DISCRETE PLANES, $\mathbb{Z}^2$-ACTIONS, JACOBI-PERRON ALGORITHM AND SUBSTITUTIONS

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Abstract. We introduce two-dimensional substitutions generating two-dimensional sequences related to discrete approximations of irrational planes. These two-dimensional substitutions are produced by the classical Jacobi-Perron continued fraction algorithm, by the way of induction of a $\mathbb{Z}^2$-action by rotations on the circle. This gives a new geometric interpretation of the Jacobi-Perron algorithm, as a map operating on the parameter space of $\mathbb{Z}^2$-actions by rotations.

0. Introduction

The aim of this paper is to discuss an explicit method to build a discrete approximation of an irrational plane in $\mathbb{R}^3$. Such an approximation can be either studied as a stepped surface [24, 25] or it can be described by a two-dimensional sequence, indexed by $\mathbb{Z}^2$, defined on a three-letter alphabet [42]. Furthermore, such a sequence is directly related to symbolic dynamics for a $\mathbb{Z}^2$-action by rotations on the unit circle [10, 11, 12].

We will show that this sequence can be generated by applying the Jacobi-Perron algorithm to the coordinates of the unit vector orthogonal to the given plane; this algorithm produces a sequence of generalized substitutions, which can be interpreted as acting on two-dimensional sequences, creating arbitrarily large parts of the symbolic sequence associated with the plane. These generalized substitutions are deeply connected to the higher-dimensional extensions of substitutions introduced in [24, 25, 3, 5], which act on unit tips (faces of unit cubes with integer vertices), generating the stepped surface.

This paper grew out of an attempt to generalize to higher dimensions well-known results for usual continued fractions. Let us first recall shortly the framework of the usual continued fractions, and its interpretation in terms of dynamics of rotations, Sturmian sequences and approximation of irrational lines; we will be more precise in the first section.

Consider a line with positive irrational direction $ay = bx$. One can approximate in an obvious way this line by a broken line made of horizontal and vertical segments with integer vertices (this is what is done, for example, to represent such a line on a computer screen [7]).

One can represent this line by a symbolic sequence with values in the alphabet $\{0, 1\}$. This family of sequences has been much studied: they are the so-called *Sturmian sequences*, and it is well-known that they are linked in a natural way with the symbolic dynamics of rotations (see [29] and see also the surveys [1, 2, 8, 9]). The dynamical system generated by the shift on such a sequence is a one-to-one extension, except on a countable set, of an irrational rotation, of angle $b/(a+b)$. Furthermore, each such sequence can be generated by an infinite sequence of substitutions, made only by two elementary substitutions σ_0, σ_1 (see [6] and the survey [16]); this sequence of substitutions can be written as $\sigma_0^{a_0}\sigma_1^{a_1}\sigma_0^{a_2}\ldots$, where $[a_0; a_1, a_2, \ldots]$ is the continued fraction expansion of the slope b/a of the line. A substitution over the alphabet $\mathcal{A}$ is a non-erasing morphism of the free monoid $\mathcal{A}^*$ endowed with the concatenation.

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We wish to extend the previous concepts to the two-dimensional case. The most natural problem is to try to approximate a pair of real numbers (α,β) (such that $1,\alpha,\beta$ are rationally independent) by rational numbers with the same denominator. Many algorithms have been proposed for this, one of the oldest being the classical Jacobi-Perron algorithm.

Recall that the Jacobi-Perron algorithm is defined on $(0,1)^2$ by the maps: $\Phi(\alpha,\beta) = (\frac{\beta}{\alpha} - \lfloor\frac{\beta}{\alpha}\rfloor, \frac{1}{\alpha} - \lfloor\frac{1}{\alpha}\rfloor)$ and $F(\alpha,\beta) = (\lfloor\frac{\beta}{\alpha}\rfloor, \lfloor\frac{1}{\alpha}\rfloor)$; with each pair of irrational and rationally independent numbers (α,β), we can associate the sequence $F(\Phi^n(\alpha,\beta))_{n\in\mathbb{N}}$, and one can easily obtain, from this sequence, a sequence of pairs of rational numbers that approximate the initial pair (α,β). For more details, see for instance [13]. The dynamical system defined by Φ on the unit square has been much studied [40], in particular its invariant measures and its unique invariant ergodic probability measure equivalent to Lebesgue measure [14, 15] (generalization of the classical Gauss measure).

Our purpose here is to show that, in the same way as classical continued fractions can be interpreted in terms of induction of rotations, this algorithm can be interpreted in terms of induction of $\mathbb{Z}^2$-actions by rotations on the circle. We will define a sequence of substitutions corresponding to Jacobi-Perron algorithm, and show how it can be used to generate double sequences coding the approximation of an irrational plane in the three-dimensional space by a discrete plane.

This paper is organized as follows.

In Section 1, we recall some classical results on discrete lines, rotations and continued fractions, and a less classical way to compute the discrete line associated with a line of equation $ax + by = 0$; these are the results and techniques we generalize in this paper.

In Section 2, we consider a plane $\mathcal{P} : ax + by + cz + h = 0$, with a, b, c strictly positive, and define the stepped surface associated with this plane, as the upper boundary of the set of unit cubes with integer vertices that intersect this plane. We show that, by projecting this stepped surface on the diagonal plane $x + y + z = 0$ along the main diagonal direction $(1,1,1)$, and considering the lattice Γ, projection of $\mathbb{Z}^3$ on this plane (this lattice is isomorphic to $\mathbb{Z}^2$), one can code the stepped surface as a two-dimensional sequence U with values in a three-letter alphabet (i.e., a map from $\mathbb{Z}^2$ to the set $\{1,2,3\}$). We then recall [10] how one can recover this sequence as a symbolic dynamics for the $\mathbb{Z}^2$-action by two rotations R_a and R_b of respective angles a and b on a circle of length $a + b + c$, and we prove the following result:

Theorem 1. *Let U be the coding of the plane $\mathcal{P} : ax + by + cz + h = 0$, with a, b, c strictly positive. We have*

$$\forall (m,n) \in \mathbb{Z}^2,\ (U_{m,n} = i \Longleftrightarrow R_a^m R_b^n(h) \in \mathcal{I}_i),$$

with $\mathcal{I}_3 = [0, c[$, $\mathcal{I}_2 = [c, c + b[$ and $\mathcal{I}_1 = [b + c, a + b + c[$.

In Section 3, we define the notion of induction for $\mathbb{Z}^n$-actions by rotations on the circle, by considering the equivalence relation generated by the action (its classes are the orbits of the action), and taking the restriction of this relation to a subset. In this framework, we prove a general theorem: under suitable arithmetic conditions, the induced equivalence relation on an interval is again generated by a $\mathbb{Z}^n$-action by rotations. This allows us to give a geometric interpretation of a generalized continued fraction algorithm.

Theorem 2. *Let $a_0, a_1, \ldots, a_n$ be $n+1$ positive real numbers such that $a_0, a_1, \ldots, a_n$ are rationally independent. Let $b_0 = \sum_{i=0}^n k_i a_i$ be a real number such that $0 < b_0 < \sum_{i=0}^n a_i$, with $k_0, \ldots, k_n$ relatively prime integers. Then, there exist numbers $b_1, \ldots, b_n$ such that the induction on an interval of length b_0 of the $\mathbb{Z}^n$-action by rotations $R_{a_1}, \ldots, R_{a_n}$ on the circle of length $a_0 + a_1 + \ldots + a_n$ is generated by n rotations $R_{b_1}, \ldots, R_{b_n}$, defined modulo b_0.*

In Section 4, we show explicitly how to apply this in the case of the Jacobi- Perron algorithm, and explain how we can recover the symbolic dynamics for the initial action from the symbolic dynamics of the $\mathbb{Z}^2$-action via a substitution, which associates with each letter a finite pattern. We give a first result in Theorem 3.

We then give an other form for this pointed substitution, which is more convenient in our framework (here, $(a_1, b_1, c_1; h_1)$ is the image of the initial $(a, b, c; h)$ by the inhomogeneous Jacobi-Perron algorithm defined in Section 4):

Theorem 4. *Let U (respectively U_1) be the coding of the plane $\mathcal{P} : ax + by + cz + h = 0$ (respectively $\mathcal{P}_1 : a_1x + b_1y + c_1z + h_1 = 0$).*

Let $B_1 = \lfloor b/a \rfloor$, $C_1 = \lfloor c/a \rfloor$, $J = [(C_1 - 1)a, c) \cup [c + b - B_1a, c + b)$, $N_1 = \inf\{n \in \mathbb{N},\ n \neq 0 |\ R_a^n(-h) \in J\}$, and $p_1 = \lfloor \frac{h_1 + ma_1 + nb_1}{a_1 + b_1 + c_1} \rfloor$, $m = N_1 - mB_1 - n_1(C_1 + 1) + p_1(B_1 + C_1) + C_1 - 1$, $n = m_1 - n_1$. We have:

- *if $U^1(m_1, n_1) = 1$, then $U(m + B_1, n_1) = 2$;*
- *if $U^1(m_1, n_1) = 2$, then $U(m, n) = 3$;*
- *if $U^1(m_1, n_1) = 3$, then: if $0 \le i < C_1$, $U(m - i, n) = 3$; $U(m - C_1, n) = 1$; if $0 \le i \le B_1 - 1$, $U(m - C_1 + i, n - 1) = 2$.*

Furthermore, this completely defines the sequence U.

This is the central result of the paper; one could rephrase it in this way: when one replaces in U^1 the letter 1 by 2, 2 by 3 and 3 by $\begin{array}{cccc} 1 & 3 & \cdots & 3 \\ 2 & \cdots & 2 & \end{array}$, (with C_1 3's and B_1 2's), the rule of placement of the images of the letters being given by $(m(m_1, n_1), n(m_1, n_1))$, the sequence obtained is exactly U. The definition of these two-dimensional substitutions, unlike the classical one-dimensional case, is not trivial; in particular, it is not immediate to prove the consistency.

In Section 5, we consider these two-dimensional substitutions from two different points of view, first as pointed substitutions, and second, as generated by local rules. Indeed, given the value of the initial sequence at the point x, we first deduce the value of the image sequence on a pointed pattern situated at a point y that can be computed from x and its value. This is however inconvenient for explicit computations. We show also that this sequence can be computed from local rules: if we know the image of the initial point, we can compute the values of adjacent points by using a finite number of patterns, and in this way, compute the image of the complete sequence. This is closer to the usual notion of substitution on one-dimensional sequences, and we will prove in later papers that it can be extended to act on a larger class of sequences.

We will show in Section 6 that one can build directly the stepped surface by the dual map of the one-dimensional extension of a substitution, using the framework of [3, 5], and recover the generalized substitutions of the previous section in a more geometric way. Finally, we give in Section 7 a few additional remarks, and directions for future researches.

This notion of substitution has to be compared with the notion of substitution tiling, which corresponds to a globally defined hierarchical structure in a geometric space (see for instance [18, 32, 33, 34]). It is proved in [22] that one can construct local rules for such tilings under some mild conditions. See also [30] for a generalization of Durand's characterization of minimal substitutive sequences [19] in this framework of substitutive tilings. For a notion of two-dimensional constant length substitutions, replacing each letter by a square of same size, see also [37, 38].

1. Discrete lines, rotations and continued fractions

We will here explain the relation between the discrete approximation of an irrational line, the dynamics of a rotation, and classical continued fractions. We summarize the detailed exposition of [1].

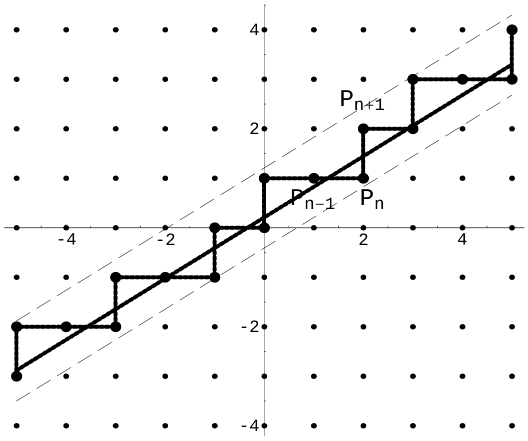

FIGURE 1. The discrete line

There are two ways to approximate a line by a broken line (see Figure 1, 2). The first one that we consider here shows in a more natural way the connections with substitutions. The point of view corresponding to the second one is the one we will generalize in the higher-dimensional case.

1.1. **Approximation of a line: the direct viewpoint.** Consider first a line L with positive irrational slope $y = \alpha x + \eta$. We want to approximate this line by a broken line with integer vertices. One of the most convenient way to do it is to progress by unit segments, either up or to the right, always going in the direction of the line (see Figure 1).

A simple computation shows that the vertices of this broken line are the elements of the set $B = \{(x, y) \in \mathbb{Z}^2 | -\alpha < y - \alpha x - \eta < 1\}$ (we ignore the special case where the line goes through a vertex; we must then take a special convention). This set can be ordered, using the natural partial order on $\mathbb{Z}^2$ given by the positive cone, in a sequence $(P_n)_{n\in\mathbb{Z}}$; since the sequence $P_{n+1} - P_n$ can take only values $(1, 0)$ and $(0, 1)$, we can *code* the broken line as a biinfinite sequence with values in the alphabet $\{0, 1\}$.

A first remark is that this sequence is linked to a rotation; indeed, let π be the projection along the line L on the vertical line of equation $\alpha x + \eta = 0$, through the intersection of L and the horizontal axis. From the formula given above for B, we see that all points of B project to the interval $-\alpha < y < 1$ on this line. Furthermore, if $\pi(P_n)$ is negative, $\pi(P_{n+1}) = \pi(P_n) + 1$, while, if $\pi(P_n)$ is positive, $\pi(P_{n+1}) = \pi(P_n) - \alpha$. Hence, the sequence associated with B is defined by a rotation of angle 1 on a circle of length $1 + \alpha$. This gives the link between the discrete line and the dynamics of the rotation.

We have approximated the line by translates of the two basic unit segments; it is however possible to approximate also using diagonal segments. If the slope is less than 1, we can use segments of direction $(1, 0)$, $(1, 1)$, and if it is greater than 1, we can use segments of direction $(0, 1)$, $(1, 1)$. It is readily seen that, in the first case, the initial symbolic sequence is obtained from the new one by replacing each 1 by 10, and in the second case, by replacing each 0 by 01. This shows that the initial sequence is obtained from the new sequence by one of two elementary substitutions σ_0, σ_1; these two substitutions are related to the induction of the initial rotation on a suitable interval. We can iterate this process, and we obtain a sequence of substitutions $\sigma_0^{a_0}\sigma_1^{a_1}\sigma_0^{a_2}\ldots$, where $[a_0; a_1, a_2, \ldots]$ is the continued fraction expansion of the slope α. In this way, we can recover the stepped line knowing the continued fraction expansion of α, as limit of an infinite sequence of substitutions (at least if the line goes through the origin; in the general case, we need also some information about η, which can be done through an Ostrowsky expansion of η with respect to the continued fraction expansion of α, giving rise to a skew extension of the usual continued fraction map [4]). Remark that, in this way, we only obtain an infinite sequence, so we only know the positive part of the stepped line; this is however always sufficient to completely specify the line.

Instead of generating the symbolic sequence, one can generate directly the discrete line in the following way: this discrete line is made of unit segments starting at points with integral coordinates; one can denote such a segment by (P, i), where P is the integral point, and $i = 0$ if the segment is horizontal, and 1 if it is vertical. Suppose that the line has slope less than 1; then, one can change of basis, taking as new basis $(e_0, e_0 + e_1)$, and consider the new discrete line in this basis.

One can recover the initial discrete line from this one in the following way. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. One first builds the new discrete line, with segments parallel to the two basis vectors e_0 and $e_0 + e_1$. The horizontal segments $(P, 0)$ of the new line are horizontal segments of the initial line, but with initial point $A.P$, because of the change of coordinates; the vertical segments $(P, 1)$ are changed to the union of an horizontal and a vertical segment, and one checks that the image of $(P, 1)$ is the union of $(A.P, 1)$ and $(A.P + (0, 1), 0)$. One defines in this way a map $E_1(\sigma_0)$, which can be extended as a linear map to the space of formal linear combinations of unit segments. For more details, see [3].

It is possible to iterate this operation; with the sequence of substitutions defined above, one associates a sequence of linear maps $E_1(\sigma_0)^{a_0} E_1(\sigma_1)^{a_1} \ldots$, and the images of the unit segment at the origin by these maps converge to the discrete line approximating the line with given slope through the origin. A similar, but more complicated, algorithm allows us to approximate a line that does not go through the origin (see for example [4]).

1.2. **Approximation of a line: the dual viewpoint.** We can consider the similar problem of approximating a line $ax + by = 0$, where $0 < a, b$. In that case, it is more convenient to consider the so-called "stair" over the line (called *stepped line*), that is, the upper boundary of the set of unit squares with integer vertices the interior of which intersect the lower half-plane defined by the line (see Figure 2).

One can here also try to approximate the line by a sequence of bases. There are two possible basic changes of basis: with the initial basis (e_0, e_1), one can associate either $(e_0 - e_1, e_1)$ or $(e_0, e_1 - e_0)$, and we choose at each step the unique basis such that both vectors are "above" the line for the natural partial order in the plane.

With each such change of basis, on can, as above, associate one of the substitutions σ_0, σ_1. We cannot in this situation use directly the linear maps $E_1(\sigma)$ defined previously, for several reasons: first, this would produce an approximation of a line with positive slope. Second, and more importantly, this would produce an approximation of the renormalized line, starting with the given initial line, which is not what we want; the situation has changed from contravariant to covariant, that is, the order of composition of substitutions has been reversed.

In such a case, one would like to use the inverse of the map $E_1(\sigma)$; however, it is readily checked that this map is not invertible. A substitute to the inverse is the transpose map, which also reverses the direction of composition (remark that it is quite natural to obtain dual maps, since the line $ax + by = 0$ can be seen as the kernel of the linear form with coordinates (a, b) in the canonical basis; hence one can consider the approximation of the line as approximation of the linear form, that is, the dual problem of the approximation of a vector). For more details, see [1].

This can be done, using the framework of [3, 5], and we can generate the stair over the irrational line using the dual of the one-dimensional extension of the substitutions we have obtained (we will give more detailed explanations in Section 6).

The aim of this paper is to recover similar results for an irrational plane in the three-dimensional space; it turns out that the dual viewpoint (approximating a linear form, or the plane representing its kernel) generalizes more easily than the direct viewpoint, as we shall see below.

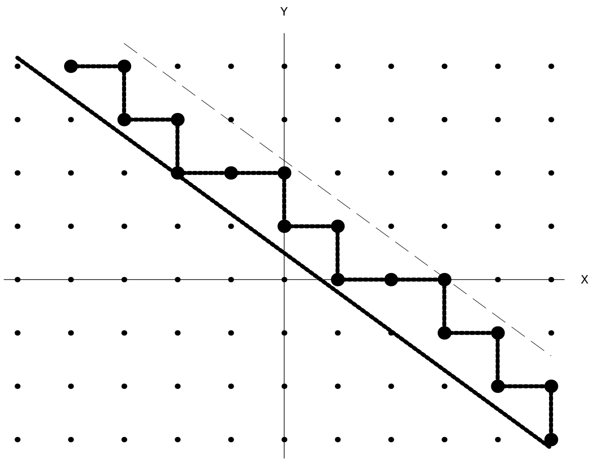

FIGURE 2. The discrete line: dual case

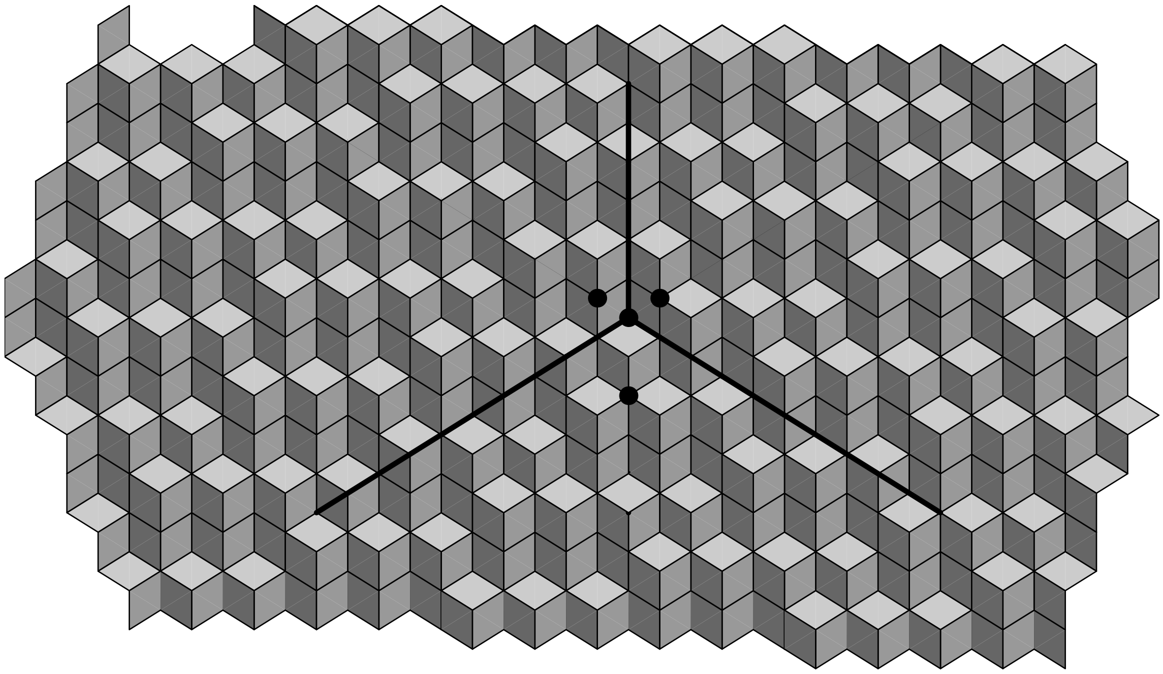

FIGURE 3. The stepped surface

2. Symbolic representation of discrete planes

Our aim in this section is to define the discrete approximation (also called discrete plane or stepped surface, see Figure 3) associated with an irrational plane and to show how we can associate with this stepped surface a symbolic sequence indexed by $\mathbb{Z}^2$. We will then explain how we can recover this symbolic sequence as symbolic dynamics of a $\mathbb{Z}^2$-action generated by two rotations on the circle [10]. Our construction can be rephrased in terms of the classical "cut and project" construction (see for instance [39]); see also [41] for a dual approach.

2.1. **Construction of the stepped surface.** We denote by $(O, \vec{e_1}, \vec{e_2}, \vec{e_3})$ the canonical basis of the space $\mathbb{R}^3$. In $\mathbb{R}^3$, we denote by $\mathcal{P}$ the plane of equation $ax + by + cz + h = 0$, with $a, b, c > 0$. We will always suppose that the plane has totally irrational direction, that is, the triple (a, b, c)

satisfies no rational relation (but we make no assumption on h). We can also assume, without loss of generality, that $c > a, b > 0$; this assumption will be used in Section 4.

We associate with the plane $\mathcal{P}$ a discrete plane P by approximating $\mathcal{P}$ by unit square faces as follows (see Figure 3). This construction corresponds to the stepped surface introduced by Ito and Ohtsuki in [24, 25].

Definition 1. *Let $\mathcal{S}$ be the set of translates of the fundamental cube with integer vertices that intersect the lower half-space $ax + by + cz + h < 0$.*

The discrete plane, *or* stepped surface, *P is defined as the boundary of $\mathcal{S}$.*

A vertex of the stepped surface is an integral point that belongs to P. We denote by V the set of vertices of the stepped surface.

Remark Some authors prefer to consider the set of unit cubes that intersect the plane $\mathcal{P}$; in that case, the boundary of this set has two connected components, and our discrete plane P is the upper component of the boundary.

The definition implies that P is also the boundary of the union of the integral unit cubes the interior of which do not intersect the lower half-space (this is the closure of the complement of $\mathcal{S}$); hence, there is a simple criterium to decide whether an integral point is a vertex of the stepped surface:

Proposition 1. *An integral point (p, q, r) belongs to the set V of vertices of the stepped surface P if and only if $0 \le ap + bq + cr + h < a + b + c$*

Proof. If $0 \le ap + bq + cr + h$, the cube of which (p, q, r) is the lowest corner (for the natural partial order in $\mathbb{R}^3$) does not intersect the plane $\mathcal{P}$, and if $ap + bq + cr + h < a + b + c$, the cube of which it is the highest corner does intersect the lower half space. Hence (p, q, r) belongs to the boundary of $\mathcal{S}$. □

2.2. **A lattice structure for the stepped surface.** We consider now the vertices of the stepped surface. These vertices can be determined by a "cut and project" method, as we have just seen, but they clearly do not form a sublattice of $\mathbb{Z}^3$, since the plane has irrational direction. It is however possible, and very important for the sequel, to impose on the set of vertices a lattice structure, by projecting them on the diagonal plane $x + y + z = 0$.

Let π be the affine projection on the plane $x + y + z = 0$ along the direction (1,1,1). Since the projection is along a rational direction, the projection Γ of the lattice $\mathbb{Z}^3$ is a lattice in the plane $x + y + z = 0$; a simple computation proves that the sublattice $\{(p, q, r) \in \mathbb{Z}^3 |\ p + q + r = 0\}$ is a sublattice of index 3 in Γ (see Figure 4).

Proposition 2. *The projection π is a bijection from V to Γ.*

Proof. Consider an arbitrary point $g \in \Gamma$; by definition, there is $(p, q, r) \in \mathbb{Z}^3$ such that $g = \pi(p, q, r)$. But it is clear that there is exactly one integer $n \in \mathbb{Z}$ such that $0 \le a(p+n) + b(q+n) + c(r+n) + h < a + b + c$, hence g is the image of exactly one element of V. □

Hence, we can parameterize the vertices of the discrete plane by a lattice; but this is not sufficient, and we want to understand the local structure of the discrete plane around a given vertex.

2.3. **Symbolic dynamics for the stepped surface.** The discrete plane P is a union of translates of unit square faces. We use the following notation:

$$E_1 = \{-\lambda \vec{e_2} - \mu \vec{e_3} | (\lambda, \mu) \in [0, 1[^2\},$$
$$E_2 = \{\lambda \vec{e_1} - \mu \vec{e_3} | (\lambda, \mu) \in [0, 1[^2\},$$
$$E_3 = \{\lambda \vec{e_1} + \mu \vec{e_2} | (\lambda, \mu) \in [0, 1[^2\}.$$

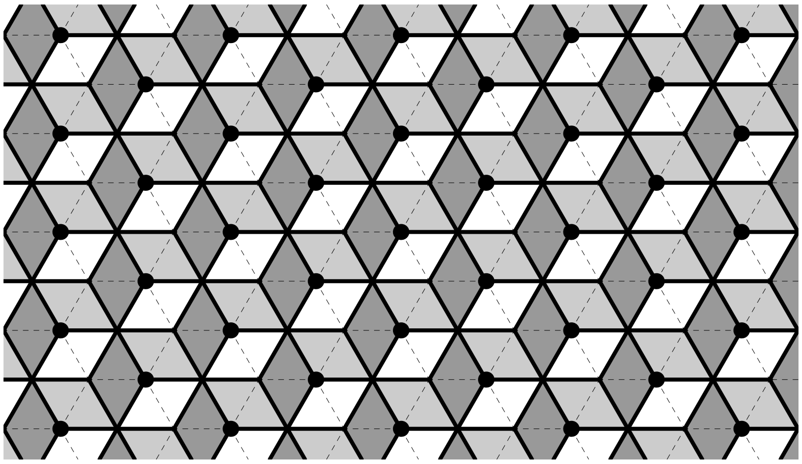

FIGURE 4. The lattice Γ

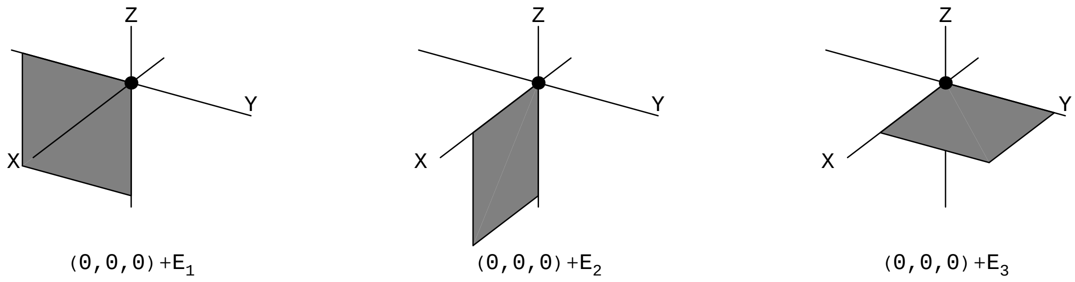

FIGURE 5. The three possible faces with distinguished vertex at the origin

We call *pointed face* of *type* i and *distinguished vertex* $(p, q, r) \in \mathbb{Z}^3$ the set of points

$$\{(p, q, r) + E_i\}.$$

Remark that, because of the signs we have used, (p, q, r) is not always the lowest vertex (for the natural partial order in $\mathbb{R}^3$) of its pointed face: this is the case only for faces of type 3; for a face of type 1, the corresponding vertex is the highest point, while for a face of type 2, it is an intermediate point (see Figure 5). This can seem a cumbersome notation, but it has two important advantages: with this definition, we will see that the pointed faces form a partition of the discrete plane (this is the reason for the semi-open interval and the signs in the definition of the faces), and that each vertex in V is the distinguished vertex of exactly one pointed face in P.

Proposition 3. *The discrete plane P is a union of pointed faces.*

Proof. It is clear from the definition that P, being the boundary of a union of cubes, is a union of squares. The only thing to check is that, because of our convention, each edge and each vertex of P belong to exactly one pointed face of P; we will prove it for the vertices.

It is not immediately clear that a vertex in V cannot be the distinguished vertex of 2 pointed faces: it is not difficult to figure out that each point in V can belong to the closure of 3, 4, 5, or 6 faces (see Figure 3, where we have indicated by dots the four cases).

Remark first that the projection from P to the plane $x+y+z=0$ is one-to-one, since each line parallel to the vector $(1,1,1)$ crosses P exactly once. Hence, the projection of P tiles this plane by three kinds of diamonds with vertices in Γ, corresponding to the three types of faces.

Let us endow the plane $x+y+z=0$ with the basis $(O,\vec{\imath},\vec{\jmath})$, where $\vec{\imath}=\vec{\pi}(\vec{e_1})$, $\vec{\jmath}=\vec{\pi}(\vec{e_2})$. The lattice Γ, having symmetry of order 6, determines a tiling by equilateral triangles. Given an element g of Γ, consider the triangle $(g, g+\vec{\imath}, g+\vec{\imath}+\vec{\jmath})$. This triangle can be completed in exactly one way in a diamond, which corresponds to the projection of a face in P.

One easily checks that, in the three possible cases, our convention have been chosen in such way that the preimage of g will be the distinguished vertex of the square corresponding to that diamond; hence, a vertex cannot be the distinguished vertex of two faces (because their projections would overlap), but must be the distinguished vertex of one face, otherwise the corresponding triangle in the plane would have no preimage. □

The following corollary is an immediate consequence of the preceding proof:

Corollary 1. *The projections of the square faces of P tile the plane by three kinds of diamonds being the projection of a face of type E_k, where $k=1,2$ or 3. Furthermore, each point of V is the distinguished vertex of exactly one pointed face, hence each point of Γ is the projection of a distinguished vertex of a face of determined type.*

Corollary 1 implies that we can **code** the tiling of the plane that we obtain by a double sequence defined over $\mathbb{Z}^2$. Indeed, we can define a sequence indexed by Γ, by associating with each element of Γ the type of the face corresponding to its preimage. But Γ, being a lattice in the plane, is isomorphic (in a non-canonical way) to $\mathbb{Z}^2$.

Definition 2. *Recall that $\vec{\imath}=\vec{\pi}(\vec{e_1})$, $\vec{\jmath}=\vec{\pi}(\vec{e_2})$. Let $U=(U_g)_{g\in\Gamma}$ be the sequence that associates with each point of Γ the type of the face whose distinguished vertex projects on g, or equivalently which codes each triangle with vertices $(g, g+\vec{\imath}, g+\vec{\imath}+\vec{\jmath})$ by the index k of the corresponding diamond $\pi(E_k)$.*

The sequence U is called the coding *of the plane $\mathcal{P}$.*

In the sequel, we will use the basis $(\vec{\imath},\vec{\jmath})$ of Γ, and assimilate Γ to $\mathbb{Z}^2$.

2.4. **Symbolic dynamics for $\mathbb{Z}^2$-actions.** If we follow the proof of Corollary 1 further, we can give an explicit description of the type of an element of V:

Proposition 4. *Let (p,q,r) be an element of V. Then:*

- *If $0\le ap+bq+cr+h<c$, (p,q,r) is the distinguished vertex of a face of type 3.*
- *If $c\le ap+bq+cr+h<b+c$, (p,q,r) is the distinguished vertex of a face of type 2.*
- *If $b+c\le ap+bq+cr+h<a+b+c$, (p,q,r) is the distinguished vertex of a face of type 1.*

Proof. To prove the proposition, remark that the given conditions determine some neighbouring vertices. For example, if $0\le ap+bq+cr+h<c$, we have $a+b\le a(p+1)+b(q+1)+cr+h<a+b+c$; hence $(p+1,q+1,r)$, and, by the same kind of proof, $(p+1,q,r)$ and $(p,q+1,r)$ are elements of V. These are the four vertices of a face of type 3, whose distinguished vertex is (p,q,r). The other cases are proved in a similar way. □

This proposition is not completely satisfying, because we need to know the coordinates (p,q,r) of an element of V. It would be preferable to use only coordinates in Γ, to give an explicit description of the sequence U, and this can be easily achieved.

We shall use the usual representation for two-dimensional sequences: the first index indicates the column number from bottom to top, whereas the second index denotes the row number, from left to right.

Definition 3. *The triple of strictly positive numbers (a,b,c) being fixed, we denote by R_a the map:*

$$R_a : [0, a+b+c[\quad \to [0, a+b+c[\qquad x \mapsto x + a \bmod a+b+c,$$

and similarly, R_b is the map:

$$R_b : [0, a+b+c[\quad \to [0, a+b+c[\qquad x \mapsto x + b \bmod a+b+c.$$

We will call these maps, by abuse of language, rotations of angle a (respectively b) on the interval $[0, a+b+c[$ (since these are conjugate, after identification of points 0 and $a+b+c$, to a circle rotation).

Theorem 1. *Let U be the coding of the plane $\mathcal{P} : ax+by+cz+h = 0$ following Definition 2. We have*

$$U_{m,n} = i \Longleftrightarrow R_a^m R_b^n(h) \in \mathcal{I}_i,$$

with $\mathcal{I}_3 = [0, c[$, $\mathcal{I}_2 = [c, c+b[$ and $\mathcal{I}_1 = [b+c, a+b+c[$.

Proof. Let (m,n) be an element of Γ, and (p,q,r) its preimage in V. We know that the type of (p,q,r) depends only on $ap+bq+cr$. But we have $(m,n) = (p-r, q-r)$, hence $ap+bq+cr = am+bn+r(a+b+c)$.

This proves that $am+bn$ and $ap+bq+cr$ are congruent modulo $a+b+c$, and the theorem follows immediately from the definition of R_a and R_b and Proposition 4; for example, for a vertex of type 3, we must have $0 \le ap+bq+cr+h < c$, that is, $0 \le ma+nb+r(a+b+c)+h < c$: this exactly means that $R_a^m R_b^n(h) \in \mathcal{I}_3$. $\square$

3. Induction of $\mathbb{Z}^2$-actions

3.1. **General framework.** We keep the same notation as in the previous sections, a, b and c being rationally independent positive real numbers.

Since the two rotations R_a and R_b on the circle of length $a+b+c$ commute, they generate a free $\mathbb{Z}^2$-action on this circle, by $(m,n).x = R_a^m R_b^n x$. Our aim is to understand better the small scale structure of this $\mathbb{Z}^2$-action (that is, the way elements of an orbit $R_a^m R_b^n x$ can approximate the initial element x), and we want to use for that purpose the tool of *induction*, as one does in the case of a unique rotation on the circle. There are however two difficulties:

First, the *induced map* T_A of a map T on a subset A is easily defined by $T_A(x) = T^{n_x}(x)$, with $n_x = \inf\{p > 0 | T^p(x) \in A\}$. But this definition uses in a fundamental way the order structure of $\mathbb{Z}$, and cannot be extended as such to $\mathbb{Z}^2$: for a $\mathbb{Z}^2$-action, it does not make sense to define a "first return map". We will explain below how we can define a notion of induction for $\mathbb{Z}^2$-action, by considering the equivalence relation related to the $\mathbb{Z}^2$-action, that is, the equivalence relation whose classes are the orbits of the action; we will consider the induced equivalence relation obtained by restriction to a subset. It is however unclear (and in fact, it is generally not the case) that such an induced equivalence relation comes from a $\mathbb{Z}^2$-action by rotations. It is a remarkable fact that it is the case for a suitable subinterval.

Second, the induced map of a rotation on an arbitrary subinterval of the circle is usually not a rotation, but an exchange of three intervals, with quite different ergodic properties (for example, most of these, contrary to rotations, are weakly mixing [26]). However, for suitable admissible subintervals, which have in particular a length that is an integral linear combination of a and $a+b+c$, it is the case that the induced map of R_a is again a rotation (on a circle of length $a+b+c$) on the given subinterval. Since the numbers a, b, c are rationally independent, it is impossible to find an interval on which the induced maps of R_a and R_b are both rotations. Hence, it is quite surprising that we can find, as we will prove below, a suitable subinterval on which the induction of the $\mathbb{Z}^2$-action is again generated by a $\mathbb{Z}^2$-action by a pair of rotations.

Note that this induction involves a non-trivial rearrangement of the orbit: an orbit for the induced action is always isomorphic to $\mathbb{Z}^2$. It is also included in an orbit of the original action, and

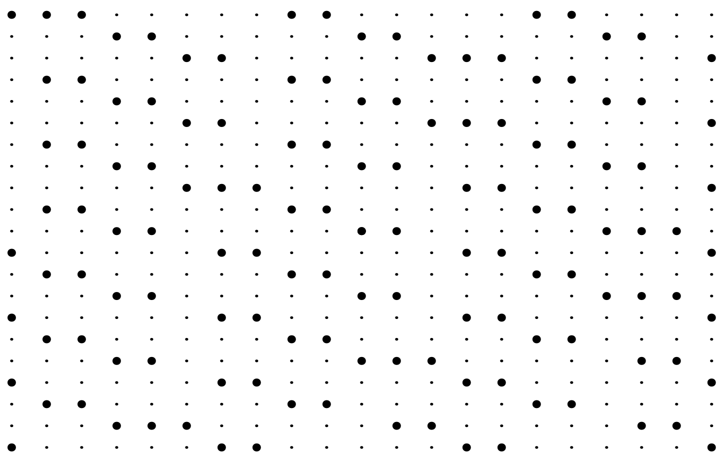

FIGURE 6. An induced orbit

this original orbit too is isomorphic to $\mathbb{Z}^2$, hence the induced orbit can be considered in a natural way as a subset of $\mathbb{Z}^2$: if I is the induction interval, consider the set of (m, n) such that $R_a^m R_b^n x \in I$. However, this subset is NOT a sublattice of $\mathbb{Z}^2$ (see Figure 6, where we have shown the points in the $\mathbb{Z}^2$-orbit that fall in a given subinterval).

We will express in Section 5 this correspondence in combinatorial terms, via the notion of substitutions, and explain how one can generate symbolic dynamics for the initial action using the symbolic dynamics for the $\mathbb{Z}^2$-action.

Note finally that, unlike the classical $\mathbb{Z}$-action, the generators of the $\mathbb{Z}^2$-action are not canonically defined (since we can find an infinite number of bases for the lattice $\mathbb{Z}^2$); hence, there is a large choice for the induction procedure. This is to relate to the fact that there seems to be no way to define a "best" two-dimensional continued fraction algorithm. In Section 4, we will define a particular induction process related to the Jacobi-Perron algorithm; other choices are obviously possible. (See [13] for other examples of two-dimensional continued fraction algorithms, such as Brun's or Selmer's algorithms.)

3.2. **Induction of $\mathbb{Z}^n$-actions: definitions.** With a $\mathbb{Z}^n$-action on a set S, one can always associate an equivalence relation on S, two points being equivalent if they belong to the same orbit.

Definition 4. *If I is a subset of S, we define the* induced equivalence relation *on I as the restriction of the original equivalence relation to the set I.*

Definition 5. *Consider a free $\mathbb{Z}^n$-action by rotations (as defined in Definition 3) on an interval S, and I a subinterval of S. We call* generator *of the induced equivalence relation, a free $\mathbb{Z}^n$-action on the set I such that both equivalence relations coincide: we say that this new $\mathbb{Z}^n$-action* generates *the induction of the initial action on the subset I.*

If the $\mathbb{Z}^n$-action has all dense orbits and I contains an open set, the classes of the equivalence relation are countable, and we can certainly find a $\mathbb{Z}^n$-action on I with the same orbits, but there is in general no natural way to exhibit generators for this $\mathbb{Z}^n$-action.

In the case of a $\mathbb{Z}$-action, one can be more explicit: we can define as above the induced map, which is a generator for the induced equivalence relation. If this $\mathbb{Z}$-action is given by an irrational rotation R_a on the circle $\mathbb{R}/\mathbb{Z}$, and we induce on a (half-open) interval of the circle, it is easy to compute explicitly the induced map on I. In the general case, it is an exchange of three intervals, which has discontinuity points. For some special interval (the so-called *admissible* intervals), in particular for intervals of length $1 - ka$, with $0 \leq k < 1/a$, the induced map turns out to be a rotation on the smaller circle obtained by identifying extremities of I; iteration of this procedure leads to the classical continued fraction algorithm. We can define a similar notion for $\mathbb{Z}^n$-actions:

Definition 6. *Consider a free $\mathbb{Z}^n$-action by rotations on an interval S, and I a subinterval of S. The interval I is said* admissible *if the induced equivalence relation has a generator by rotations.*

We can in fact generalize almost exactly the above result for free $\mathbb{Z}^n$-actions by rotations on the unit circle:

Theorem 2. *Let $a_0, a_1, \ldots, a_n$ be $n+1$ positive real numbers such that $a_0, a_1, \ldots, a_n$ are rationally independent. Let $b_0 = \sum_{i=0}^n k_i a_i$ be a real number in $(0,1)$, with $k_0, \ldots, k_n$ relatively prime integers. Then, there exist numbers $b_1, \ldots, b_n$ (with $b_0, b_1, \ldots, b_n$ rationally independent), such that the induction on an interval of length b_0 of the $\mathbb{Z}^n$-action by rotations $R_{a_1}, \ldots, R_{a_n}$ on the circle of length $a_0 + a_1 + \ldots + a_n$ is generated by the free $\mathbb{Z}^n$-action by n rotations $R_{b_1}, \ldots, R_{b_n}$ defined on a circle of length b_0.*

Proof. Lift the $\mathbb{Z}^n$-action by rotations $R_{a_1}, \ldots, R_{a_n}$ to the universal cover $\mathbb{R}$ of the circle of length $a_0+a_1+\ldots+a_n$; we obtain a $\mathbb{Z}^{n+1}$- action by translations (we add the translation by $a_0+a_1+\ldots+a_n$). The irrationality condition means that the lattice $\mathbb{Z}^{n+1}$ acts without fixed point on $\mathbb{R}$.

Consider any indivisible element $b_0 = \sum_{i=0}^n k_i a_i$ of the lattice $\mathcal{L}$ with basis $(a_0, a_1, \ldots, a_n)$ (an element of the lattice is called *divisible* if it is a nontrivial product by an integer of an element of the lattice; indivisibility is equivalent to the fact that the coordinates $k_0, k_1, \ldots, k_n$ are relatively prime integers). It can be completed by numbers $b_1, \ldots, b_n$ in a basis for the lattice $\mathcal{L}$. If we quotient $\mathbb{R}$ by the translation by b_0, the other translations define rotations of the same angle b_i on a circle of length b_0; this is exactly what we want to prove. □

We could prove that we have in fact an equivalent condition: a subinterval is admissible if and only if its length is an indivisible element of the lattice; but we will not need this fact.

3.3. **An algorithm for induction of $\mathbb{Z}^2$-actions by rotations.** We consider now a $\mathbb{Z}^2$-action by rotations R_a, R_b on a circle of length $a+b+c$. An *induction algorithm* is a way to define an admissible subinterval, and to give explicit generators for the $\mathbb{Z}^2$-action obtained by induction on the admissible subinterval.

Here is a way to proceed (see Figure 7): suppose that $a, b < c$. Consider the interval

$$J := [(\lfloor c/a \rfloor - 1)a, c + b - \lfloor b/a \rfloor a).$$

The length $|J|$ of this interval is $a+b+c-(\lfloor c/a \rfloor + \lfloor b/a \rfloor)a$, hence it satisfies the condition of Theorem 2. One checks immediately that the numbers $c - \lfloor c/a \rfloor a$ and $b - \lfloor b/a \rfloor a$ are generators for the induced action on a circle of length $|J| = c - \lfloor c/a \rfloor a + b - \lfloor b/a \rfloor a + a$; they satisfy again the induction hypothesis ($c - \lfloor c/a \rfloor a$, $b - \lfloor b/a \rfloor a < a$), hence we can iterate. We will show below that this algorithm is a dynamical version of the classical Jacobi-Perron algorithm.

4. Jacobi-Perron algorithm

4.1. **The classical Jacobi-Perron algorithm and a linear version.** The usual Jacobi-Perron algorithm, which will be, in our terms, the projective Jacobi-Perron algorithm, is usually defined on the unit square in the following way (see for instance [13]):

Definition 7. *The projective Jacobi-Perron algorithm is defined on the unit square $X = [0,1) \times [0,1)$ by the transformation Φ:*

$$\Phi(\alpha, \beta) = \begin{cases} \left(\frac{\beta}{\alpha} - \lfloor \frac{\beta}{\alpha} \rfloor, \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor\right) & \text{if } (\alpha, \beta) \in X - I \\ (0, \beta) & \text{if } (\alpha, \beta) \in I, \end{cases}$$

where I is given by $I = \{(0, \beta);\ \beta \in [0,1)\}$.

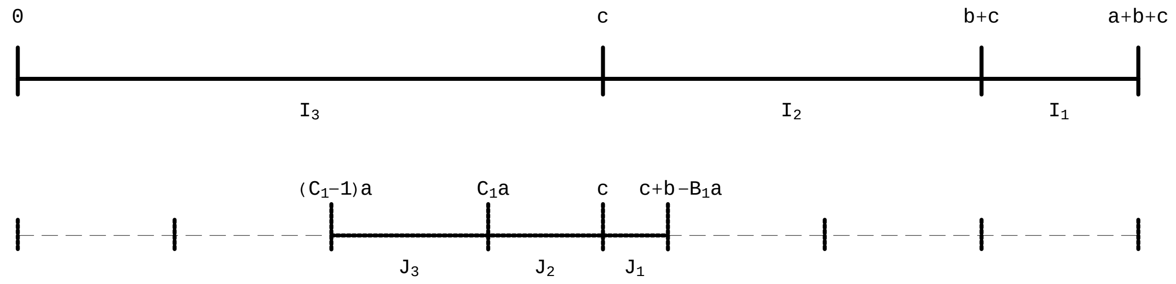

FIGURE 7. Induction on the interval J

This is a map that is piecewise rational, and we can consider it as a projective map, coming from a piecewise linear map in three dimensions; we can define in this way a linear algorithm.

Definition 8. *The linear Jacobi-Perron algorithm is defined on the positive cone* $\{(a,b,c) \in \mathbb{R}^3 | 0 \leq a, b < c\}$ *by the transformation* F*:*

$$F(a,b,c) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

Since we will always suppose that the numbers are irrational, we do not bother to define F if $a = 0$. Remark that, if we renormalize the last coordinate to 1, we recover the initial piecewise rational transformation.

4.2. **The induction algorithm.** We consider a $\mathbb{Z}^2$-action on the interval $I := [0, a+b+c)$ by two rotations R_a and R_b. This is the action used in Section 2 to obtain the symbolic sequence related to the discrete plane $ax + by + cz + h = 0$.

Following Theorem 1, to obtain the symbolic sequence, we partition the interval $[0, a+b+c)$ into 3 subintervals: $I_1 = [b+c, a+b+c)$, of length a, $I_2 = [c, b+c)$, of length b, and $I_3 = [0, c)$, of length c (these intervals are naturally linked to the generators of the action: I_1 and $I_2 \cup I_3$ are the continuity intervals of the map R_a (considered as an exchange of two intervals), while I_3 and $I_1 \cup I_2$ are the continuity intervals of R_{a+b}).

Let

$$J := [(\lfloor c/a \rfloor - 1)a, c + b - \lfloor b/a \rfloor a).$$

We will induce on the subinterval J obtained by subtracting as many times as possible the interval I_1 from I_2 and I_3 (see Figure 7). More precisely, the union $\cup_{-\lfloor b/a \rfloor \leq k < \lfloor c/a \rfloor} R_a^k(I_1)$ is an interval of the circle, whose complement is an interval naturally partitioned into three intervals J_1, J_2, J_3 of respective lengths $b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a$: we have

$$J_3 = [(\lfloor c/a \rfloor - 1)a, \lfloor c/a \rfloor a),$$

$$J_2 = [\lfloor c/a \rfloor a, c),$$

$$J_1 = [c, c + b - \lfloor b/a \rfloor a).$$

We recognize the Jacobi-Perron algorithm, and Theorem 2 shows that the induced $\mathbb{Z}^2$-action is given by $F(a,b,c)$ (Definition 8). It is then clear that we can iterate the process. The aim of this section is to formulate this in more precise terms.

4.3. **The inhomogeneous Jacobi-Perron algorithm.** We are interested in the discrete plane associated with $ax+by+cz+h=0$, or equivalently with the symbolic dynamics of the orbit of h for the $\mathbb{Z}^2$-action associated with the triple (a,b,c) on the circle $[0,a+b+c)$. We define h_1 by

$$h_1 = R_a^n(h) - (\lfloor c/a \rfloor - 1)a,$$

where n is the smallest nonnegative integer such that $R_a^n(h)$ belongs to J. Recall that $J = [(\lfloor c/a \rfloor - 1)a, c+b-\lfloor b/a \rfloor a)$. Hence n satisfies:

$$n = \begin{cases} \lfloor \frac{-h}{a} \rfloor + (\lfloor c/a \rfloor - 1), \text{ if } h \in [0, (\lfloor c/a \rfloor - 1)a), \\ 0, \text{ if } h \in J, \\ \lfloor \frac{a+b+c-h}{a} \rfloor + (\lfloor c/a \rfloor - 1), \text{ otherwise.} \end{cases}$$

We thus define an inhomogeneous (projective and linear) version of the Jacobi-Perron algorithm.

In the Jacobi-Perron algorithm, one substracts the rotation vector a as much as possible from the other two quantities b and c. This algorithm will be shown to act on the whole system of all the orbits under the $\mathbb{Z}^2$-action associated with the triple (a,b,c) on the circle $[0,a+b+c)$. The inhomogeneous Jacobi-Perron algorithm will act on the orbit of a given point $h \in [0,a+b+c)$ as follows: the original point h is translated by multiples of the rotation vector a so that first, the image of h lands into the iduction interval J of length $a_1+b_1+c_1$. It is then translated (again by multiples of the rotation vector a) so that it lands into the interval $[0,a_1+b_1+c_1)$, on which the algorithm is defined.

Definition 9. *The inhomogeneous projective Jacobi-Perron algorithm is defined on* $\{(\alpha,\beta,\kappa) \in (0,1)\times(0,1)\times(0,1+\alpha+\beta)\}$ *by the transformation* $\tilde{\Phi}$*:*

if $0<\kappa<\lfloor 1/\alpha \rfloor - 1$*, then*

$$\tilde{\Phi}(\alpha,\beta;\kappa) = \left(\frac{\beta}{\alpha} - \lfloor \frac{\beta}{\alpha} \rfloor, \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor; \frac{\kappa}{\alpha} + \lfloor \frac{-\kappa}{\alpha} \rfloor)\right);$$

if $\lfloor 1/\alpha \rfloor - 1 \le \kappa < \beta + 1 - \lfloor 1/\beta \rfloor$*, then*

$$\tilde{\Phi}(\alpha,\beta;\kappa) = \left(\frac{\beta}{\alpha} - \lfloor \frac{\beta}{\alpha} \rfloor, \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor; \frac{\kappa}{\alpha} - \lfloor 1/\alpha \rfloor + 1\right);$$

if $\beta + 1 - \lfloor 1/\beta \rfloor \le \kappa < 1+\alpha+\beta$*, then*

$$\tilde{\Phi}(\alpha,\beta;\kappa) = \left(\frac{\beta}{\alpha} - \lfloor \frac{\beta}{\alpha} \rfloor, \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor; \frac{\kappa}{\alpha} + \lfloor \frac{1+\alpha+\beta-\kappa}{\alpha} \rfloor\right).$$

In other words, the inhomogeneous Jacobi-Perron algorithm is a skew product of its homogeneous version.

Definition 10. *We will use in the sequel the following notation:*

$$B_1 = \lfloor b/a \rfloor, \ C_1 = \lfloor c/a \rfloor,$$

$$N_1 = \min\{n \in \mathbb{N}, \ n \neq 0; \ h + na \in J\} - (C_1 - 1).$$

The inhomogeneous linear Jacobi-Perron algorithm is defined on the positive cone $\{(a,b,c;h) \in \mathbb{R}^3 | 0 < a, b < c, 0 < h < a+b+c\}$ *by the transformation* $\tilde{F}$*:*

$$\tilde{F}(a,b,c;h) = (a_1,b_1,c_1;h_1) = (b - B_1 a, c - C_1 a, a; h + N_1 a).$$

Remark If $h=0$, we have $h_1=0$, and this property is conserved by iteration. Hence, the homogeneous algorithm defines the symbolic dynamics of the plane $ax+by+cz=0$.

4.4. **Generation of symbolic sequences.** We can recover the symbolic dynamics of the orbit of h from the symbolic dynamics of h_1 for the induced action.

Theorem 3. *Let U be the symbolic sequence given by the orbit of h for the action defined by the triple (a,b,c) defined as in Theorem 1:*

$$\forall (m,n) \in \mathbb{Z}^2,\ U(m,n) = i \Longleftrightarrow ma + nb + h \in I_i \text{ modulo } (a+b+c),$$

where $I_1 = [b+c, a+b+c[$, $I_2 = [c, b+c[$, $I_3 = [0,c[$. Let $U^1 = (U^1(m_1,n_1))_{(m_1,n_1)\in\mathbb{Z}^2}$ be the symbolic sequence similarly defined by $(a_1,b_1,c_1;h_1) = \tilde{F}(a,b,c;h)$. Let

$$p_1 = \lfloor \frac{h_1 + m_1 a_1 + n_1 b_1}{a_1 + b_1 + c_1} \rfloor,$$

and

$$\left\{ \begin{array}{ll} m & = N_1 - m_1 B_1 - n_1(C_1+1) + p_1(B_1+C_1) + C_1 - 1 \\ n & = m_1 - n_1. \end{array} \right. \tag{1}$$

We have:

- *if $U^1(m_1,n_1) = 1$, then $U(m,n) = 2$,*
- *if $U^1(m_1,n_1) = 2$, then $U(m,n) = 3$,*
- *if $U^1(m_1,n_1) = 3$, then: if $0 \le i < C_1$, $U(m-i,n) = 3$; $U(m-C_1,n) = 1$; if $C_1 < i \le B_1 + C_1$, $U(m-i,n) = 2$.*

Furthermore, this completely defines the sequence U, i.e., for any $(m,n) \in \mathbb{Z}^2$, there exists a unique (m_1,n_1) such that (m,n) satisfies (1).

Proof. Recall the notation:

$$B_1 = \lfloor b/a \rfloor,\ C_1 = \lfloor c/a \rfloor,$$
$$N_1 = \min\{n \in \mathbb{N},\ n \neq 0;\ h + na \in J\} - C_1 + 1,$$
$$a_1 = b - B_1 a,\ b_1 = c - C_1 a,\ c_1 = a; h_1 = h + N_1 a,$$
$$J = [(C_1 - 1)a, c + b - B_1 a).$$

It suffices to use the fact that, up to a translation of $(C_1-1)a$ (because we take the initial point of interval J as origin of coordinates for the induced action), the orbit of h_1 is just the intersection of the orbit of h with J, that is, the orbit of h_1 is the intersection of the orbit of h with $[0, a_1+b_1+c_1)$. To prove this, the only difficulty is to obtain the index, in the "big" orbit, of the point of index (m_1,n_1) in the induced orbit. This is not trivial, as we remarked in Section 3.1: we are now making explicit the rearrangement of the orbit implied by the induction.

For this purpose, we lift the orbit of h_1 to the universal cover $\mathbb{R}$. The point of coordinates (m_1,n_1) can be written as

$$h_1 + m_1 a_1 + n_1 b_1 - p_1(b_1 + a_1 + c_1),$$

where

$$p_1 = \lfloor \frac{h_1 + m_1 a_1 + n_1 b_1}{a_1 + b_1 + c_1} \rfloor.$$

This means that we consider the action on $\mathbb{R}$ by a group of three rotations, and we quotient by the rotation of length the induction interval. Indeed p_1 is exactly the number of times we must subtract the length of this interval, after advancing from h_1 by $ma_1 + nb_1$.

We can now replace a_1, b_1, c_1 by their respective values, and express the resulting value in terms of the basis $(a, b, a+b+c)$ of the lattice:

$$\begin{array}{ll} & h_1 + m_1 a_1 + n_1 b_1 - p_1(a_1 + b_1 + c_1) \\ = & h + N_1 a + m_1(b - B_1 a) + n_1(c - C_1 a) - p_1(a + b + c - (B_1 + C_1)a) \\ = & h + a(N_1 - m_1 B_1 - n_1(C_1+1) + p_1(B_1 + C_1)) + \\ + & b(m_1 - n_1) + (a+b+c)(-p_1 + n_1). \end{array}$$

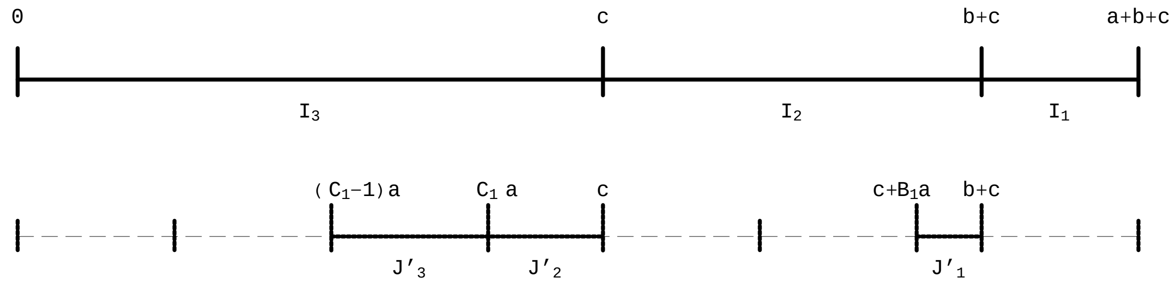

FIGURE 8. Induction on the interval J'

Hence, we get

$$h_1 + m_1 a_1 + n_1 b_1 - p_1(b_1 + a_1 + c_1) = h + ma + nb - p(a + b + c),$$

with

$$m = N_1 - m_1 B_1 - n_1(C_1 + 1) + p_1(B_1 + C_1),\ n = m_1 - n_1,\ p = p_1 - n_1.$$

Indeed we proved that $R_a^m R_b^n(h) \in [0, a_1 + b_1 + c_1)$ if and only if there exists $(m_1, n_1) \in \mathbb{Z}^2$ such that (m, n) satisfies the preceding relation. This implies that $R_a^m R_b^n(h) \in J$ if and only if there exists $(m_1, n_1) \in \mathbb{Z}^2$ such that (m, n) satisfies

$$m = N_1 - m_1 B_1 - n_1(C_1 + 1) + p_1(B_1 + C_1) + C_1 - 1,\ n = m_1 - n_1,\ p = p_1 - n_1.$$

Note that such a (m_1, n_1) always exists and is unique since

$$\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} -B_1 & -(C_1+1) & (B_1 + C_1) \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ n_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} N_1 + C_1 - 1 \\ 0 \\ 0 \end{pmatrix},$$

and the determinant of this matrix equals 1.

We then remark that by construction $J_1 \subset I_2$, and $J_2 \subset I_3$, $J_3 \subset I_3$ (see Figure 7). This completely defines U for the part of the orbit that belongs to J. Hence $U^1(m_1, n_1) = i$ implies $U(m, n) = i$, for $i \in \{1, 2, 3\}$.

Observe now that the complement of J is partitioned by the intervals $R_a^{-i} J_3$, for $0 < i \leq B_1 + C_1$, and each of these intervals is included in one of the I_n. This implies that the sequence U is completely defined in this way. Indeed, if $R_a^m R_b^n(h)$ is not in J, there is an iterate $R_a^k R_a^m R_b^n(h)$, with $k \leq B_1 + C_1$, that belongs to J; using this point, the value $U(m, n)$ is defined by the third condition in the statement of the theorem.

□

4.5. **A second generating process.** The aim of the next section is to use Theorem 3 to construct two-dimensional substitutions. In combinatorial terms, Theorem 3 means that the sequence U is deduced from the sequence U^1 by replacing 1 by 2, 2 by 3, and 3 by the one-dimensional word $2^{B_1} 1 3^{C_1}$ (a more precise meaning to this statement will be discussed in Section 5). It will be more convenient in order to replace the letter 3 by a two-dimensional word, to induce on a non-connected set, i.e., on the set (see Figure 8)

$$J' := J_3 \cup J_2 \cup R_a^{B_1}(J_1) = [(C_1 - 1)a, c) \cup [c + b - B_1 a, c + b).$$

Note that we choose to first induce on J since we needed for algebraic reasons (Theorem 2) the connectedness of J.

We thus deduce from Theorem 3:

Theorem 4. *Let U and U^1 be defined as in Theorem 3. Let*

$$p_1 = \lfloor \frac{h_1 + m_1 a_1 + n_1 b_1}{a_1 + b_1 + c_1} \rfloor,$$

and

$$\left\{ \begin{array}{ll} m & = N_1 - m_1 B_1 - n_1(C_1+1) + p_1(B_1+C_1) + C_1 - 1 \\ n & = m_1 - n_1. \end{array} \right. \tag{2}$$

We have:

- *if $U^1(m_1,n_1) = 1$, then $U(m+B_1,n) = 2$,*
- *if $U^1(m_1,n_1) = 2$, then $U(m,n) = 3$,*
- *if $U^1(m_1,n_1) = 3$, then: if $0 \le i < C_1$, $U(m-i,n) = 3$; $U(m-C_1,n) = 1$; if $0 \le i \le B_1 - 1$, $U(m - C_1 + i, n-1) = 2$.*

Furthermore, this completely defines the sequence U, i.e., for any $(m,n) \in \mathbb{Z}^2$, there exists a unique (m_1,n_1) such that (m,n) satisfies (2).

5. Two-dimensional substitutions

5.1. **Pointed substitutions.** Let us recall that a substitution over the alphabet $\mathcal{A}$ is a non-erasing morphism of the free monoid $\mathcal{A}^*$ endowed with the concatenation. Substitutions are usually used to substitute a finite word or a sequence, and also, as iteration devices which generate infinite sequences [31]. We want to extend this notion to the two-dimensional case. We are not able to endow the set of two-dimensional finite words (whatever its definition could be) with an algebraic structure, as the concatenation over $\mathcal{A}^*$. If we restrict ourselves to square factors or rectangular factors (this corresponds to picture languages [21]), some results have been established in this direction. This is the case in particular for the notion of two-dimensional substitutions of constant length which correspond to two-dimensional automatic sequences [37, 38]. We are interested here in the non-constant length case.

Theorem 4 shows how to deduce the sequence U from the sequence U^1: we can summarize simply by saying that we replace 1 by 2, 2 by 3 and 3 by $\begin{array}{c} 13^{C_1} \\ 2^{B_1} \end{array}$. This, however, would be sufficient for a one-dimensional sequence and for a usual substitution, but not for a two-dimensional sequence; in fact, we need to define the position of the pattern that replaces a given letter. We thus need to introduce the notion of *pointed substitution* as a map that sends a letter i located in position (m,n) to a pointed pattern, depending only on i, located in position (m',n') given as a function of (m,n) and i. (Note that similar substitutions have been introduced in [24, 25, 3, 5] in the framework of the stepped surface: these substitutions act on unit tips.) Hence, the sequence U is the image of U^1 by a pointed substitution that is completely determined by $(a_1,b_1,c_1;h_1)$, defined via the inhomogeneous Jacobi-Perron algorithm. To make this statement more precise, we need to introduce a suitable formalism.

We want a two-dimensional substitution to act on two-dimensional words and sequences. By word, we mean roughly speaking a finite set of letters in $\{1,2,3\}$, located in some position in $\mathbb{Z}^2$ with no overlaps, which may be not connected. More precisely, a *pointed word* is defined as a map with finite support from $\mathbb{Z}^2 \times \{1,2,3\}$ to $\{0,1\}$. We shall need to consider some words with overlaps. Hence, it is more convenient to consider linear combinations of words with coefficients in $\mathbb{R}$, so as to get a vector space. Hence we introduce the following definition.

Definition 11. *Let $\mathcal{F}$ be the vector space of maps from $\mathbb{Z}^2 \times \{1,2,3\}$ to $\mathbb{R}$, that take value zero everywhere except for a finite set.*

For $x \in \mathbb{Z}^2$ and $i \in \{1,2,3\}$, note (x,i) the element of $\mathcal{F}$ which takes value 1 at (x,i), and 0 elsewhere; the set $\{(x,i),\ x \in \mathbb{Z}^2,\ i \in \{1,2,3\}\}$ is a basis of $\mathcal{F}$. We call such an element a pointed letter.

The support *of an element of $\mathcal{F}$ is the set of (x,i) on which it is not zero.*

We say that an element of $\mathcal{F}$ is a pointed pattern *if it takes value 0 and 1 and for every $x \in \mathbb{Z}^2$, there exists at most one $i \in \{1,2,3\}$ such that it takes 1 at (x,i). One can represent a pointed pattern as a two-dimensional pointed word located in point $x \in \mathbb{Z}^2$.*

If M and N are two pointed patterns, we will say for simplicity that M contains *N (we denote it $N \subset M$) if the support of M contains that of N.*

Similarly, we say that a pointed pattern M is included in the two-dimensional sequence V (we note it $M \subset V$) if the support of M is included in the support of $\mathcal{V}$, where $\mathcal{V}$ denotes the map from $\mathbb{Z}^2 \times \{1,2,3\}$ to $\mathbb{R}$ that takes value 1 for every $(x, V(x))$, and 0 otherwise.

We say that two pointed patterns M and N are disjoint ($M \cap N = \emptyset$) if their supports are disjoint.

We are now able to define a notion of pointed substitution $\Sigma_{(a_1,b_1,c_1;h_1)}$ on $\mathcal{F}$. This notion can be compared with the formalism developed by C. Radin (see for instance [32, 18, 33, 34]) for tiling spaces which are generated by substitution rules acting on polygonal tiles, as the well-known example of the Penrose tiling.

Definition 12. *Let us use the notation of Theorem 4. We define the* pointed substitution

$$\Sigma_{(a_1,b_1,c_1;h_1)} : \mathcal{F} \to \mathcal{F}$$

as the linear map defined on the basis of $\mathcal{F}$

$$\{((m_1,n_1),i);\ (m_1,n_1) \in \mathbb{Z}^2,\ i \in \{1,2,3\}\}$$

by

$$\begin{cases} \Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),1) = & ((m+B_1,n),2) \\ \Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),2) = & ((m,n),3) \\ \Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),3) = & \sum_{0\le k\le C_1-1}((m-k,n),3) + ((m-C_1,n),1) + \\ & \sum_{0\le k\le B_1-1}((m-C_1+k,n-1),2), \end{cases}$$

with

$$\begin{cases} m = m(m_1,n_1) = & N_1 - mB_1 - n(C_1+1) + p_1(B_1+C_1) + C_1 - 1, \\ n = n(m_1,n_1) = & m_1 - n_1, \\ p_1 = p_1(m_1,n_1) = & \left[\frac{h_1+m_1a_1+n_1b_1}{a_1+b_1+c_1}\right], \end{cases}$$

and

$$\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),i) = 0,$$

otherwise.

The map $(m_1,n_1) \mapsto (m,n)$ is called the placing rule *associated with the substitution $\Sigma_{(a_1,b_1,c_1;h_1)}$.*

Remark

Note that when $h = 0$, then $h_1 = 0$, $N_1 = 0$ and the image of $((0,0),3)$ contains $((0,0),3)$. Furthermore, the parameter N_1 acts as a translation of the image of a letter.

Let $\mathcal{U}$ (respectively $\mathcal{U}^1$) denote the map from $\mathbb{Z}^2 \times \{1,2,3\}$ to $\mathbb{R}$ that takes value 1 for every $(x, U(x))$ (respectively $(x, U^1(x))$), and 0 otherwise.

We can define in a natural way $\Sigma_{(a_1,b_1,c_1;h_1)}$ on $\mathcal{U}^1$. More precisely, we can translate the induction process (Theorem 4) into the following combinatorial terms:

Theorem 5. *We have:*

(1) $\forall (x,i) \in \mathcal{U}^1,\ \Sigma_{(a_1,b_1,c_1;h_1)}(x,i) \subset \mathcal{U}$.
(2) $\forall (y,j) \in \mathcal{U},\ \exists!(x,i) \in \mathcal{U}^1,\ \Sigma_{(a_1,b_1,c_1;h_1)}(x,i)$ *contains* (y,j).
(3) *If* $(x,i),(x',j) \in \mathcal{U}^1$ *with* $(x,i) \neq (x',j)$,

$$\textit{then } \Sigma_{(a_1,b_1,c_1;h_1)}(x,i) \cap \Sigma_{(a_1,b_1,c_1;h_1)}(x',j) = \emptyset.$$

Remarks In other words, this theorem means that the image of the two-dimensional sequence U^1 under the action of $\Sigma_{(a_1,b_1,c_1;h_1)}$ is exactly the two-dimensional sequence U.

Note that, in contrast to one-dimensional substitutions, it is non-trivial to prove that the definition of a pointed substitution is consistent: the images of pointed patterns could overlap, or not cover the whole image sequence. Indeed, the image by $\Sigma_{(a_1,b_1,c_1;h_1)}$ of a pointed pattern is not always a pointed pattern or the image of a connected pointed pattern may not be connected. But we know that if the pointed pattern is contained in $\mathcal{U}^1$, then its image is still a pointed pattern and it is contained in the double sequence $\mathcal{U}$.

Remark that the substitution depends on the inhomogeneous part h and that it may not be consistent anymore if one considers a pointed pattern included in a sequence $\mathcal{U}^1$ associated with a different h (although the languages, i.e., the sets of factors are the same).

5.2. **Local rules.** It is however very inconvenient to use a pointed substitution, since we need at each step *global information*, including a_1, b_1, c_1, h_1, and not only B_1, C_1, N_1. In particular we are not able to iterate it in order to generate a double sequence. It is much more convenient to be able to use a *local information*, i.e., local rules (and this is exactly what is done when one computes one-dimensional substitutions: one does not compute the exact position of a given pattern, but only uses the fact that patterns follow each other). Roughly speaking, a local rule says how to place the image of a (pointed) letter with respect to the images of the letters belonging to a finite neighbourhood. The idea here is that in fact, the relative position of patterns contains all the information in a_1, b_1, c_1, h_1; hence we must rely only on this relative position. A local information is sufficient for iteration. Fortunately, it is possible to give such local rules for the two-dimensional substitutions we use. We obtain seven local rules involving only one letter adjacent to the one we consider. In order to give a more precise meaning to the notion of local rule, we need to introduce the concept of pattern.

We define a *pattern* as a pointed pattern up to translation. More precisely, a local pattern is an equivalence class for a given pointed pattern for the following relation:

$$M \sim N \text{ if and only if there exists } t \in \mathbb{Z}^2 \text{ such that}$$

$$\forall (x,i) \in \mathbb{Z}^2 \times \{1,2,3\} : (x,i) \subset M \text{ if and only if } (x+t,i) \subset N.$$

In other words, a pattern is a word considered without a precise location in $\mathbb{Z}^2$.

Consider the following seven patterns: we give a representative of each of these patterns below. We denote by $\mathcal{C}$ the union of the equivalence classes of those pointed patterns.

- $33 :\ ((0,0),3) + ((1,0),3)$
- $13 :\ ((0,0),1) + ((1,0),3)$
- $\begin{smallmatrix} & 3 \\ 2 & \end{smallmatrix} :\ ((0,0),2) + ((1,1),3)$
- $\begin{smallmatrix} 3 \\ 3 \end{smallmatrix} :\ ((0,0),3) + (0,1),3)$
- $\begin{smallmatrix} & 1 \\ 3 & \end{smallmatrix} :\ ((0,0),3) + ((1,1),1)$
- $21 :\ ((0,0),2) + ((1,0),1)$
- $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} :\ ((0,0),3) + ((0,1),2)$

We mean by *local rule* the following: suppose for instance

$$((m,n),U_{m,n}),(m+1,n),U_{m+1,n}))\in\mathcal{C};$$

then one knows where to place the image of $((m+1,n),U_{m+1,n})$ with respect to the one of $((m,n),U_{m,n})$. Theorem 6 below gives a more precise meaning to this. Note that the patterns in $\mathcal{C}$ do not depend on the coefficients (B_1,C_1,N_1), contrary to the local rules.

Theorem 6. *Let $v\in\mathbb{Z}^2$. Let T_v denote the translation defined on the basis $\{(x,i)\}$ by $T_v(x,i)=(x+v,i)$. The following relations hold for every pointed local pattern included in $\mathcal{U}^1$, with the notation of Definition 12:*

- $\Sigma_{(a_1,b_1,c_1;h_1)}[(m_1,n_1),3)+((m_1+1,n_1)),3)]$
  $=\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),3)+T_{(-B_1,1)}[\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),3)]$
- $\Sigma_{(a_1,b_1,c_1;h_1)}[((m_1-1,n_1),1)+((m_1,n_1),3)]$
  $=\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),3)+T_{(B_1-C_1-1,1)}[\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),1)]$
- $\Sigma_{(a_1,b_1,c_1;h_1)}[((m_1-1,n_1-1),2)+((m_1,n_1),3)]$
  $=\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),3)+T_{(1,0)}[\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),1)]$
- $\Sigma_{(a_1,b_1,c_1;h_1)}[((m_1,n_1),3)+(m_1,n_1+1),3)]$
  $=\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),3)+T_{(-C_1-1,-1)}[\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),3)]$
- $\Sigma_{(a_1,b_1,c_1;h_1)}[((m_1,n_1),3)+((m_1+1,n_1+1),1)]$
  $=\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),3)+T_{(-C_1-1,0}[\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),1)]$
- $\Sigma_{(a_1,b_1,c_1;h_1)}[((m_1,n_1),2)+((m_1+1,n_1),1)]$
  $=\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),2)+T_{(0,1)}[\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),1)]$
- $\Sigma_{(a_1,b_1,c_1;h_1)}[((m_1,n_1),3)+((m_1,n_1+1),2)]$
  $=\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),3)+T_{(-C_1-1,-1}[\Sigma_{(a_1,b_1,c_1;h_1)}((m_1,n_1),2)]$.

Note that these rules correspond to the ones given in [25], up to a rotation and to a permutation of the letters. The following picture gives a representation of the local rules in the case $B_1=2$, $C_1=4$. We have distinguished the letters and their respective images by overlining one of the two letters. This convention is useful in particular for what concerns the first and the fourth rules, where the same letter occurs twice.

- $\overline{3}3 \mapsto \begin{array}{l}13333\\ 22\overline{13333}\\ \overline{22}\end{array}$
- $\overline{1}3 \mapsto \begin{array}{l}13333\\ 22\overline{2}\end{array}$
- $\begin{array}{l}3\\ \overline{2}\end{array} \mapsto \begin{array}{l}13333\overline{3}\\ 22\end{array}$
- $\begin{array}{l}\overline{3}\\ 3\end{array} \mapsto \begin{array}{l}13333\\ \overline{13333}22\\ \overline{22}\end{array}$
- $\begin{array}{l}\overline{1}\\ 3\end{array} \mapsto \begin{array}{l}\overline{2}13333\\ 22\end{array}$
- $\overline{2}1 \mapsto \begin{array}{l}2\\ \overline{3}\end{array}$
- $\begin{array}{l}\overline{2}\\ 3\end{array} \mapsto \begin{array}{l}13333\\ \overline{3}22\end{array}$

Proof. Let us prove the first assertion of Theorem 6 for instance. The proof of the other assertions follows the same scheme. Suppose that $U^1(m_1,n_1)=U^1(m_1+1,n_1)=3$. Let us use the notation of Definition 12. We thus have $p_1(m_1+1,n_1) = p_1(m_1,n_1)$, $m(m_1+1,n_1) = m(m_1,n_1) - B_1$, $n(m_1+1,n_1) = n(m_1,n_1)+1$. □

The next step is to prove that these rules completely define the image of the sequence U^1. Let us use the terminology of [24, 25].

Definition 13. *Let M be a pointed pattern. We say that M is $\mathcal{C}$-*covered *if:*

$$\forall (x,i),(x',i') \subset M,\ \exists n \in \mathbb{N},\ \exists (y_0,j_0),\ (y_1,j_1),\dots,(y_n,j_n) \in \mathbb{Z}^2 \times \{1,2,3\}$$

$$\textit{such that } (y_0,j_0) = (x,i),\ (y_n,j_n) = (x',i'),\ \textit{and for } 0 \le k \le n-1,$$

$$((y_k,j_k) + (y_{k+1},j_{k+1})) \in \mathcal{C}.$$

One easily deduces from the local rules the following.

Lemma 1. *If a pointed pattern $M \subset \mathcal{U}^1$ is $\mathcal{C}$-covered, then $\Sigma_{(a_1,b_1,c_1;h_1)}(M)$ is also a $\mathcal{C}$-covered pointed pattern.*

Proof. This is a direct consequence of the fact that the image of any pointed pattern in $\mathcal{C}$ is $\mathcal{C}$-covered. This is just a case study.

□

We are now able to extend the definition of the two-dimensional substitution $\Sigma_{(a_1,b_1,c_1:h_1)}$ to any $\mathcal{C}$-covered pointed pattern included in $\mathcal{U}^1$ by using only local rules and forgetting the placing rule (Definition 12). This implies in particular that we can easily consider limits of iterations of the two-dimensional substitutions starting from any $\mathcal{C}$-covered pointed pattern.

5.3. **Iteration of the two-dimensional substitutions in the homogeneous case.** Consider the homogeneous case, that is, the plane $ax+by+cz=0$. We have given a meaning to the fact that $\mathcal{U}$ is the image of $\mathcal{U}^1$ under the action of the substitution $\Sigma_{(a_1,b_1,c_1;0)}$. We can now iterate this procedure, using again the Jacobi-Perron algorithm.

Recall that a,b,c are independent over $\mathbb{Q}$. We associate with (a,b,c) the sequence $(B_n,C_n)_{n\in\mathbb{N}}$ of integers defined by:

$$F^n(a,b,c) = (a_n,b_n,c_n),\ B_{n+1} = \lfloor b_n/a_n \rfloor,\ C_{n+1} = \lfloor c_n/a_n \rfloor.$$

Recall that we have for every n the following admissibility conditions:

$$0 \le B_n \le C_n \text{ and if } B_n = C_n \text{ then } B_{n+1} \ne 0.$$

We can iterate this process in order to generate the sequence U by the composition of such two-dimensional substitutions. More precisely, we eventually define arbitrarily large parts of the sequence U even if we only know the initial letter of each sequence, or in the worst case, if we only know a finite path surrounding $((0,0),3)$. Indeed $((0,0),3)$ belongs to $\mathcal{U}^n$ for every integer n (where $\mathcal{U}^n$ is associated as previously with the plane $a_nx+b_ny+c_nz=0$). We have $\forall n,\ h_n=0$. Hence $\Sigma_{(a_1,b_1,c_1;0)}\Sigma_{(a_2,b_2,c_2;0)}\cdots\Sigma_{(a_n,b_n,c_n;0)}((0,0);3)$ belongs to $\mathcal{U}$ for every n. Furthermore, since the sequence of pointed patterns $\Sigma_{(a_1,b_1,c_1;0)}\Sigma_{(a_2,b_2,c_2;0)}\cdots\Sigma_{(a_n,b_n,c_n;0)}((0,0);3)$ is increasing, one can give a meaning to its limit, and

$$\lim_{n\to\infty} \Sigma_{(a_1,b_1,c_1;0)}\Sigma_{(a_2,b_2,c_2;0)}\cdots\Sigma_{(a_n,b_n,c_n;0)}((0,0);3)$$

covers an infinite part of the sequence $\mathcal{U}$; we shall return to the infinite sequence we obtain in the last section.

6. Dual substitutions: algebraic construction of stepped surfaces

In this section, we introduce an explicit construction for the stepped surface and emphasize the connection between the notion of two-dimensional substitutions we introduced and the dual maps for one-dimensional extensions of substitutions discussed in [3, 5], and alluded to in Subsection 1.2. Indeed, the work done so far can be reformulated in the initial framework of stepped surfaces.

We will introduce a family of one-dimensional substitutions; with these, one associates, (following the formalism of [3, 5]) a linear map on an infinite dimensional space, the one-dimensional geometric realization. One then defines the dual map $E_1(\sigma)^*$ of these linear maps; using these dual maps, one first recovers the substitutions of the previous section, and second, generates in a constructive way the stepped surface.

We will for the sake of simplicity restrict ourselves to the homogeneous case: that is, we will consider only the case $h = h_1 = 0$, or equivalently, we will construct stepped surfaces for irrational planes through the origin. Our results generalize to the inhomogeneous case, but the notation becomes quite cumbersome.

In this section, we will consider a rationally independent triple (a, b, c), and the triple (a_1, b_1, c_1) obtained from it by the Jacobi-Perron algorithm, and take as previously the notation $B_1 = \lfloor b/a \rfloor$, $C_1 = \lfloor c/a \rfloor$, so that $a_1 = b - B_1 a$, $b_1 = c - C_1 a$, $c_1 = a$.

In Section 6.1, we will reformulate the pointed substitutions of the previous section to obtain directly the stepped surface. In the next two subsection, we will briefly review the formalism of one-dimensional extensions of substitutions and their dual maps. In Section 6.4, we apply this to the present case, and in Section 6.5, we will prove that we can so recover in a completely formal way the previous results; the last subsection gives another interpretation in terms of tilings of the line.

6.1. A combinatorial construction of the stepped surface.

In Section 5, we defined pointed substitutions taking pointed letters to pointed patterns. If we denote, as in Section 2, $\mathcal{P}$ (respectively $\mathcal{P}^1$) the plane $ax + by + cz = 0$ (respectively $a_1 x + b_1 y + c_1 z = 0$), P (respectively P^1) the corresponding stepped surface, we recover easily both stepped surfaces from the symbolic sequences U, U^1. The pointed substitution, sending sequence U^1 to U, can then be conjugate to a map that sends each face of P^1 to a disjoint union of faces of P, in such a way that any face of P is in the image of exactly one face of P^1.

A straightforward computation, using Theorem 4, gives the exact formula for this map, where we denote as above by $(X, Y, Z) + E_i$ the face of type i with distinguished vertex (X, Y, Z) and we denote disjoint unions by a formal sum:

Theorem 7. *If $(X, Y, Z) + E_i$ is a face of P_1, the pointed substitution sends it to a union of faces of P according to the following:*

- $(X, Y, Z) + E_1$ *is sent to* $(-B_1 X - C_1 Y + Z + B_1 + C_1 - 1, X, Y) + E_2$
- $(X, Y, Z) + E_2$ *is sent to* $(-B_1 X - C_1 Y + Z + C_1 - 1, X, Y) + E_3$
- $(X, Y, Z) + E_3$ *is sent to* $[(-B_1 X - C_1 Y + Z, X + 1, Y + 1) + E_1] + \sum_{i=0}^{B_1 - 1} [(-B_1 X - C_1 Y + Z + i, X, Y + 1) + E_2] + \sum_{i=0}^{C_1 - 1} [(-B_1 X - C_1 Y + Z + i, X, Y) + E_3]$.

Proof. We will only prove the last and most complicated formula, the first two are proved in the same way. Suppose that the face $(X, Y, Z) + E_3$ belongs to P_1. Then, by Proposition 4, we have $a_1 X + b_1 Y + c_1 Z \in J_3 = [0, c_1[$ (this equality is valid in $\mathbb{R}$, not modulo $a_1 + b_1 + c_1$).

This action comes by induction on J from the action defined by (a, b, c). We can then obtain the position of this point, and a part of its orbit for the initial action, with respect to the natural partition for this initial action. Namely, taking into account the shift of $(C_1 - 1)a$ coming from the change of origin, we obtain that

$$\text{for } 0 \leq i \leq C_1 - 1,\ a_1 X + b_1 Y + c_1 Z + (C_1 - 1)a - ia \in I_3,$$

the next preimage (for which we must take into account that we have made a complete turn $(a+b+c)$)

$$a_1X + b_1Y + c_1Z + (C_1-1)a - C_1a + (a+b+c) \in I_1,$$

and images of this last point by $R_b^{-1}R_a^i$, for $0 \le i \le B_1 - 1$,

$$a_1X + b_1Y + c_1Z + (C_1-1)a - C_1a + (a+b+c) - b + ia \in I_2,$$

(see Figure 7). Replacing a_1, b_1, c_1 by their respective values $b - B_1a, c - C_1a, a$ shows that

$$a(-B_1X - C_1Y + Z) + b(X+1) + c(Y+1) \in I_1,$$

hence $(-B_1X - C_1Y + Z, X+1, Y+1) + E_1$ is a face of P, and similarly for the other faces; as it was proved in Theorem 4, each face is obtained in this way exactly once. □

The aim of the next section is to show that, using the formalism of [3, 5], we can recover in a purely algebraic way this formula from the following ordinary one-dimensional substitutions associated with the Jacobi-Perron algorithm.

Definition 14. *We denote by σ_{B_1,C_1} the substitution over the three- letter alphabet $\{1,2,3\}$ defined by:*

$$\sigma_{B_1,C_1}(1) = 3, \qquad \sigma_{B_1,C_1}(2) = 13^{B_1}, \qquad \sigma_{B_1,C_1}(3) = 23^{C_1}.$$

6.2. **The one-dimensional extension of a substitution.** We denote by $\{1,2,3\}^*$ the free monoid on 3 letters, and by f the natural map (*abelianization*) from $\{1,2,3\}^*$ to $\mathbb{Z}^3$ (if W is an element of $\{1,2,3\}^*$ (a word), $f(W)$ is the vector that counts the number of occurrence of each letter in W).

Let σ be a substitution on three letters. We will denote by M the matrix associated with the abelianization of σ ($M_{i,j}$ is the number of occurrences of the letter i in the word $\sigma(j)$). There is an obvious commutative diagram:

$$\begin{array}{ccc} \{1,2,3\}^* & \xrightarrow{\sigma} & \{1,2,3\}^* \\ f\downarrow & & \downarrow f \\ \mathbb{Z}^3 & \xrightarrow{M} & \mathbb{Z}^3. \end{array}$$

We will take the notation

$$\sigma(i) = W^{(i)} = W_1^{(i)} \dots W_{l_i}^{(i)} = P_n^{(i)} W_n^{(i)} S_n^{(i)},$$

for $1 \le n \le l_i$, where l_i is the length of $\sigma(i)$, $P_n^{(i)}$ is the prefix of length $n-1$ of $\sigma(i)$ (the empty word for $n=1$), and $S_n^{(i)}$ is the suffix of length $l_i - n$ of $\sigma(i)$ (the empty word for $n = l_i$).

It is natural to associate with each finite word $W = w_1w_2\dots w_n$ on 3 letters a path in the three-dimensional space, starting from 0 and ending in $f(W)$, with vertices in $f(w_1\dots w_i)$ for $i = 1\dots n$: we start from 0, advance by $\vec{e}_i$ if the first letter is i, and so on. This allows us to define a map on paths, coming from the substitution, by taking the path for W to the path for $\sigma(W)$. In fact, this map can be defined in a consistent way for all paths with adjacent integer vertices, and made in a linear map, in the following way:

Definition 15. *We denote by $((X,Y,Z),i) \in \mathbb{Z}^3 \times \{1,2,3\}$ an* elementary path *(that is, a segment from (X,Y,Z) to $(X,Y,Z)+\vec{e}_i$); we denote by $\mathcal{G}$ the real vector space of formal finite weighted sums of elementary paths. We call* one-dimensional extension *of σ, and denote by $E_1(\sigma)$, the linear map defined on $\mathcal{G}$ by:*

$$E_1(\sigma)((X,Y,Z),i) = \sum_{n=1}^{l_i}((M.(X,Y,Z) + f(P_n^{(i)}), W_n^{(i)}).$$

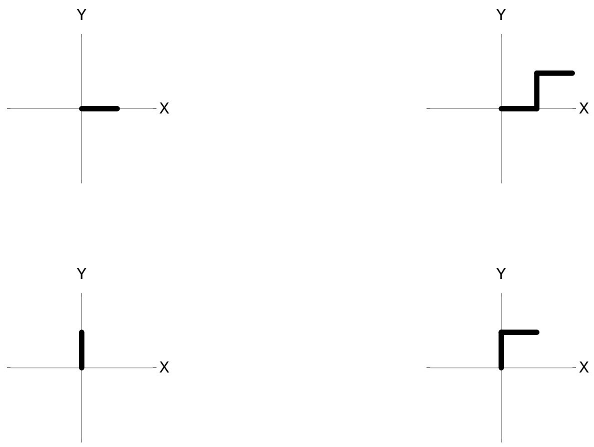

FIGURE 9. The one-dimensional extension of $0 \mapsto 010$, $1 \mapsto 10$

It is easily checked that this formula is such that σ takes the continuous path corresponding to a word W to the continuous path corresponding to $\sigma(W)$; indeed, the first path ends in $f(W)$, while the second ends in $f(\sigma(W)) = M.f(W)$, and the formula ensures that if we extend the first path, the image of the extension will start at the end of the second path (see Figure 9).

6.3. **The dual substitution.** For the sequel, we need the matrix M to be invertible in the set of integral matrices, and a sufficient condition is that its determinant is 1, hence the next definition:

Definition 16. *A substitution σ is called* unimodular *if its abelianized matrix M has determinant 1.*

From now on, we suppose that σ is a unimodular substitution; it is readily checked that this is the case of the Jacobi-Perron substitutions defined above (Definition 14).

We want to study the dual map $E_1^*(\sigma)$ of $E_1(\sigma)$, as a linear map on $\mathcal{G}$.

Definition 17. *We denote by $\mathcal{G}^*$ the space of dual maps with finite support (that is, dual maps that give value 0 to all but a finite number of the vectors of the canonical basis).*

The space $\mathcal{G}^*$ has a natural basis $((X,Y,Z),i^*)$, the map that gives value 1 to $((X,Y,Z),i)$ and 0 to all other vectors. It is possible to give a geometric meaning to this dual space by a kind of Poincaré duality: we represent the element $((X,Y,Z),i^*)$ by the upper face perpendicular to the direction $\vec{e}_i$ of the unit cube with lowest vertex (X,Y,Z) (see [3, 5] for more details, and a more general framework).

The map $E_1(\sigma)$ has a dual map, and it is easy to prove that, if the map M is not degenerated, it preserves the space $\mathcal{G}^*$ [3]; when σ is unimodular, it is easy to compute explicitly this dual map:

Proposition 5. *If σ is unimodular, the dual map $E_1^*(\sigma)$ is defined on $\mathcal{G}^*$ by*

$$E_1^*(\sigma)((X,Y,Z),i^*) = \sum_{j\in\{1,2,3\}} \sum_{W_n^{(j)}=i} (M^{-1}.[(X,Y,Z) - f(P_n^{(j)})], j^*).$$

Proof. Let us compute $E_1^*(\sigma)$. Using the natural bilinear product over $\mathcal{G}^* \times \mathcal{G}$, one gets:

$$< E_1^*(\sigma)((X,Y,Z),i^*))|((X',Y',Z'),j) >=< (X,Y,Z),i^*)|E_1(\sigma)((X',Y',Z'),j) >$$

$$=< ((X,Y,Z),i^*)| \sum_{n=1}^{l_j} (M.(X',Y',Z') + f(P_n^{(j)}), W_n^{(j)} > .$$

The product is nonzero if and only if there exists n such that $W_n^{(j)} = i$, $M.(X', Y', Z') + f(P_n^{(j)}) = (X, Y, Z)$, that is, $(X', Y', Z') = M^{-1}([(X, Y, Z) - f(P_n^{(j)})], j^*)$. □

6.4. **Computation of the dual substitution of the Jacobi-Perron substitution.** We can apply the preceding formula to the Jacobi-Perron substitution.

The matrix M and its inverse are given in that case by:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & B_1 & C_1 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} -B_1 & -C_1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The first two letters occur exactly once in all the images, so in the corresponding images by the dual map, the sum reduces to exactly one element, and the prefix is empty, making it very easy to compute. The letter 3 occurs once in $\sigma(1)$, B_1 times in $\sigma(2)$, and C_1 times in $\sigma(3)$, hence the image $E_1^*(\sigma)((***), 3^*)$ consists in a sum of one element of type $((***), 1^*)$, B_1 elements of type $((***), 2^*)$ and C_1 elements of type $((***), 3^*)$, as for the pointed substitution computed in Theorem 7. The exact formula is:

Proposition 6. *The dual substitution $E_1^*(\sigma)$ is defined by:*

- $E_1(\sigma)^*((X, Y, Z), 1^*) = ((-B_1X - C_1Y + Z, X, Y), 2^*)$
- $E_1(\sigma)^*((X, Y, Z), 2^*) = ((-B_1X - C_1Y + Z, X, Y), 3^*)$
- $E_1(\sigma)^*((X, Y, Z), 3^*) = ((-B_1X - C_1Y + Z, X, Y), 1^*) + \sum_{i=0}^{B_1-1}((-B_1X - C_1Y + Z + 1 + i, X - 1, Y), 2^*) + \sum_{i=0}^{C_1-1}((-B_1X - C_1Y + Z + 1 + i, X, Y - 1), 3^*)$.

6.5. **Geometric interpretation.** It seems at first sight that we do not recover exactly the formula given in Theorem 7. But the discrepancy only comes from a difference of convention: recall that, by definition, $(X, Y, Z) + E_1$ represents a face of type 1 whose upper vertex is (X, Y, Z); this choice was made to make easier the proofs in Section 2, so that (X, Y, Z) belongs to the face; but by definition, $((X, Y, Z), 1^*)$ represents the upper face, orthogonal to the direction 1, of the unit cube whose lowest vertex is (X, Y, Z). This choice was imposed upon us in the paper [5] for coherence reasons.

Hence, in our notation, $((X, Y, Z), 1^*)$ and $(X + 1, Y + 1, Z + 1) + E_1$ correspond to the same face of P. In the same way, we have $((X, Y, Z), 2^*) = (X, Y + 1, Z + 1) + E_2$ and $((X, Y, Z), 3^*) = (X, Y, Z + 1) + E_3$.

We can rephrase in these notation the preceding proposition:

Proposition 7. *The dual substitution $E_1^*(\sigma)$ is defined by:*

- $E_1(\sigma)^*((X, Y, Z) + E_1) = (-B_1X - C_1Y + Z + B_1 + C_1 - 1, X, Y) + E_2$
- $E_1(\sigma)^*((X, Y, Z) + E_2) = (-B_1X - C_1Y + Z + C_1 - 1, X, Y) + E_3$
- $E_1(\sigma)^*((X, Y, Z) + E_3) = [(-B_1X - C_1Y + Z, X + 1, Y + 1) + E_1] + \sum_{i=0}^{B_1-1}[(-B_1X - C_1Y + Z + i, X, Y + 1) + E_2] + \sum_{i=0}^{C_1-1}[(-B_1X - C_1Y + Z + i, X, Y) + E_3]$.

We can check that we recover the initial formula in Theorem 7.

6.6. **Dynamical interpretation of the Jacobi-Perron substitution.** The meaning of the one-dimensional substitution σ_{B_1,C_1} is not completely clear. However, we remark that the induction defines a tiling of the initial interval I by intervals of length a_1, b_1, c_1. If we take the negative orientation, which is consistent with our way to number the intervals, we see that interval 1 is tiled exactly by one interval of type 3, interval 2 is tiled by 23^{B_1} and interval 3 is tiled by 13^{C_1}; if we iterate, we get a new substitution σ_{B_2,C_2}, and the new tiling, in the reverse order, is now given by $\sigma_{B_2,C_2}\sigma_{B_1,C_1}(123)$.

It should be interesting to study the property of the sequence of tilings generated in this way.

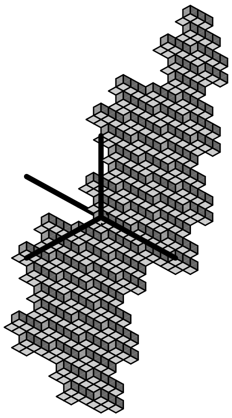

FIGURE 10. Iteration starting from the faces at the origin

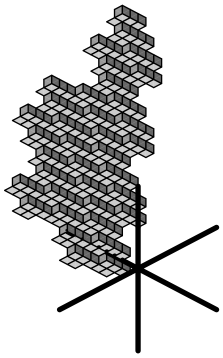

FIGURE 11. Iteration starting from the face $((1,-1,-1),3^*)$

7. Additional remarks

We have proved above that, by iterating the sequence of pointed substitutions given by Jacobi-Perron algorithm, we generate an infinite part of the discrete surface.

Note that in some cases, this limit is strictly included in the sequence U, as shown in the next example (where the sequence $(B_n, C_n)_{n\in\mathbb{N}}$ is purely periodic of period $[(1,1),(1,2),(0,1)]$). Figure 10 shows the iteration of the substitution starting from the three faces at the origin; these however do not generate everything, and the face denoted by $((1,-1,-1),3^*)$ is contained in its own image, hence it never appears in the iterated images of the faces at the origin. Figure 11 shows an iterated image of this face. The last figure shows that, together, these four faces generate a neighbourhood of the origin. One can in fact prove that, in this way, one generates the complete plane.

One could ask what is the shape of the piece of the stepped surface generated after n iterations. This is not known in the general case; however, in the periodic case, one can prove that, if we renormalize by a suitable matrix (restriction to the plane of the abelianization of the substitution), the shape converges to a particular fractal set. Several papers have been devoted to this study, specially [24, 25, 3, 5]. The particular case where all partial quotient are equal to 1 in the Jacobi-Perron algorithm gives the substitution $1 \mapsto 3$, $2 \mapsto 13$, $3 \mapsto 23$ (Definition 4); up to a change of

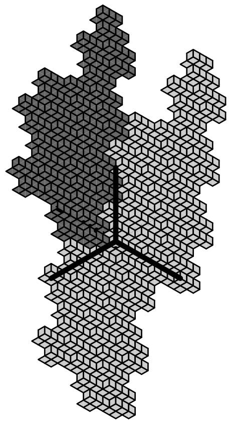

FIGURE 12. A neighbourhood of the origin obtained by iteration on 4 faces

direction and the exchange of 1 and 3, this is the same as the Rauzy substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$ which was first studied by Rauzy in [35] and in several other papers [17, 27, 28].

An interesting question is the extent to which one can generalize these notions of substitution. It has been addressed in [5] for extensions of substitutions. It seems that substitutions defined by local rules can also be used in a quite general context, contrary to pointed substitutions, which are quite rigidly restricted to symbolic sequences associated with discrete planes.

Our discussion is not restricted to Jacobi-Perron algorithm; in fact, any classical algorithm can be used. The algorithm of Brun is particularly interesting, since it has an explicitly defined natural extension, and we plan to return to this topic in a future paper.

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