

2 Discrete lines and Sturmian words

A standard arithmetic discrete line is made of horizontal and vertical steps. One can code such a standard line by using the *Freeman code* over the two-letter alphabet $\{0, 1\}$ as follows: one codes horizontal steps by a 0, and vertical ones by a 1. One thus gets a Stumian word $(u_n)_{n\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$.

A natural algorithm for the recognition of finite factors of Sturmian words can be obtained by desubstituting according to the two substitutions

$$\sigma_0(0) = 0,\ \sigma_0(1) = 10,\ \sigma_1(0) = 01,\ \sigma_1(1) = 1,$$

and to the choice of the isolated letter. For more details, see, e.g., [8, 9]. Such a desubstitution/recoding process can be translated in terms of continued fraction algorithm and Ostrowski numeration.

3 Generalized substitutions

We now introduce similar objects in the higher-dimensional case.

The *free group* over $\{1, \ldots, d\}$ is denoted by F_d. A *morphism* of F_d is thus a map $\sigma : F_d \rightarrow F_d$ such that, for any u, v in F, $\sigma(uv) = \sigma(u)\sigma(v)$.

A morphism is said to be *non-negative* if it maps each letter of $\{1, \ldots, d\}$ to a word over $\{1, \ldots, d\}$, and it is said to be *non-erasing* if it does not map any letter to the empty word. Positive non-erasing morphisms are usually called *substitutions*. The *incidence matrix* M_σ of a morphism σ of F_d is the $d \times d$ matrix whose entry at i-th row and j-th column is the number of occurrences of the letter i in $\sigma(j)$. A morphism is said to be *unimodular* if its incidence matrix belongs to the linear group $GL(d, \mathbb{Z})$, that is, has determinant ± 1. According to the formalism developed in [2, 4], it is possible to associate with any unimodular morphism of the free group σ a so-called *generalized substitution* acting on unions of faces as follows:

$$E_1^*(\sigma)(\boldsymbol{x}, i^*) = \sum_{j|\sigma(j)=pis} (M_\sigma^{-1}(\boldsymbol{x} - \boldsymbol{f}(p)), j^*) - \sum_{j|\sigma(j)=pi^{-1}s} (M_\sigma^{-1}(\boldsymbol{x} - \boldsymbol{f}(p) + \boldsymbol{e}_i), j^*).$$

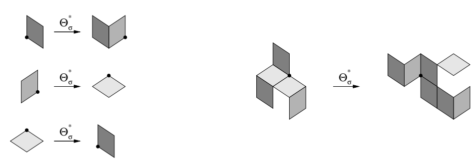

Fig. 3. Generalized substitution associated with a morphism (acting on the faces $(\mathbf{0}, i^*)$)

It is well-known that the image of a discrete plane under the action of $E_1(\sigma)^*$, when σ is a substitution, is again a discrete plane plane according to [2, 5].

Proposition 1 ([2, 5]). *Let σ be a unimodular substitution. Let $\boldsymbol{\alpha} \in \mathbb{R}_+^d$ be a nonzero vector. The generalized substitution Θ_σ^* maps without overlaps the discrete plane $\mathcal{P}_{\boldsymbol{\alpha},\rho}$ onto $\mathcal{P}_{{}^tM_\sigma \boldsymbol{\alpha},\rho}$.*

We now extend the domain of definition of generalized substitutions to more general geometric objects.

A *stepped surface* (also called *functional discrete surface*) is defined as a union of pointed faces such that the orthogonal projection π onto the antidiagonal plane $(\boldsymbol{e}_1 + \ldots + \boldsymbol{e}_d)^\perp$ induces an homeomorphism from the stepped surface onto the antidiagonal plane.

One interest of this notion relies in the fact it is possible to recognize whether a set of points in $\mathbb{Z}^d$ is contained in a stepped surface by considering only a finite neighbour of each point [6]. Furthermore, generalized substitutions act not only on stepped planes but also on stepped surfaces according to [1].

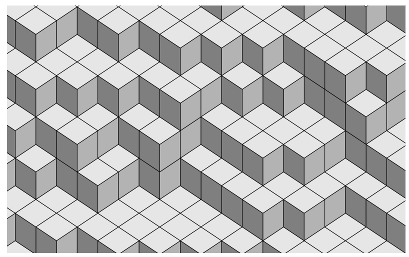

Fig. 4. A piece of a stepped surface

4 Desubstitution

The point now is to try to desubstitute according to generalized substitutions. We will use the following key property

$$E_1^*(\sigma \circ \mu) = E_1^*(\mu) \circ E_1^*(\sigma),$$

that holds for any two morphisms of the free group σ, μ, applied to an invertible substitution σ and to its inverse σ^{-1}. Consequently, desubstitution according to $E_1^*(\sigma)$ consists in the action of $E_1^*(\sigma^{-1})$.

Let σ be a unimodular substitution. A stepped surface is said to be σ-*tilable* if it is a union of translates of $E_1(\sigma)^*(\mathbf{0}, i^*)$, for $1 \leq i \leq d$. The following question is thus natural: can we desubstitute a σ-tilable stepped surface?

Theorem 1. *Let σ be an* invertible *substitution. Let $\mathcal{S}$ be a σ-tilable stepped surface. We assume that there exists a non-zero vector $\boldsymbol{\alpha}$ with non-negative entries such that $M_\sigma^{-1}(\boldsymbol{\alpha})$ has non-negative entries. Then, $E_1^*(\sigma^{-1})(\mathcal{S})$ is a stepped surface.*

Our proof is based on a geometrical approach, using the generation of functional stepped surfaces by *flips*. A flip is a classical notion in the study of dimer tilings and lozenge tilings associated with the triangular lattice. It consists in a local reorganization of tiles that transforms a tiling into another one. Such a reorganization can also be seen in the 3-dimensional space on the functional stepped surface itself. Suppose indeed that a functional stepped surface contains

3 faces that form the lower faces of a unit cube with integer vertices. By replacing these three faces by the upper faces of this cube, one obtains another functional stepped surface. According to [1], any functional stepped surface can be obtained from an arithmetic discrete plane by a sequence of flips, possibly infinite but locally finite, in the sense that, for any bounded neighborhood of the origin in the antidiagonal plane, there is only a finite number of flips whose domain has a projection which intersects this neighborhood.

We thus can come back via the notion of flips to the case of discrete planes. We then use the following:

Proposition 2. *Let σ be a unimodular morphism of the free group. Let $\boldsymbol{\alpha} \in \mathbb{R}_+^d$ be a nonzero vector such that*

$$^tM_\sigma \boldsymbol{\alpha} \geq 0.$$

Then, $E_1^*(\sigma)$ maps without overlaps the discrete plane $\mathcal{P}_{\boldsymbol{\alpha},\rho}$ onto $\mathcal{P}_{^tM_\sigma \boldsymbol{\alpha},\rho}$, that is, the stepped plane $\mathcal{P}_{\boldsymbol{\alpha},\rho}$ is σ-tilable, if and only if

$$^tM_{\sigma^{-1}} \boldsymbol{\alpha} \geq 0.$$

5 Brun's algorithm

We will see that Proposition 2 can be reformulated in arithmetic terms, when applied to particular substitutions associated with Brun's multidimensional continued fraction algorithm.

Definition 1. *The* d-dimensional Brun map T *is defined over* $[0,1]^d \backslash \{0\}$ *by:*

$$T(\alpha_1, \ldots, \alpha_d) = \left(\frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_d}{\alpha_i} \right),$$

where i is equal to the smallest index i' such that $\alpha_{i'} = \max_j \alpha_j$.

It is convenient to provide a matrix viewpoint on Brun expansions. For $a \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$, let us define the following $(d+1) \times (d+1)$ matrix:

$$B_{a,i} = \begin{pmatrix} a & & 1 & \\ & I_{i-1} & & \\ 1 & & 0 & \\ & & & I_{d-i} \end{pmatrix},$$

where I_p stands for the $p \times p$ identity matrix and all the unspecified coefficients are equal to zero. Note that $B_{a,i}$ has integer entries and determinant -1, and thus belongs to the linear group $GL(d+1, \mathbb{Z})$. Consider now $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d \backslash \{\mathbf{0}\}$. For $i = \min\{j \mid \alpha_j = ||\boldsymbol{\alpha}||_\infty\}$ and $a = \lfloor \alpha_i^{-1} \rfloor$, an easy computation shows that

$$(1, \boldsymbol{\alpha}) = ||\boldsymbol{\alpha}||_\infty B_{a,i}(1, T(\boldsymbol{\alpha})),$$

where $(1, \boldsymbol{u}) = (1, u_1, \ldots, u_n)$ for $\boldsymbol{u} = (u_1, \ldots, u_n)$. In particular, if $\boldsymbol{\alpha}$ has Brun expansion $(a_n, i_n)_n$, this yields, for every suitable n:

$$(1, \boldsymbol{\alpha}) = \mu_n M_n(1, T^{n+1}(\boldsymbol{\alpha})),$$

where $\mu_n = ||T^0(\boldsymbol{\alpha})||_\infty \times \ldots \times ||T^n(\boldsymbol{\alpha})||_\infty$ and $M_n = B_{a_0,i_0} \ldots B_{a_n,i_n}$.

With each step of the algorithm, we can associate an invertible substitution $\beta_{a,i}$.

Definition 2 (Brun morphism). *Let $\beta_{a,i}$ be the morphism of free group over $\{1,\ldots,d+1\}$ defined, for $a \in \mathbb{N}$ and $i \in \{1,\ldots,d\}$, by:*

$$\beta_{a,i} \ : \ \begin{cases} 1 & \mapsto \quad 1^a(i+1) \\ (i+1) & \mapsto \quad 1 \\ j & \mapsto j \quad \textit{for } j \neq 1, i+1. \end{cases}$$

It is easily checked that $\beta_{a,i}$ has incidence matrix $B_{a,i}$. In particular, $\beta_{a,i}$ is unimodular since $B_{a,i} \in GL(d+1,\mathbb{Z})$. Note that $\beta_{a,i}$ is a substitution, that is, a positive and non-erasing morphism. Let us note furthermoe that the $B_{a,i}$'s are symmetric matrices. The substitution $\beta_{a,i}$ is moreover an automorphism:

$$\beta_{a,i}^{-1} \ : \ \begin{cases} 1 & \mapsto \quad (i+1) \\ (i+1) & \mapsto (i+1)^{-a}1 \\ j & \mapsto \quad j \end{cases} \quad \text{and} \quad M_{\beta_{a,i}^{-1}} = B_{a,i}^{-1} = \begin{pmatrix} 0 & & 1 & & \\ & I_{i-1} & & & \\ 1 & & -a & & \\ & & & & I_{d-i} \end{pmatrix}.$$

One gets

Proposition 3. *For any $\boldsymbol{\alpha} \in [0,1]^d \backslash \{\mathbf{0}\}$ and $\rho \in \mathbb{R}$,*

$$\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho} = E_1^*(\beta_{a,i})(\mathcal{P}_{||\boldsymbol{\alpha}||_\infty(1,T(\boldsymbol{\alpha})),\rho}),$$

or, equivalently:

$$E_1^*(\beta_{a,i}^{-1})(\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}) = \mathcal{P}_{||\boldsymbol{\alpha}||_\infty(1,T(\boldsymbol{\alpha})),\rho},$$

where $i = \min\{j \mid \alpha_j = ||\boldsymbol{\alpha}||_\infty\}$ and $a = \lfloor \alpha_i^{-1} \rfloor$.

Thus, we can relate the action of the Brun map T on a vector $\boldsymbol{\alpha}$ to the action of a dual map $E_1^*(\beta_{a,i}^{-1})$ on the stepped plane $\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}$. Hence, by identifying the vector $\boldsymbol{\alpha}$ with the hyperplane with normal vector $(1,\boldsymbol{\alpha})$, it is possible to define the Brun's expansion of a discrete plane, and even of a stepped surface.

We will use here the unimodularity and the weak convergence of Brun's algorithm. The main result of our lecture is the following:

Theorem 2. *If a stepped surface $\mathcal{S}$ has the same Brun expansion as a totally irrational vector $\boldsymbol{\alpha} \in [0,1]^d \backslash \{0\}$, then it is a stepped plane of normal vector $\boldsymbol{\alpha}$.*

We conclude this lecture by discussing the application of this theorem to the recognition problem for discrete planes.

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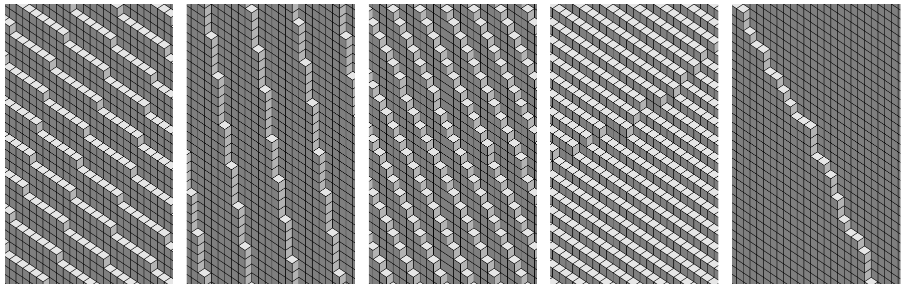

Fig. 5. An examples of Brun's expansion for a nonplane stepped surface

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