Hybrid and Timed Systems

modeling, theory, verification

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Shanghai Computer Science Summer School 2010

Introductory equations

Hybrid Systems

- **Hybrid Systems** = Discrete+Continuous
- **Hybrid Automata** = A model of Hybrid systems
- **Original motivation**= Physical plant + Digital controller
- **New applications** = biology, economy, numerics, circuits
- **Hybrid community** = Control scientists + Applied mathematicians + Some computer scientists

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Timed Systems

- **Timed Systems** = Discrete behavior+Continuous Time
- **Timed Automata** = A subclass of Hybrid automata
- **The starting point** = A beautiful result by Alur & Dill.
- **Applications**= Real-time digital system, etc...
- **Timed community** = Computer scientists

Global Outline

1. Hybrid Automata
2. Timed Automata
3. Back to Hybrid: Decidable Subclasses

Part I

Hybrid Automata

Outline

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The first (cyber-physical) example

Notation

For $x = x(t)$ we write $\dot{x} = \dot{x}(t) = x'(t) = dx/dt$.

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A strange creature. . .

A bad syntax

Some mathematicians prefer to write

$$\dot{x} = f(x, q)$$

where

$$\begin{aligned} f(x, \mathrm{Off}) &= -x \\ f(x, \mathrm{On}) &= H - x \end{aligned}$$

with some switching rules on q.

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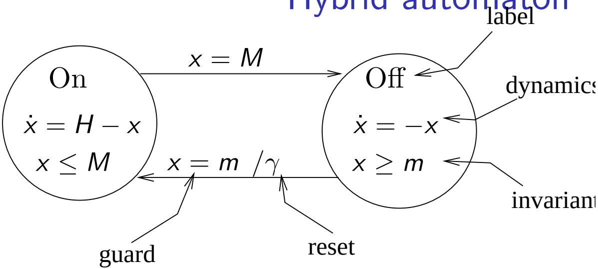
with some switching rules on q.

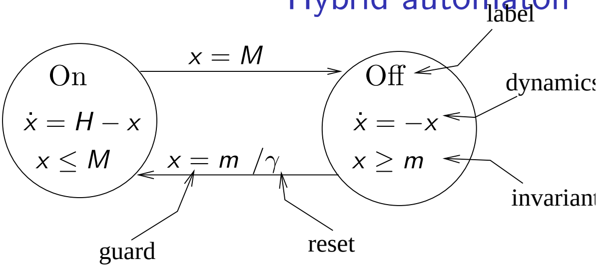
But we are computer scientists

and draw an *automaton*

Hybrid automaton

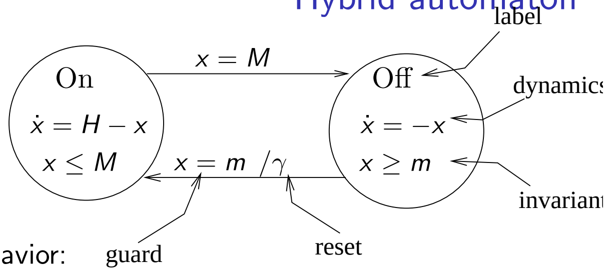


Hybrid automaton

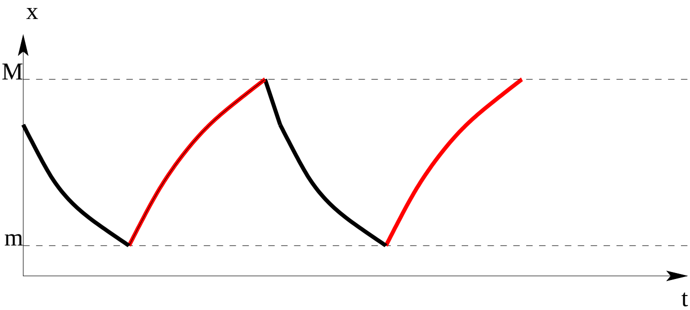


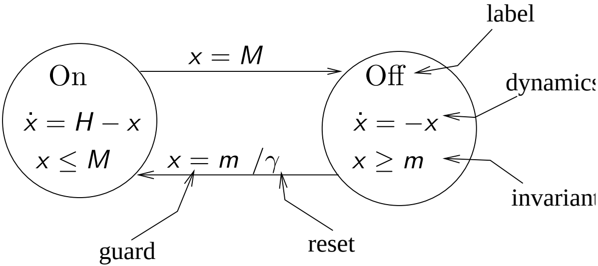
A formal definition: It is a tuple . . .

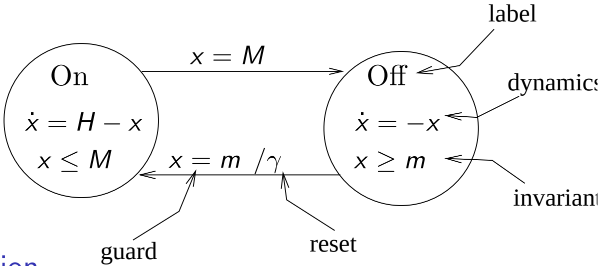
Hybrid automaton



Its behavior:



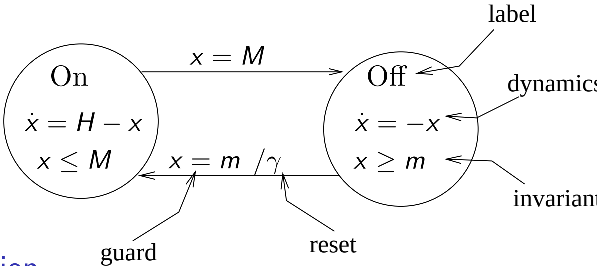

label
On
$x = M$
Off
dynamics
$\dot{x} = H - x$
$\dot{x} = -x$
$x \leq M$
$x = m \ /\gamma$
$x \geq m$
invariant
guard
reset



Definition

A hybrid automaton is $H = (Q, X, \Sigma, Dyn, I, \Delta)$ with

- Q finite set of locations
- $X = \mathbb{R}^n$, continuous state space
- Dyn, dynamics on X for every $q \in Q$
- I, invariant, staying condition in X
- Δ, finite set of transitions $\delta = (p, q, a, g, r)$



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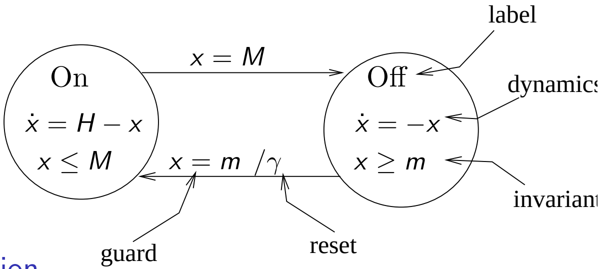

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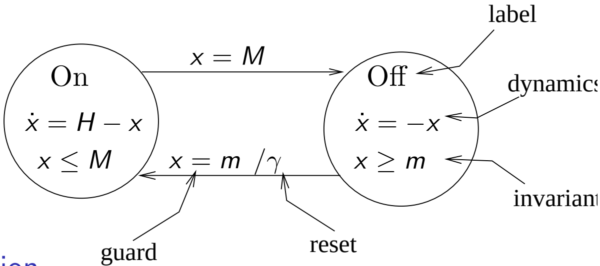

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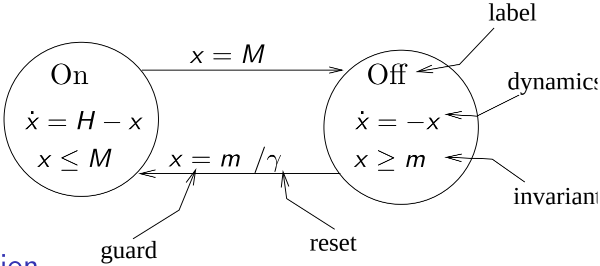

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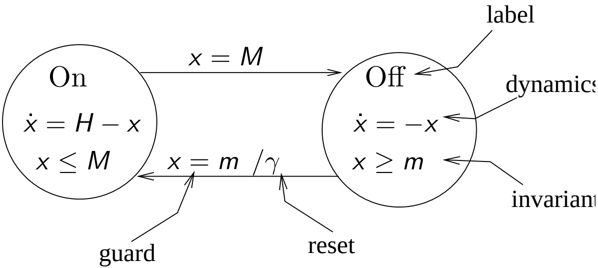
  - $p, q \in Q$, from p to q
  - $a \in \Sigma$ a label
  - g a guard; $g(\mathbf{x})$ required to take δ
  - r a reset (or jump); $\mathbf{x} := r(\mathbf{x})$ when taking δ

Trajectory-based semantics



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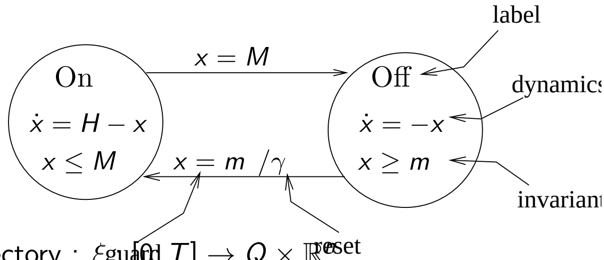
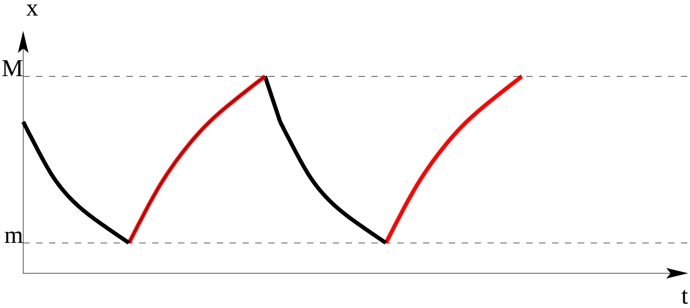


A trajectory : $\xi : [0, T] \to Q \times \mathbb{R}^n$



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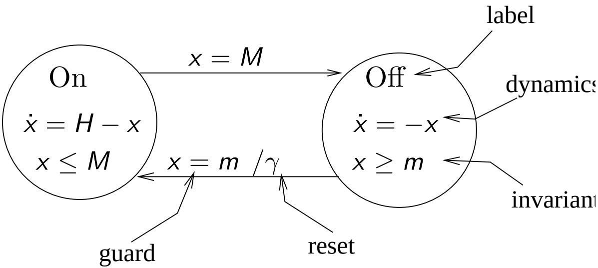

Transition system (S, T) of a HA

- **States:** $S = Q \times \mathbb{R}^n$
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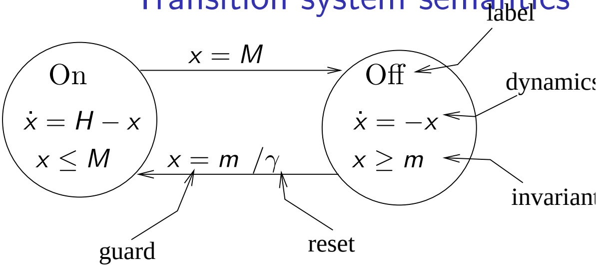

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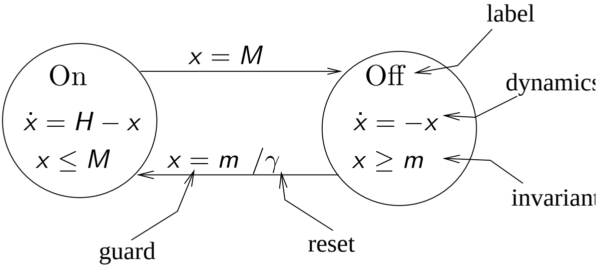
- **Transitions:** $T = T_{\text{flow}} \cup T_{\text{jump}}$
  - $(q, \mathbf{x}_1) \stackrel{\text{flow}}{\to} (q, \mathbf{x}_2) \Leftrightarrow$
    we can go from $\mathbf{x}_1$ to $\mathbf{x}_2$ in ODE $\dot{\mathbf{x}} = f_q(\mathbf{x})$
  - $(q_1, \mathbf{x}_1) \stackrel{\text{jump}}{\to} (q_2, \mathbf{x}_2) \Leftrightarrow$ if we can jump.

Transition system semantics



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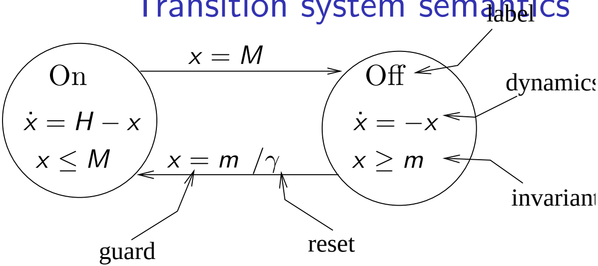

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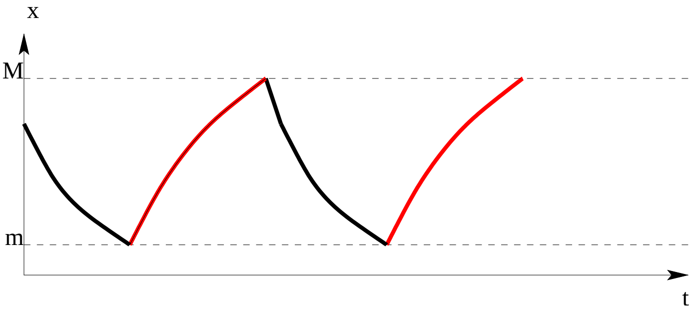
- **Runs:** sequences of states and transitions.

$$(\mathrm{On}, 0) \xrightarrow{\mathsf{flow}} (\mathrm{On}, M) \xrightarrow{\mathsf{jump}} (\mathrm{Off}, M) \xrightarrow{\mathsf{flow}} (\mathrm{Off}, m) \xrightarrow{\mathsf{jump}} (\mathrm{On}, m) \cdots$$

The two semantics are almost equivalent

- Trajectories: the system is filmed continuously



- Runs (in transition system): photos are taken when something happens

$$(\mathrm{On}, 0) \stackrel{\text{flow}}{\rightarrow} (\mathrm{On}, M) \stackrel{\text{jump}}{\rightarrow} (\mathrm{Off}, M) \stackrel{\text{flow}}{\rightarrow} (\mathrm{Off}, m) \stackrel{\text{jump}}{\rightarrow} (\mathrm{On}, m) \cdots$$

Semantic issues

- Some mathematical complications (notion of solution, existence and unicity not so evident).
- Zeno trajectories (infinitely many transitions in a finite period of time).
  - can be forbidden
  - one can consider trajectories up to the first anomaly (Sastry et al., everything OK)
  - one can consider the complete Zeno trajectories (very funny : Asarin-Maler 95)

Variants

- Discrete-time ($x_{n+1} = f_q(x_n)$) or continuous-time $\dot{x} = f(x)$
- Deterministic (e.g. $\dot{x} = f(x)$) or non-deterministic (e.g. $\dot{x} \in F(x)$)
- Eager or lazy.
- With control and/or disturbance (e.g. $\dot{x} = f(x, u, d)$)
- Various restrictions on dynamics, guards and resets: "Piecewise trivial dynamics". LHA, RectA, PCD, PAM, SPDI ... They are still highly non-trivial.

Classes of Hybrid Automata

Why classes?

Because HA are too reach; it is impossible to establish, decide, analyze properties of all HA.

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How to define a class of HA

- dimension, discrete or continuous time, eager or lazy
- what kind of dynamics
- what kind of guards/invarians/jumps

Special classes of Hybrid Automata 1

The famous one

Linear Hybrid Automata

$$\dot{\mathbf{x}} = c_1 \xrightarrow{\mathbf{x} \in P_1 / \mathbf{x} := A_1\mathbf{x} + b_1} \dot{\mathbf{x}} = c_2, \qquad \dot{\mathbf{x}} = c_2 \xrightarrow{\mathbf{x} \in P_2 / \mathbf{x} := A_2\mathbf{x} + b_2} \dot{\mathbf{x}} = c_1$$

Class specification

Dynamics $\dot{\mathbf{x}} = \mathbf{c}$; polyhedral guards and invariants; linear resets.

Special classes of Hybrid Automata 1'

Zoology of variables

- Memory cell: $\dot{x} = 0$ (but x can be reset)
- Clock: $\dot{x} = 1$.
- Stopwatch : in some locations $\dot{x} = 1$, in others $\dot{x} = 0$.
- Skewed clock: $\dot{x} = c$ (the same for every location).

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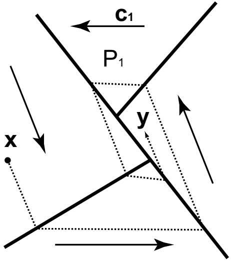
One can define subclasses of LHA like that:

> *LHA with 3 stopwatches and 1 skewed clock, with resets only to* 0 *and guards only* $x < c$.

Special classes of Hybrid Automata 2

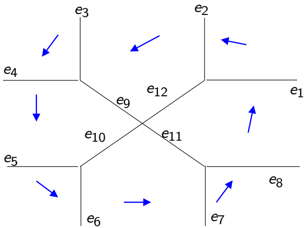
My favorite class

PCD = Piecewise Constant Derivatives



$\dot{\mathbf{x}} = \mathbf{c}_i$ for $\mathbf{x} \in P_i$

PCD is a linear hybrid automaton (LHA)



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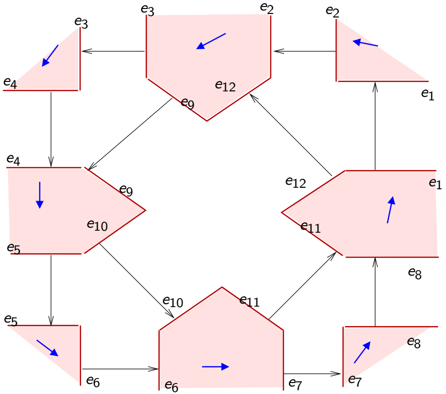

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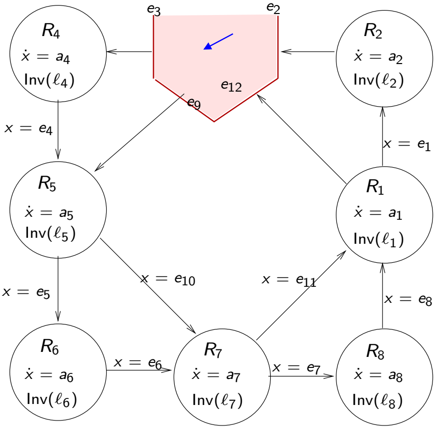

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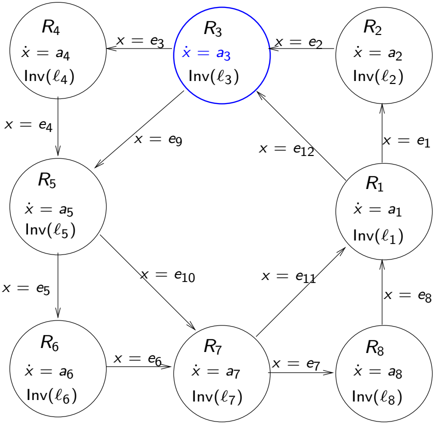

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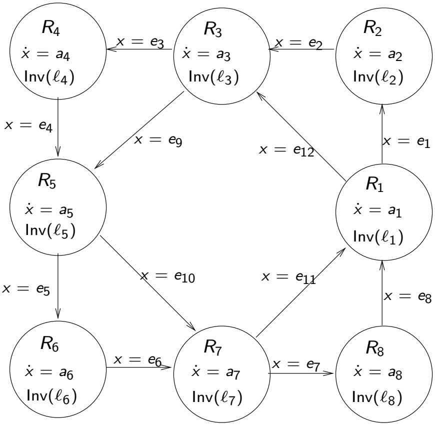
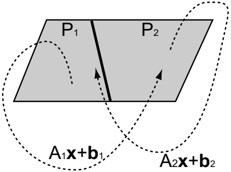


Special classes of Hybrid Automata 3

The most illustrative

Piecewise Affine Maps



$$\mathbf{x} := A_i\mathbf{x} + \mathbf{b}_i \text{ for } \mathbf{x} \in P_i$$

Special classes of Hybrid Automata 3

The most illustrative

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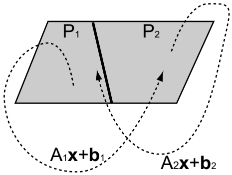

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Remark: PAM are discrete time.

How to model?

Different systems

- a control system
- a scheduler with preemption
- a genetic network

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The same class of models

A network of interacting Hybrid automata

Hybrid tools

Try to play with some tools, see at:

`http://wiki.grasp.upenn.edu/hst/index.php?n=Main.HomePage`

Maybe start with UPPAAL (timed) and PHAVER (hybrid).

Modeling exercise 1

Genetic network

We consider expression of two genes A and B, i.e. production of two proteins P and Q

- The proteins are degraded with rate k.
- P catalyzes expression of B:
  - Production of Q is proportional to the concentration of P with a coefficient a.
  - Concentration of P crosses a threshold $s \Rightarrow$ production of Q constant $= as$.
- Q inhibits expression of A:
  - Production of P equals $d - b\cdot$(concentration de Q).
  - Concentration of Q crosses a threshold $r \Rightarrow$ production of P blocks.

Modeling exercise 2

Scheduling

Schedule two jobs on one CPU and one printer with a total execution time up to 16 minutes.

- Job 1 : Compute (10 min); Print (5 min)
- Job 2 : Download (3 min); Compute (1 min); Print (2 min)

Try it :

1. without preemption;
2. with preemptible computing.

What to do with a hybrid model

- Simulate
  - With Matlab/Simulink
  - With dedicated tools
- Analyze with techniques from control science:
  - Stability analysis
  - Optimal control
  - etc..
- Analyze with your favorite techniques. The most important invention is the model.

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  - We will try verification

Outline

Verification and reachability problems

- Is automatic verification possible for HA?

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- *Safety:* are we sure that HA never enters a bad state?
- It can be seen as reachability : verify that

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- *Safety:* are we sure that HA never enters a bad state?
- It can be seen as reachability : verify that

$$\neg \mathrm{Reach}(Init, Bad)$$

- It is a natural and challenging mathematical problem.
- Many works on decidability
- Some works on approximated techniques

The reachability problem for a class C

Problem

Given

- *a hybrid automaton* $\mathcal{H} \in C$
- *two sets* $A, B \subset Q \times \mathbb{R}^n$

find out whether there exists a trajectory of $\mathcal{H}$ *starting in* A *and arriving to* B*.*
All parameters rational.

Exact methods: The curse of undecidability

Bad news

- Koiran et al.: Reach is undecidable for 2d PAM.
- AM95: Reach is undecidable for 3d PCD.
- HPKV95 Many results of the type : "3clocks + 2 stopwatches = undecidable"

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They are really bad

- Reachability is undecidable for very simple HA.
- Thus, other verification problems are also undecidable.

Undecidability Proofs — Preliminaries

Proof method:

simulation of Minsky Machine, Turing Machine etc.

Undecidability Proofs — Preliminaries

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simulation of Minsky Machine, Turing Machine etc.

Details: proof schema

- Reachability undecidable for Minsky Machines (well-known).
- A class of HA can simulate MM (to prove).
- Reach for MM $\preceq$ Reach for HA.
- Conclude that Reach for HA is undecidable.

Minsky Machines

Definition

- A counter: values in $\mathbb{N}$; operations: $C++$, $C--$; test $C>0$?
- A Minsky machine has 2 counters
- Its program has finitely many lines like that:

$$
\begin{array}{lll}
q_1: & D++; & \text{goto } q_2 \\
q_2: & C--; & \text{goto } q_3 \\
q_3: & \text{if } C>0 & \text{then goto } q_2 \text{ else } q_1
\end{array}
$$

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Theorem (Minsky)

Reachability is undecidable for Minsky machines.

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Theorem (Minsky)

Reachability is undecidable for Minsky machines.

Fact

Any algorithm can be programmed on a Minsky machine. But they are slooooooow.

A typical undecidability theorem

Theorem (Koiran, Cosnard, Garzon)

Reach *is undecidable for 2d PAM.*

A typical undecidability theorem

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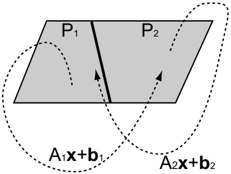
Reminder

A 2 dimensional PAM:



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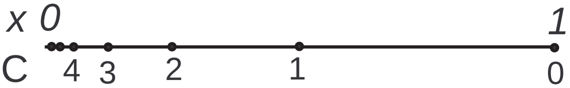
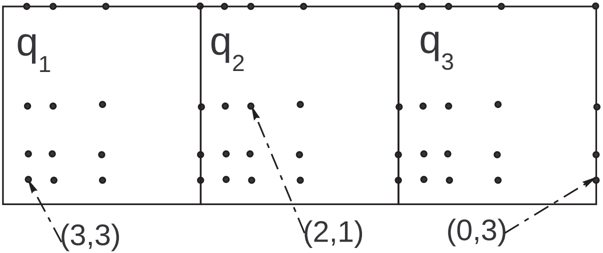
Simulating a counter by a PAM



Counter	PAM
State space $\mathbb{N}$	State space $[0;1]$
State $C = n$	$x = 2^{-n}$
$C++$	$x := x/2$
$C--$	$x := 2x$
$C > 0?$	$x < 0.75?$

Encoding a state of a Minsky Machine



Minsky Machine	**PAM**
State space $\{q_1, \dots, q_k\} \times \mathbb{N} \times \mathbb{N}$	State space $[1; k+1] \times [0; 1]$
State $(q_i, C = m, D = n)$	$x = i + 2^{-m}, y = 2^{-n}$

Simulating a Minsky Machine

Minsky Machine	**PAM**
State space $\{q_1, \ldots, q_k\} \times \mathbb{N} \times \mathbb{N}$	State space $[1; k+1] \times [0; 1]$
State $(q_i, C = m, D = n)$	$x = i + 2^{-m}, y = 2^{-n}$
q_1 : $D++$; goto q_2	$\begin{cases} x := x+1 \\ y := y/2 \end{cases}$ if $1 < x \leq 2$
q_2 : $C--$; goto q_3	$\begin{cases} x := 2(x-2)+3 \\ y := y \end{cases}$ if $2 < x \leq 3$
q_3 : if $C > 0$ then goto q_2 else q_1	$\begin{cases} x := x-1 \\ y := y \end{cases}$ if $3 < x < 4$ $\begin{cases} x := x-2 \\ y := y \end{cases}$ if $x = 4$

. . . finally we have proved:

Theorem (Koiran et al.)

Reach *is undecidable for 2d PAMs.*

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Reach *is undecidable for LHA.*

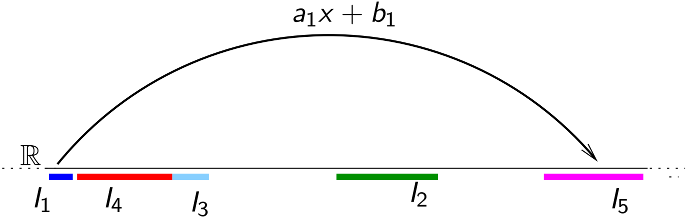
Jumps are not necessary for undecidability:

Theorem (A., Maler, Pnueli)

Reach *is undecidable for 3d PCD.*

Unrelated: a difficult problem

- 1d piecewise affine maps (PAMs): $f : \mathbb{R} \to \mathbb{R}$
  $f(x) = a_i x + b_i$ for $x \in I_i$



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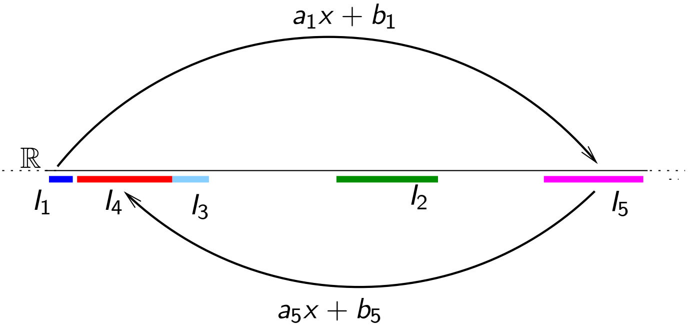

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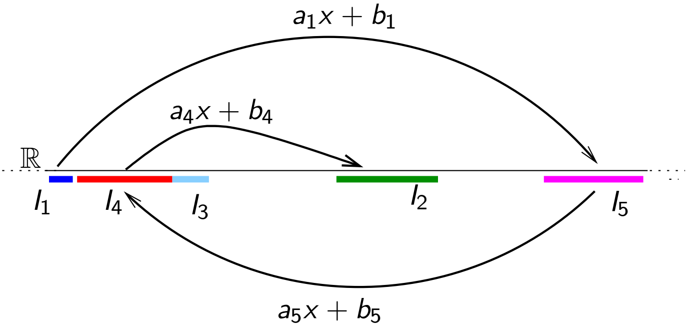

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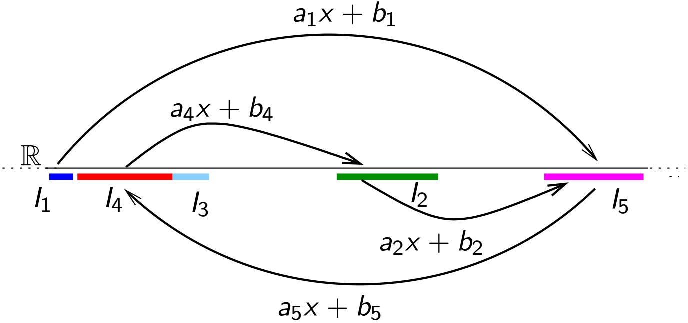

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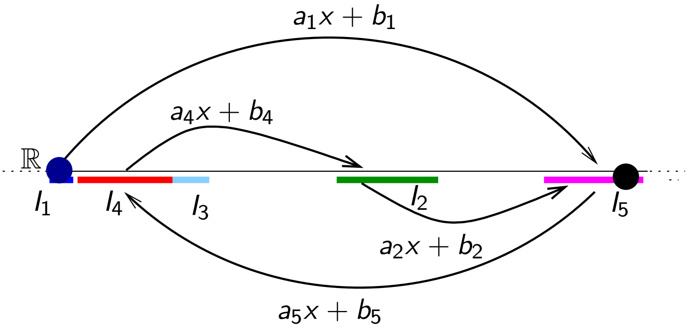

Old Open Problem

Is reachability decidable for 1d PAM?

Conclusions of Day 1

We have learned today

- What is a Hybrid Automaton.
- How to read yet another definition of HA and its semantics.
- How to model things using HA.
- Famous classes of HA.
- Safety verification as reachability problem.
- How to prove undecidability by simulation of Minsky Machines.
- Even the simplest classes of HA have undecidable reachability.

How to verify HA: theory and practice

- In practice approximate/bounded methods should be used for safety verification.
- Several tools, many methods.
- General principles are easy, implementation difficult.

Abstract algorithm - important

A generic verification algorithm A

Forward breadth-first search

F=Init
repeat
    F=F $\cup$ SuccFlow(F) $\cup$ SuccJump(F)
until (F$\cap$ Bad $\neq \emptyset$)| fixpoint | tired
say "reachable" | "unreachable" | "timeout"

Most verification methods and tools are variants of it.

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There are variants:

- forward/backward
- breadth first/depth first/best first/etc.

Most verification methods and tools are variants of it.

How to implement it

Needed data structure for representation of subsets of $\mathbb{R}^n$, and algorithms for efficient computing of

- unions, intersections;
- inclusion tests;
- SuccFlow;
- SuccJump.

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Needed data structure for representation of subsets of $\mathbb{R}^n$, and algorithms for efficient computing of

- unions, intersections;
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- SuccJump.

It could be *exact* or *over-approximate*.

Some trivial results

Theorem

If for a class of HA the Algorithm A can be implemented (exactly), then

- Reach *is semi-decidable;*
- *bounded* Reach *in n steps is decidable;*
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Fact

Suppose for a class of HA the Algorithm A can be implemented approximately. Then we can build a verification tool saying:

- *"Unreachable".*
- *"Maybe reachable".*
- *" Timeout".*

One important implementation for linear hybrid automata

We want to represent subsets of $Q \times \mathbb{R}^n$.

Objects and data structure

- A polyhedron P defined by $Ax \leq \vec{b}$. Data structure $(A, \vec{b})$.
- Several polyhedra $P_1, \ldots, P_n$. Data structure $(A_1, \vec{b}_1), \ldots, (A_n, \vec{b}_n)$,
- The same with the discrete state: $(q_1, P_1), \ldots, (q_n, P_n)$.

Fact

The data structure $(q_1, A_1, \vec{b}_1), \ldots, (q_n, A_n, \vec{b}_n)$ representing union of polyhedra in $Q \times \mathbb{R}^n$ allows implementing semi-algorithm A for LHA.

We will prove it.

Towards the proof

(for simplicity, I forget about q)

Definition

Linear constraint C on $\mathbb{R}^n$ is $\sum a_i x_i \leq b$ (or $<$).

Polyhedra and logic

Our data structure can be written in logic

- A polyhedron: $\bigwedge_i C_i$.
- A union of polyhedra - a DNF: $\bigvee_i \bigwedge_j C_{ij}$

Example: what is $(x > 0 \wedge y > 0 \wedge x + y < 3) \vee x > 7$?

A richer formalism

Definition

A first order logic of linear constraints (FOLC).

$$F ::== \sum a_i x_i \leq c | \neg F | F \vee F | \exists x F$$

(we can also express $<, =, \wedge, \forall$)

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$$Q(u, v) = \exists x \forall y (x < 3 \vee y > 2 \vee 3u + x + v < 17)$$

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Theorem

FOLC admits quantifier elimination
That is, every formula $\Leftrightarrow$ a DNF of linear constraints (a union of polyhedra).

How to remove quantifiers?

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FOLC admits quantifier elimination

Let us remove one quantifier.

Lemma (Fourier-Motzkine)

$\exists y \bigwedge_j C_j$ *can be written as* $\bigwedge_j C'_j$.

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The same thing in geometric language

Projection of a polyhedron in $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ is a polyhedron.

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To prove theorem apply FM several times.

Understanding Fourier-Motzkine

Project a 3D polyhedron to the plane (x, y).

$$F = \exists z \left\{ \begin{array}{rcl} 2x + y + z & \leq & 10 \\ 3x + 2y & \leq & 11 \\ -x - y + z & \leq & 4 \\ 6x - y - z & \leq & 9 \end{array} \right\}$$

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Let us isolate z: $\left\{ \begin{array}{rcl} z & \leq & 10 - 2x - 2y \\ 3x + 2y & \leq & 11 \\ z & \leq & 4 + x + y \\ 6x - y - 9 & \leq & z \end{array} \right.$

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Eliminate z: no z remain; lower bounds $\leq$ upper bounds

$$F \Leftrightarrow \left\{ \begin{array}{rcl} 3x + 2y & \leq & 11 \\ 6x - y - 9 & \leq & 10 - 2x - 2y \\ 6x - y - 9 & \leq & 4 + x + y \end{array} \right.$$

Terminating the proof - recall what we are doing

We know that

Theorem

Every FOLC formula $\Leftrightarrow$ a DNF of linear constraints (a union of polyhedra).

We want prove that

Fact

DNF of linear constraints (union of polyhedra) is a good data structure for verification of LHA

Terminating the proof - boolean operation and tests

Let D_1 and D_2 be two DNFs (union of polyhedra).

Union $D_1 \cup D2$ is already a DNF.

Terminating the proof - boolean operation and tests

Let D_1 and D_2 be two DNFs (union of polyhedra).

Union $D_1 \cup D2$ is already a DNF.

Intersection $D_1 \cap D_2$ can be transformed to DNF using boolean logic.

Emptiness test To know if $D_1 = \emptyset$, eliminate quantifiers from $\exists \mathbf{x} D_1(\mathbf{x})$

Inclusion test to know if $D_1 \subset D_2$, eliminate quantifiers from $\forall \mathbf{x}(D_1(\mathbf{x}) \Rightarrow D_2(\mathbf{x}))$.

Terminating the proof - successors

Let D be a DNF (union of polyhedra)

Flow successor For a state with dynamics $\dot{x} = \mathbf{c}$ and invariant I, flow successor of D is:

$$D'(\mathbf{x}) = \exists \mathbf{y} \exists t \, (D(\mathbf{y}) \wedge t > 0 \wedge \mathbf{x} = \mathbf{y} + \mathbf{c}t \wedge I(\mathbf{x}) \wedge I(\mathbf{y}$$

Eliminate quantifiers, and obtain the DNF for the successor.

Jump successor For a transition with guard G and the reset R, jump successor of D is:

$$D''(\mathbf{x}) = \exists \mathbf{y} \, (D(\mathbf{y}) \wedge G(\mathbf{y}) \wedge \mathbf{x} = R(\mathbf{y}))\,.$$

Eliminate quantifiers, and obtain the DNF for the successor.

Look at the pictures!

Concluding

We are done

We have implemented (exactly) all the operations for LHA, using unions of polyhedra (DNFs of constraints) as data structure.

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Remarks

- Fourier-Motzkine is very costly.
- Our method is very costly.
- More efficient geometric method exist for some operations.
- They are still costly.

Theorems for linear hybrid automata

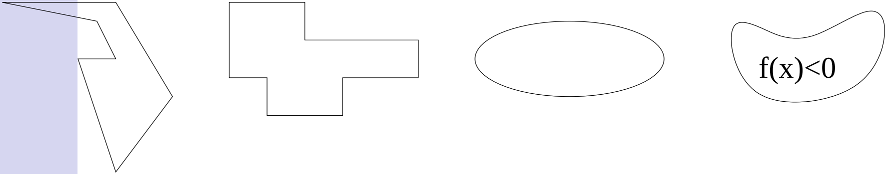
Theorem

For Linear Hybrid Automata

- *Reach is semi-decidable (recursively enumerable)*
- Reach_n *is decidable*

Known implementations

- Polyhedra (HyTech - exact. PHAVER - exact and approximate)
- "Griddy polyhedra" (d/dt)
- Zonotopes (Le Guernic)
- Ellipsoids (Kurzhanski, Bochkarev)
- Level sets of functions (Tomlin)
- Support functions(Le Guernic)



Does it work?

Up to 10-20 dimensions. Sometimes.

Using advanced verification techniques

- Searching for better data-structures and basic operations
- Abstraction and refinement
- Combining model-checking and theorem proving
- Acceleration
- Bounded model-checking

Part II

Timed Automata

Outline

Outline

Definition of TA

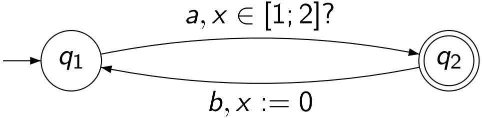
Definition

Timed automata are a subclass of hybrid automata:

- **Variables** $x_1, \ldots, x_n$, called clocks.
- **Dynamics** $\dot{x}_i = 1$, for all clocks, in all locations.
- **Guards and invariants** Conjunctions of $x_i < c$ (or $\leq, =, . \geq)$)with $c \in \mathbb{N}$
- **Resets** $x_i := 0$ for some clocks.

An example of a timed automaton

- Timed automaton (we forget to write $\dot{x} = 1$):



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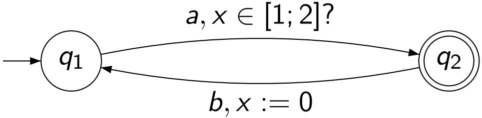

- Its run

$$(q_1, 0) \xrightarrow{1.83} (q_1, 1.83) \xrightarrow{a} (q_2, 1.83) \xrightarrow{4.1} (q_2, 5.93) \xrightarrow{b} (q_1, 0) \xrightarrow{1} (q_1, 1)$$

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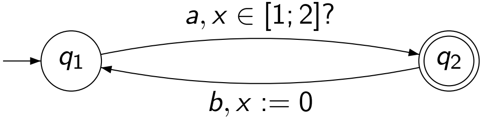

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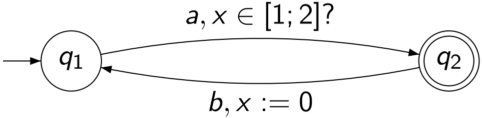

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- Its *trace* $1.83\, a\, 4.1\, b\, 1\, a$ a *timed word*
- Its *timed language*: set of all the traces starting in q_1, ending in q_2:

$$\{t_1\, a\, s_1\, b\, t_2\, a\, s_2\, b \ldots t_n\, a \mid \forall i . t_i \in [1; 2]\}$$

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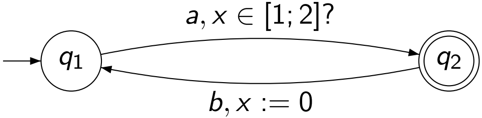

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Observation

Clock value of x: time since the last reset of x.

Some simple exercises

Draw timed automata for specifications:

- Request a arrives every 5 minutes.

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- The same, but a arrives every 5 to 7 minutes.

Meditation on TA

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Very restricted: only time progress remains from all physics.

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Time and events together. Interesting . . . ...

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Classical formulation

Empty language problem is decidable for TA

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Classical formulation

Empty language problem is decidable for TA

Both are the same

Non-empty language $\Leftrightarrow$ Reach(Init,Fin)

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- What does it mean: the same behavior?
- How to invent it?

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- Build a finite region automaton (its states are regions)
- Test reachability in this region automaton.

Two difficulties

- What does it mean: the same behavior? Bisimulation.
- How to invent it? A&D invented it using ideas of Berthomieu (Time Petri nets). In fact it is rather natural.

Region equivalence

Definition

Two states of a TA are region equivalent: $(q, \mathbf{x}) \approx (p, \mathbf{y})$ if

- Same location: $p = q$
- Same integer parts of clocks: $\forall i\,(\lfloor x_i \rfloor = \lfloor y_i \rfloor)$
- Same order of fractional parts of clocks $\forall i, j\,(\{x_i\} < \{x_j\} \Leftrightarrow \{y_i\} < \{y_j\})$

Look at the picture!

Region equivalence

Definition

Two states of a TA are region equivalent: $(q, \mathbf{x}) \approx (p, \mathbf{y})$ if

- Same location: $p = q$
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An issue

- Infinitely many equivalence classes.

Region equivalence

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Two states of a TA are region equivalent: $(q, \mathbf{x}) \approx (p, \mathbf{y})$ if

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- Same order of fractional parts small of clocks $\forall \text{small} i, j \, (\{x_i\} < \{x_j\} \Leftrightarrow \{y_i\} < \{y_j\})$
- Or they are both big : $\forall i \, ((x_i > M) \Leftrightarrow (y_i > M))$

Look at the picture!

An issue, and a solution

finitely many equivalence classes.

- Solution: when a variable is BIG, we don't care about it.

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An issue

finitely many equivalence classes.

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Definition

Equivalence classes of $\approx$ are called regions.

Region equivalence is a bisimulation

very informal

Equivalent states can make the same transitions, and arrive to equivalent states.

Region equivalence is a bisimulation

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Equivalent states can make the same transitions, and arrive to equivalent states.

Let us formalize it:

Lemma

Suppose $(q,\mathbf{x}) \approx (p,\mathbf{y})$. *Then*

- **Jump** *If* $(q,\mathbf{x}) \xrightarrow{a} (q',\mathbf{x}')$ *then* $(p,\mathbf{y}) \xrightarrow{a} (p',\mathbf{y}')$ *with* $(q',\mathbf{x}') \approx (p',\mathbf{y}')$.
- **Time** *If* $(q,\mathbf{x}) \xrightarrow{t} (q',\mathbf{x}')$ *then* $(p,\mathbf{y}) \xrightarrow{\hat{t}} (p',\mathbf{y}')$ *with* $(q',\mathbf{x}') \approx (p',\mathbf{y}')$ *(the time can be different!)*.

Reading a timed word

Iterating the previous lemma we get

Lemma

Suppose $(q,\mathbf{x}) \approx (p,\mathbf{y})$*, and* $q \xrightarrow{w} (q',\mathbf{x}')$ *(with some timed word* w*), then* $(p,\mathbf{y}) \xrightarrow{\hat{w}} (p',\mathbf{y}')$ *with* $(q',\mathbf{x}') \approx (p',\mathbf{y}')$ *(the timing in* $\hat{w}$ *can be different from* w*).*

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Corollary

The same set of regions is reachable from elements of one region.

Decision algorithm

- Build a region automaton RA
  - States are regions.
  - There is a transition $r_1 \xrightarrow{a} r_2$ if some (all) element of r_1 can go to some element of r_2 on a.
  - There is a transition $r_1 \xrightarrow{\tau} r_2$ if some (all) element of r_1 can go to some element of r_2 on some $t > 0$

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  - There is a transition $r_1 \xrightarrow{\tau} r_2$ if some (all) element of r_1 can go to some element of r_2 on some $t > 0$
- Check whether some final region in RA is reachable from initial region.

Outline

Closure property

Definition

Timed regular language is a language accepted by a TA

Closure property

Definition

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Theorem

Timed regular languages are closed under $\cap, \cup$, projection, but not complementation.

Closure property

Definition

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Theorem

Timed regular languages are closed under $\cap, \cup$, *projection, but not complementation.*

Fact

Determinization impossible for timed automata.

Decidability properties

Definition

Timed regular language (TRL) is a language accepted by a TA

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Decidable for TRL (represented by TA): $L = \emptyset$, $w \in L$, $L \cap M = \emptyset$.

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Proof.

Immediate from Alur&Dill's theorem. □

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Undecidable for TRL (represented by TA): L universal (contains all the timed words), $L \subset M$, $L = M$.

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Proof.

Encoding of runs of Minsky Machine as a timed languages. $\square$

Reminder: regular expressions

Definition

Regular expressions: $E ::= 0 \mid \varepsilon \mid a \mid E + E \mid E \cdot E \mid E^*$

Theorem (Kleene)

Finite automata and regular expression define the same class of languages.

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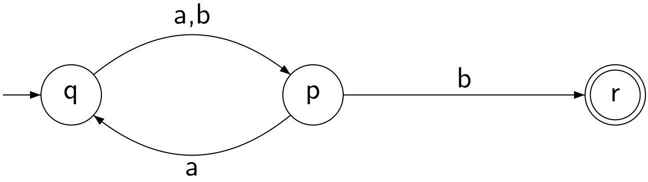
Example



$$((a+b)a)^*(a+b)b$$

Timed regular expressions

A natural question

How to define regular expressions for timed languages?

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Timed regular expressions

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How to define regular expressions for timed languages?

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Semantics:

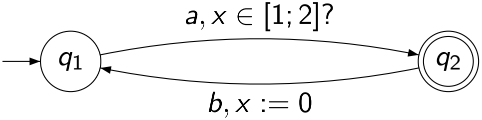
$$\|\underline{\mathbf{t}}\| = \mathbb{R}_{\geq 0} \quad \|a\| = \{a\} \qquad \|0\| = \emptyset \quad \|\varepsilon\| = \{\varepsilon\}$$

$$\|E_1 \cdot E_2\| = \|E_1\| \cdot \|E_2\| \qquad \|E_1 + E_2\| = \|E_1\| \cup \|E_2\|$$

$$\|\langle E \rangle\|_I = \{\sigma \in \|E\| \mid \ell(\sigma) \in I\} \qquad \|E^*\| = \|E\|^*$$

$$\|E_1 \wedge E_2\| = \|E_1\| \cap \|E_2\| \qquad \|[a \mapsto z]E\| = [a \mapsto z]\|E\|$$

A good example and a theorem



$$\{L = \{t_1\, a\, s_1\, b\, t_2\, a\, s_2\, b \ldots t_n\, a \mid \forall i . t_i \in [1;2]\}\}$$

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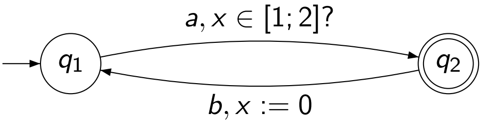


$$\{L = \{t_1\, a\, s_1\, b\, t_2\, a\, s_2\, b \ldots t_n\, a \mid \forall i . t_i \in [1;2]\}$$

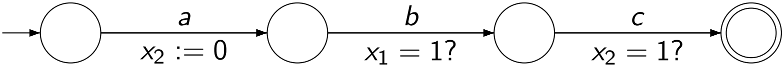
An expression for L : $\left(\langle \underline{\mathbf{t}} a \rangle_{[1;2]} \underline{\mathbf{t}} b\right)^*$

Theorem (A., Caspi, Maler)

Timed Automata and Timed regular expressions (with $\wedge$ and $[a \mapsto z]$) define the same class of timed languages

A nasty example

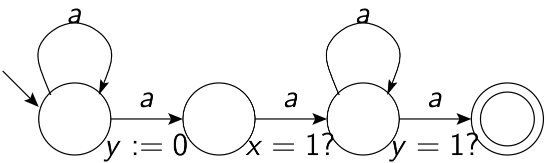
Intersection needed [ACM]



$$\{t_1 a t_2 b t_3 c \mid t_1 + t_2 = 1, t_2 + t_3 = 1\} = \underline{\mathbf{t}} a \langle \underline{\mathbf{t}} b \underline{\mathbf{t}} c \rangle_1 \wedge \langle \underline{\mathbf{t}} a \underline{\mathbf{t}} b \rangle_1 \underline{\mathbf{t}} c$$

Another nasty example

Renaming needed [Herrmann]



$$[b \mapsto a]\Big(((\underline{\mathbf{t}}a)^* \langle \underline{\mathbf{t}} b (\underline{\mathbf{t}}a)^* \rangle_1 \wedge \langle (\underline{\mathbf{t}}a)^* \underline{\mathbf{t}} b \rangle_1 (\underline{\mathbf{t}}a)^*\Big).$$

Outline

Model-checking etc.

Reminder: decidability for TA

- We can decide: Reach, $L \neq \emptyset$, $L \cap M = \emptyset$, $w \in L$
- Undecidable: $L =$ all the words; $L \subset M$, $L = M$

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Verification problem

Given a system S and a property P, verify that S satisfies P.

Verification approaches

For simple safety properties:

- Represent S by a TA A_S.
- Represent P as ¬Reach(Init,Bad).
- Apply reachability algorithm.

For all kind of properties

(even with ω-behaviors)

- Represent S by a TA A_S.
- Represent $\neg P$ by a TA $A_{\neg P}$.
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Or express P in a temporal logic and use some model-checking.

A simple verification example

Exercise

How to verify this?

System A bus passes every 7 to 9 minutes. A taxi passes every 6 to 8 minutes. At noon a bus and a taxi passed.

Property Between 12:05 and 12:30, within 5 minutes after every bus, a taxi passes.

Reachability in practice: no regions

Fact

Real verification tools, e.g. UPPAAL, *do not use the region automaton. They apply a variant of the algorithm we know.*

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Algorithm B

F=Init
repeat
    F=F $\cup$ SuccFlow(F) $\cup$ SuccJump(F)
    Widen(F)
until (F$\cap$ Final $\neq \emptyset$)| fixpoint
say "reachable" | "unreachable"

Zones and DBMs

What is needed to implement Algorithm B

Data structure and basic algorithms for subsets of $Q \times \mathbb{R}^n$

Zones and DBMs

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Data structure and basic algorithms for subsets of $Q \times \mathbb{R}^n$

Definition

Let $x_0 = 0$; let $x_1, \ldots, x_n$ - clocks.

- *Zone:* polyhedron defined by a conjunction of constraints $x_i - x_j \leq d_{ij}$ (or $<$) wirh $d_{IJ} \in \mathbb{N}$.
- *Difference bound matrix (DBM) for a zone:* $D = (d_{ij})$.

Fact

A zone is a union of regions.

Zones and verification of TA

Fact

Using DBMs, the following tests and operations on zones are easy ($O(n) - O(n^3)$):

- $Z_1 = Z_2$?; $Z = \emptyset$?; $Z_1 \cap Z_2$.
- $SuccFlow(Z)$ *and* $Succ_\delta(Z)$ *- both are zones.*

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See Cormen, graph algorithms.

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Corollary

Unions of zones, represented $(q_1, D_1), \ldots (q_n, D_n)$*, are suitable to implement Algorithm B*

Termination

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Theorem

Algorithm B is correct and terminates (and used in practice)

Part III

Back to Hybrid automata: decidability

Outline

Outline

Reduction to TA : simple cases

Fact

Reachability is decidable for the following subclasses of HA, it is reduced to TA reachability.

- *Like TA, rational constants.*

Reduction to TA : simple cases

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Reachability is decidable for the following subclasses of HA, it is reduced to TA reachability.

- *Like TA, rational constants.*
  Reduction: *Multiply all the guards by the common denominator K, you obtain a timed automaton with the same reachability (location to location).*

Reduction to TA : simple cases

Fact

Reachability is decidable for the following subclasses of HA, it is reduced to TA reachability.

- *Like TA, rational constants.*
- *Like TA, but the rate of each clock = arbitrary rational: $\dot{x}_i = r_i$ (the same everywhere).*

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- *Like TA, but the rate of each clock = arbitrary rational: $\dot{x}_i = r_i$ (the same everywhere).*
  Reduction: *Change of variables $\bar{x}_i = x_i/r_i$ (and corresponding change guards) transform the system into a TA with the same reachability.*

Reduction to TA : simple cases

Fact

Reachability is decidable for the following subclasses of HA, it is reduced to TA reachability.

- *Like TA, rational constants.*
- *Like TA, but the rate of each clock = arbitrary rational:* $\dot{x}_i = r_i$ *(the same everywhere).*
- **Initialized skewed-clock automata** *Like TA, but in a state* q *we have that* $\dot{x}_i = r_{iq}$ *(may depend on the state).* **Restriction:** *when we change rate, we forget the value. Formally, for any transition* $p \to q$*, either* $r_{ip} = r_{iq}$ *or* x_i *is reset.*

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  Reduction*: Change of variables* $\bar{x}_i = x_i / r_{iq}$ *at state* q*. It works because of the restriction.*

Rectangular Hybrid Automata

Let us generalize

We want to extend the previous example to the largest possible decidable class.

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Definition

The class of Rectangular Hybrid automata is defined as follows:

- Variables $x_1, \dots x_n$.
- Dynamics at each state q : inclusion $\dot{x}_i \in [a_{iq}, b_{iq}]$ (for each i)
- Invariant at each state q, and guard of each transition : $x_i \in [a_., b_.]$
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Fact

Reachability is undecidable for RHA.

Initialized Rectangular Hybrid Automata

To obtain reachability one needs a restriction:

Definition (When we change rate, we forget the value)

Initialized RHA should reset x_i on each transition that changes its rate.

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Theorem (Henzinger et al.)

Reachability is decidable for Initialized RHA.

Probably the "largest" known decidable class of HA!

Outline

o-minimal automata

They have a complex, sometimes nonlinear dynamic, but they also forget the variable, when its equation changes.

Outline

Topological decision method: 2D only

Reach is decidable for

- MP94: 2d PCD + Key idea
- CV96: 2d multi-polynomial systems.
- ASY01: 2d “non-deterministic PCD” (wait a minute)

SPDI

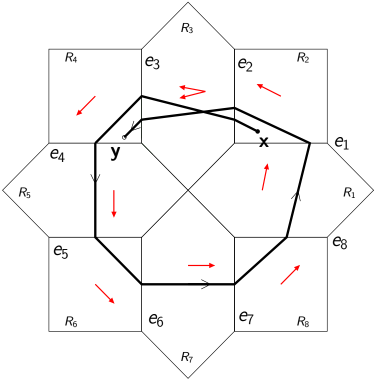
Simple Polygonal Differential Inclusion =
the non-deterministic version of PCD=

- A partition of the plane into polygonal regions
- A constant differential inclusion for each region

$$\dot{x} \in \angle_{\mathbf{a}}^{\mathbf{b}} \text{ if } \mathbf{x} \in R_i$$

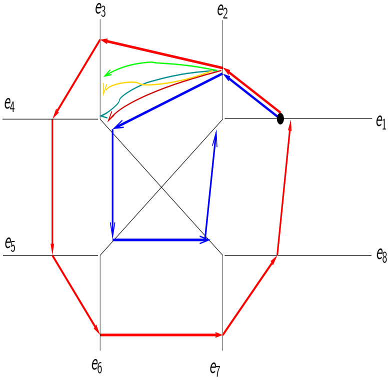
SPDI

Simple Polygonal Differential Inclusion =



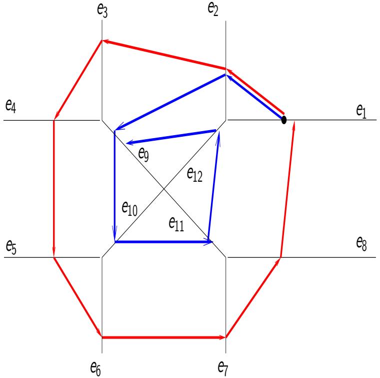
Difficulties

Too many trajectories (even locally)



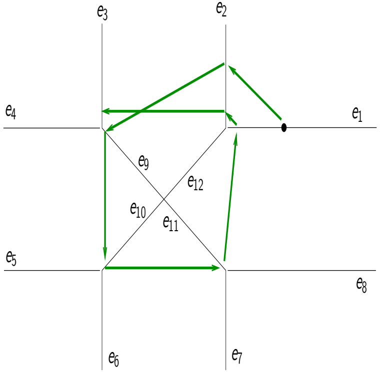
Difficulties

Too many signatures



Difficulties

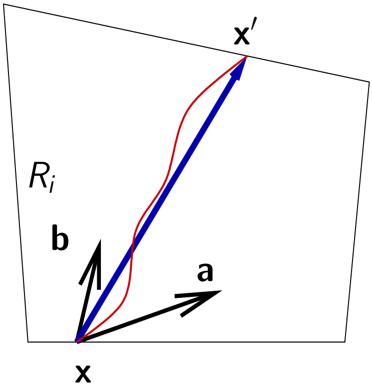
Self-crossing trajectories



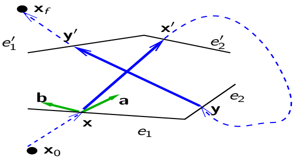
Plan of solution

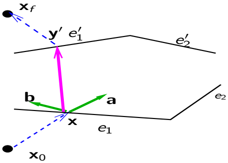
- Simplify trajectories
- Enumerate types of signatures
- Test reachability for each type using accelerations

Simplification 1: Straightening



Simplification 2: Removing self-crossings





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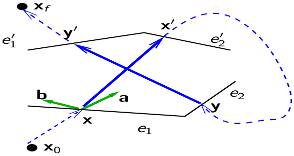
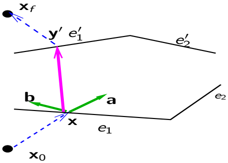


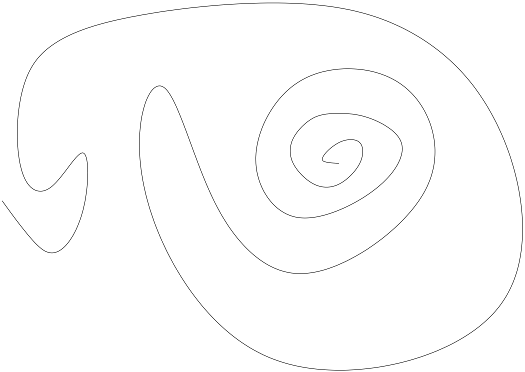


Bottom line: Reach$(x, y) \Leftrightarrow \exists$ a *simple piecewise straight* trajectory from x to y

Key topological remark

Fact (Jordan, Poincaré-Bendixson, applied by Maler-Pnueli)

Simple curves on the plane are very simple.



Signatures of simplified trajectories

- Lemma (Representation Theorem:)

  Any edge signature can be represented as

$$\sigma = r_1(s_1)^{k_1} r_2(s_2)^{k_2} \dots r_n(s_n)^{k_n} r_{n+1}$$

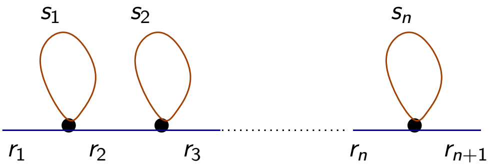
  - Properties
    - r_i is a seq. of pairwise different edges;
    - s_i is a simple cycle;
    - r_i and r_j are disjoint
    - s_i and s_j are different

  Proof based on Jordan's theorem (MP94)

Classification of signatures

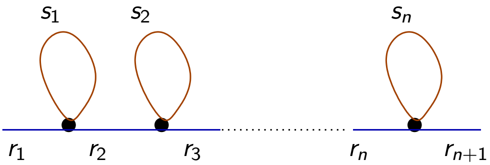
Any edge signature belongs to a type

$$r_1(s_1)^* r_2(s_2)^* \dots r_n(s_n)^* r_{n+1}$$



There are finitely many types!

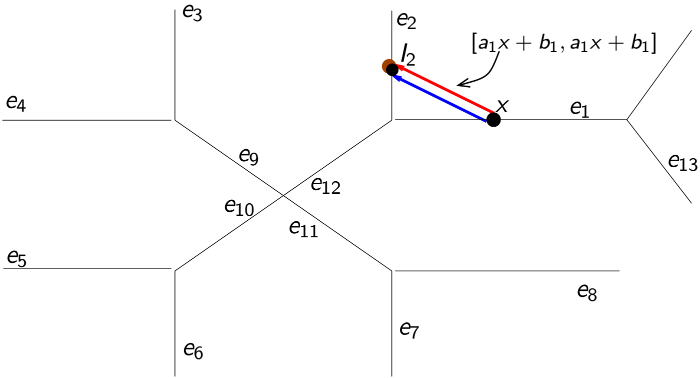
How to explore one type?



Recipe: compute successors and accelerate cycles.

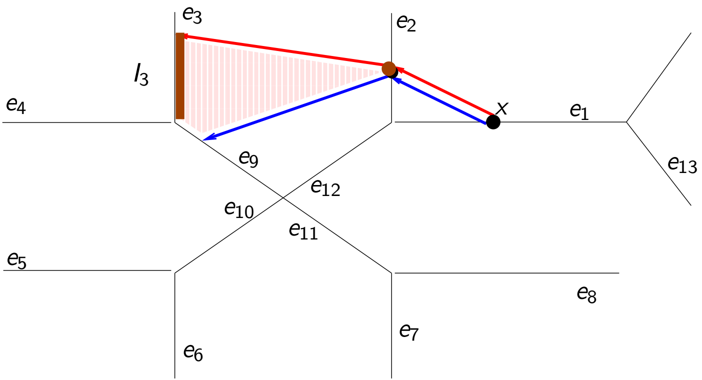
Successors (by σ)

One step ($\sigma = e_1 e_2$)



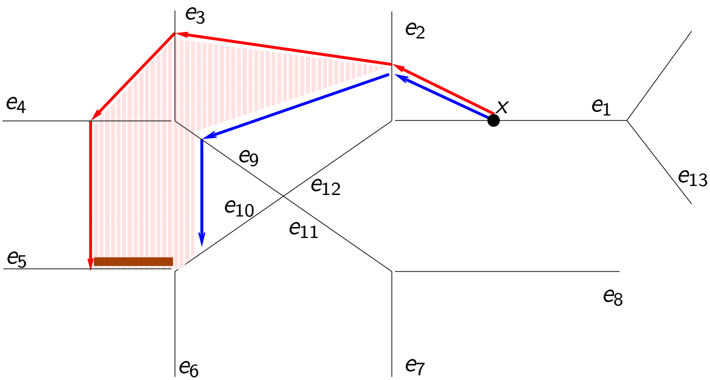
Successors (by σ)

Several steps ($\sigma = e_1 e_2 e_3$)



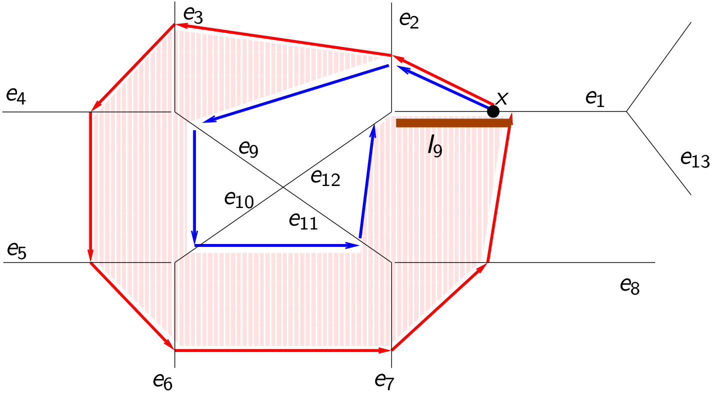
Successors (by σ)

Several steps ($\sigma = e_1 e_2 e_3 e_4 e_5$)

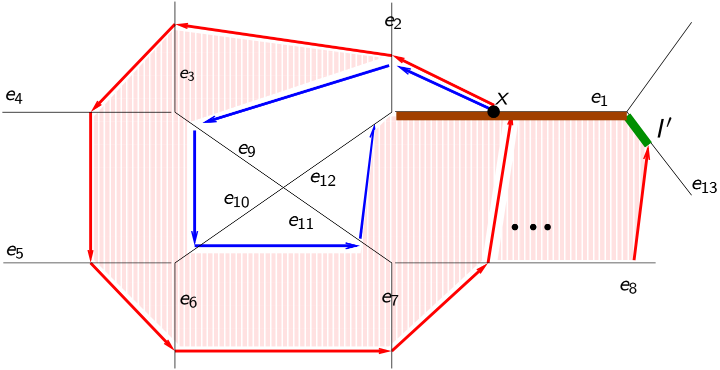


Successors (by σ)

One cycle ($\sigma = s = e_1 e_2 \cdots e_8 e_1$)



Successors (by σ)



5 One cycle iterated: ≈*solution of fixpoint equation (acceleration)* ($\mathrm{Succ}_\sigma(I) = I$)

The calculus of TAMF

- **Fact:** All successors are TAMF
- Affine function (AF):
  $f(x) = ax + b$ with $a > 0$
- Affine multi-valued function (AMF):
  $\tilde{F}(x) = [f_1(x), f_2(x)]$
- Truncated affine multi-valued function (TAMF): $F(x) = \tilde{F}(x) \cap J$ if $x \in S$

Lemma

AF, AMF and TAMF are closed under composition.

Lemma

Fixpoint equations $F(I) = I$ can be explicitely solved (without iterating)

Reachability Algorithm

for each type of signature τ **do**
test whether $x \xrightarrow{\tau} y$

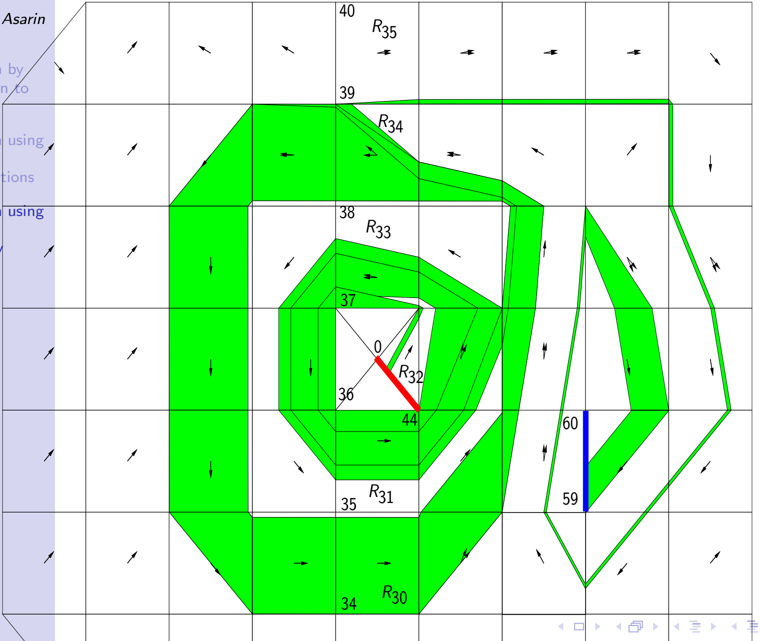
To test $x \xrightarrow{\tau} y$ for $\tau = r_1(s_1)^* r_2(s_2)^* \dots r_n(s_n)^* r_{n+1}$ compute Succ_r and accelerate $(\mathrm{Succ}_s)^*$

Main result for SPDI

Theorem (A., Schneider, Yovine)

Reachability is decidable for SPDI.

SPeeDI the tool



Part IV

Conclusions and perspectives

Outline

Hybrid: conclusions for a pragmatical user

- A useful and proper model of cyber-physical systems: HA. Modeling languages available.
- Simulation possible with old and new tools
- No hope for exact analysis
- In simple cases approximated analysis (and synthesis) with guarantee is possible using verification paradigm. Tools available
- (Not discussed) Some control-theoretical techniques available (stability, optimal control etc).

Hybrid: perspectives for a researcher

- Obtain new decidability results (nobody cares for undecidability).
- Find better data structures and algorithms for approximated verification.
- Apply modern model-checking techniques
- Create hybrid theory of formal languages
- etc.

Timed: Conclusions for a pragmatical user

- A useful and proper model of computer systems immersed in physical time : TA.
- Modeling and specification languages available.
- Efficient simulation, verification and synthesis tools available.

Timed: perspectives for a researcher

- Develop a theory of timed languages. Algebra, logic, topology etc. (see my text `http://hal.archives-ouvertes.fr/hal-00157685`)
- Improve verification techniques.
- Study rich and decidable specification formalisms (logical, algebraic, etc.) for timed languages.
- etc.