RATIONAL SELF-AFFINE TILES ASSOCIATED TO STANDARD AND NONSTANDARD DIGIT SYSTEMS

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Abstract. We consider digit systems $(A, \mathcal{D})$, where $A \in \mathbb{Q}^{n\times n}$ is an expanding matrix and the *digit set* $\mathcal{D}$ is a suitable subset of $\mathbb{Q}^n$. To such a system, we associate a self-affine set $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ that lives in a certain representation space $\mathbb{K}_A$. If A is an integer matrix, then $\mathbb{K}_A = \mathbb{R}^n$, while in the general rational case $\mathbb{K}_A$ contains an additional solenoidal factor. We give a criterion for $\mathcal{F}$ to have positive Haar measure, *i.e.*, for being a *rational self-affine tile*. We study topological properties of $\mathcal{F}$ and prove some tiling theorems. Our setting is very general in the sense that we allow $(A, \mathcal{D})$ to be a *nonstandard* digit system. A *standard* digit system $(A, \mathcal{D})$ is one in which we require $\mathcal{D}$ to be a complete system of residue class representatives w.r.t. a certain naturally chosen residue class ring. Our tools comprise the Frobenius normal form and character theory of locally compact abelian groups.

1. Introduction

This paper is a contribution to the theory of self-affine tiles whose foundations were established in the early 1990s by Bandt [3], Kenyon [12], Gröchenig and Haas [8], as well as Lagarias and Wang [16, 17, 18] and which has gained a lot of attention in the past decades.

We recall the definition of a self-affine tile. Let $A \in \mathbb{R}^{n\times n}$ be an expanding matrix (*i.e.*, all its eigenvalues lie outside the unit circle) with integer determinant, and let $\mathcal{D} \subset \mathbb{R}^n$ be a *digit set* with $|\mathcal{D}| = |\det A|$. Then we call the pair $(A, \mathcal{D})$ a *digit system*. By Hutchinson [9], there exists a unique nonempty compact subset $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ of $\mathbb{R}^n$ that satisfies the set equation

$$A\mathcal{F} = \bigcup_{d\in\mathcal{D}} (\mathcal{F} + d). \tag{1.1}$$

If $\mathcal{F}$ has positive Lebesgue measure, it is called a *self-affine tile*. Of special interest are the so-called integral self-affine tiles (see [16]), which are obtained when the matrix and the digits have integer coefficients. By Bandt [3], the Lebesgue measure of an integral self-affine tile $\mathcal{F}$ is certainly positive if $(A, \mathcal{D})$ is a *standard digit system*, meaning that $\mathcal{D}$ is a complete set of residue class representatives of $\mathbb{Z}^n/A\mathbb{Z}^n$. One very famous example of such an integral self-affine tile is Knuth's *twin dragon* (see [14, p. 206]), whose boundary is a fractal set. Lagarias and Wang [16, 17] also regarded the matter of $(A, \mathcal{D})$ being *nonstandard*. In this case, for a given matrix $A \in \mathbb{Z}^{n\times n}$ it is a highly nontrivial problem to characterize all digit sets $\mathcal{D}$ for which $\mathcal{F}(A, \mathcal{D})$ has positive Lebesgue measure (cf. [2, 19]).

Rational self-affine tiles are introduced by the second and thirds authors in [26]. They constitute a natural generalization of integral self-affine tiles to rational matrices which no longer need to have an integer determinant. In particular, a rational self-affine tile is defined in terms of an expanding matrix in $\mathbb{Q}^{n\times n}$ with irreducible characteristic polynomial and a digit set taken from a $\mathbb{Z}$-module defined in terms of this matrix. In the present paper we extend the theory of rational self-affine tiles by taking arbitrary expanding rational matrices (this includes the "reducible case", as is referred to in [26]), and allowing nonstandard digit systems, as well as defining a representation space in

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a somewhat more general way. We provide results in the spirit of Lagarias and Wang [16, 17] for this setting.

Let $A \in \mathbb{Q}^{n\times n}$ be an expanding matrix and let

$$\mathbb{Z}^n[A] := \bigcup_{k=1}^{\infty} \left(\mathbb{Z}^n + A\mathbb{Z}^n + \cdots + A^{k-1}\mathbb{Z}^n\right)$$

be the smallest nontrivial A-invariant $\mathbb{Z}$-module containing $\mathbb{Z}^n$. We first define a *digit system* $(A,\mathcal{D})$ where A acts as a *base* and $\mathcal{D} \subset \mathbb{Z}^n[A]$ is some finite *digit set*, and explore its properties. In this digit system, the set $\mathbb{Z}^n[A]$ plays the role of $\mathbb{Z}^n$ in the sense that $(A,\mathcal{D})$ can be used to represent elements of $\mathbb{Z}^n[A]$. We will always assume that the digit set $\mathcal{D}$ satisfies $|\mathcal{D}| = |\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]|$, which turns out to be a natural size for a digit set (such digit systems are studied for instance in [10]). After that, in order to set up the definition of our tiles, we introduce a representation space of the form $\mathbb{K}_A := \mathbb{R}^n \times \mathbb{Z}^n((A^{-1}))$, where $\mathbb{Z}^n((A^{-1}))$ is a valuation ring of certain Laurent series of powers of A^{-1} with coefficients in $\mathbb{Z}^n$. The ring $\mathbb{Z}^n((A^{-1}))$ is a solenoid, and in the one-dimensional case it is isomorphic to the ring of b-adic numbers for some $b \in \mathbb{N}$ (which is even a field when b is a prime number). We establish a suitable "diagonal" embedding φ that maps the elements of $\mathbb{Z}^n[A]$ into $\mathbb{K}_A$ in a natural way. This allows us to define the main object of study of this paper: the *rational self-affine tile* $\mathcal{F} = \mathcal{F}(A,\mathcal{D}) \subset \mathbb{K}_A$, which arises as the unique nonempty compact solution of the set equation

$$A\mathcal{F} = \bigcup_{d\in\mathcal{D}} (\mathcal{F} + \varphi(d)),$$

and can be interpreted as the set of "fractional parts" of expansions in base A with digits in $\mathcal{D}$ (embedded in $\mathbb{K}_A$). Since we want to study tilings of $\mathbb{K}_A$ induced by $\mathcal{F}$, we require rational self-affine tiles to have positive measure, which always holds when $\mathcal{D}$ is a complete set of residue class representatives of $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$, and in this case we call $(A,\mathcal{D})$ a *standard digit system*. However, we also allow $(A,\mathcal{D})$ to be *nonstandard*, and give a criterion in terms of the digits to guarantee positive measure of $\mathcal{F}$ in this general setting. We prove some topological properties of rational self-affine tiles, as well as the existence of two tilings given by translations of $\mathcal{F}$: the first one has a translation set defined in terms of the digits $\mathcal{D}$, and the second one is a multiple tiling where the translation set is a lattice obtained from the ring $\mathbb{Z}^n[A]$ by embedding it into the space $\mathbb{K}_A$. Before arriving to the proof of the existence of the multiple tiling, we present a careful analysis of the character group of the locally compact abelian group $\mathbb{K}_A$. The article [24] deals with a special one-dimensional case of what we present in this paper. It is accessible to a general audience and serves as an introduction to the topic.

Our theory presents three main features that make its study difficult and rich. First of all, we treat the n-dimensional case: when dealing with matrices, computing quotients is not always so simple, and we make use of some machinery of linear algebra (like the Frobenius normal form) to solve some of these issues. Secondly, we deal with a space that has a A^{-1}-adic factor, and on it we define an ultrametric. More challenges arrive when we study the character group of the representation space. Finally, we consider nonstandard digit sets. This implies that sometimes we lose the group structure when considering certain digit expansions, which makes it harder to define tilings.

2. Setting and definitions

In this section we introduce digit sets and present a way to compute their cardinality. After that we define the representation space, and finally arrive at the definition of a rational self-affine tile.

What we are doing generalizes well-known facts on "ordinary" α-ary expansions. In particular, let α be an integer with $|\alpha| > 1$. Every real number has an expansion

$$\pm(d_m d_{m-1} \ldots d_0 . d_{-1} d_{-2} \ldots)_\alpha := \pm \sum_{j=-\infty}^{m} d_j \alpha^j$$

in base α, where every d_j is a *digit* taken from the set $\{0, 1, \dots, |\alpha| - 1\}$ (the sign "$\pm$" is required only for positive α). We can associate to this digit system the set of *integer expansions*, that is, the set of numbers that can be expressed using only nonnegative powers of the *base* α; it equals the lattice $\mathbb{Z}$. Besides that there is the set of *fractional parts*, namely, the numbers that can be expressed using only negative powers of the base α (excluding the added "$-$" sign at the beginning). This set is a compact interval given by $J = [0, 1]$ for $\alpha > 0$ and $J = \left[\frac{\alpha}{1-\alpha}, \frac{1}{1-\alpha}\right]$ for $\alpha < 0$. The collection $\{J + z \mid z \in \mathbb{Z}\}$ forms a tiling of $\mathbb{R}$, and this property geometrically reflects the fact that almost every real has a unique α-ary expansion. In our general setting, $\mathbb{Z}^n[A]$ will play the role of $\mathbb{Z}$, the rational self-affine tile $\mathcal{F}$ will play the role of $[0, 1]$, and $\mathbb{K}_A$ will play the role of $\mathbb{R}$.

2.1. **Digit systems with rational matrices.** Many generalizations of α-ary expansions (also known as *radix expansions* or *radix representations*) have been studied. Kempner [11] and later Rényi [22] proposed expansions with respect to nonintegral real bases. Knuth [13] introduced complex bases and related them to fractal sets. Tilings of $\mathbb{R}^n$ arising form radix expansions were studied by Vince [27], and Kovács [15] considered digit systems in finite dimensional Euclidean spaces. Akiyama et al. [1] introduced a type of expansion where the base is a rational number, and generalizations have been studied in [4] and [25]. In [26], the authors considered digit expansions where the base is an algebraic number, which is equivalent to taking expansions w.r.t. rational matrices with irreducible characteristic polynomial. The setting from the present paper is a generalization of the one in [26] for arbitrary expanding rational matrices.

Following Kovács [15], given an integer matrix $A \in \mathbb{Z}^{n\times n}$, we can consider expansions in base A with digits in a certain set $\mathcal{D} \subset \mathbb{Z}^n$, meaning we look at ways of expanding a vector $x \in \mathbb{R}^n$ in the form

$$x = \sum_{j=-\infty}^{k} A^j d_j, \quad d_j \in \mathcal{D}, \tag{2.1}$$

and this is linked to the study of integral self-affine tiles that we have mentioned before (see [16]). We need the assumption that A is expanding in order for this series to converge.

If we consider a rational matrix $A \in \mathbb{Q}^{n\times n}$, then $\mathbb{Z}^n[A]$ is the natural generalization of $\mathbb{Z}^n$ (we refer to [10] for more on rational matrix digit systems). This leads to the following definition.

Definition 2.1 (Digit system)**.** Let $A \in \mathbb{Q}^{n\times n}$ be an expanding matrix and let $\mathcal{D} \subset \mathbb{Z}^n[A]$ be such that $|\mathcal{D}| = |\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]|$. Then we say that $(A, \mathcal{D})$ constitutes a *digit system*, where A is the *base* and $\mathcal{D}$ is the *digit set*. When $\mathcal{D}$ is a complete set of residue class representatives of $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$, we say that $(A, \mathcal{D})$ is a *standard digit system* (following [16, p. 163]). Otherwise, we say that $(A, \mathcal{D})$ is a *nonstandard digit system.*

In connection with digit systems, finite expansions are desirable. For digit systems $(A, \mathcal{D})$, the *finiteness property*, stating that every vector $x \in \mathbb{Z}^n[A]$ has a finite expansion of the form $x = A^k d_k + \cdots + A d_1 + d_0$ has been studied extensively (see [10] and the references given there). It is easily seen that the requirement that $\mathcal{D}$ is a complete system of residue class representatives of $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$ is a necessary, but in general not sufficient, condition for $(A, \mathcal{D})$ to have the finiteness property. Eventually periodic expansions have also been investigated.

Computing the size of a digit set for a given expanding matrix $A \in \mathbb{Q}^{n\times n}$ amounts to computing the order of the quotient group $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$, and this is not always straightforward. From here onwards, set

$$a := |\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]|, \quad b := |\mathbb{Z}^n[A^{-1}]/A^{-1}\mathbb{Z}^n[A^{-1}]|. \tag{2.2}$$

We now show how to make use of the Frobenius normal form of A to compute a and b, and we prove that $|\det A| = \frac{a}{b}$, which will be crucial later. Let $A \in \mathbb{Q}^{n\times n}$ with characteristic polynomial χ_A be given. Consider the space $\mathbb{Q}^n$ regarded as a finitely generated $\mathbb{Q}[t]$-module with the action of t given by multiplication by A, that is, if $v \in \mathbb{Q}^n$ then $t \cdot v := Av$, and the action can be linearly extended to all elements in $\mathbb{Q}[t]$. According to the structure theorem for finitely generated modules over principal ideal domains (see [5, Chapter 12, Theorem 6]), there exists an isomorphism of the form

$$\mathbb{Q}^n \simeq \bigoplus_{i=1}^{k} \mathbb{Q}[t]/(p_i),$$

where $p_i \in \mathbb{Q}[t]$ are the so-called *invariant factors* of $\mathbb{Q}^n$, with the divisibility properties $p_1 \mid p_2 \mid \ldots \mid p_k \mid \chi_A$. The polynomials p_i are assumed to be monic, and with this assumption they are unique. This implies that A is similar to a block diagonal matrix $F = \operatorname{diag}(C_1, \ldots C_k)$, where C_i is the companion matrix of p_i $(1 \leqslant i \leqslant k)$. F is the well-known *Frobenius normal from* of A, also called *rational canonical form*, see [5, Section 12.2]. Using this notation, we get the following result which shows how to compute the value of a.

Proposition 2.2. *Let $A \in \mathbb{Q}^{n\times n}$ be given, let p_i $(1 \leqslant i \leqslant k)$ be the corresponding invariant factors, and consider the integer polynomials $q_i = c_i p_i \in \mathbb{Z}[t]$, where each $c_i \in \mathbb{Z}$ is chosen so that q_i has coprime coefficients. Let $q_i^* \in \mathbb{Z}[t]$ be the reciprocal polynomial of q_i, namely $q_i^*(t) := t^{\deg(q_i)} q_i(t^{-1})$. Then*

$$a = \prod_{i=1}^{k} |q_i(0)|, \quad b = \prod_{i=1}^{k} |q_i^*(0)|, \tag{2.3}$$

and $|\det A| = \frac{a}{b}$.

Proof. Let $F = \operatorname{diag}(C_1, \ldots, C_k)$ be the Frobenius normal form of A. Then it is clear that

$$a = \prod_{i=1}^{k} |\mathbb{Z}^{\deg(q_i)}[C_i]/C_i\mathbb{Z}^{\deg(q_i)}[C_i]|. \tag{2.4}$$

Let $C \in \mathbb{Q}^{m\times m}$ be the companion matrix of some polynomial $p \in \mathbb{Q}[t]$ and let $q = cp \in \mathbb{Z}[t]$, where $c \in \mathbb{Z}$ is chosen so that q has coprime coefficients. We claim that

$$\mathbb{Z}^m[C] \simeq \mathbb{Z}[t]/(q). \tag{2.5}$$

To prove this, let $v \in \mathbb{Z}^m[C]$ be given; then it can be expressed as $v = \sum_{j=0}^{L} C^j v_j$ with $v_j \in \mathbb{Z}^m$, $L \geqslant 0$. For $1 \leqslant j \leqslant m$, denote by $\mathbf{e}_j \in \mathbb{Q}^m$ the j-th canonical basis vector. Clearly, each v_j can be expressed in the canonical basis with integer coefficients. Because C is a companion matrix, it follows that $C\mathbf{e}_j = \mathbf{e}_{j+1}$ for $1 \leqslant j < m$, so all this yields that v is of the form

$$v = \sum_{j=0}^{\ell} b_j C^j \mathbf{e}_1, \tag{2.6}$$

for some $\ell \in \mathbb{N}$ minimal and $b_0, \ldots, b_\ell \in \mathbb{Z}$. If $\ell \geqslant m$, we will show that we can take $b_m, \ldots, b_\ell \in \{0, \ldots, |q^*(0)| - 1\}$ (here, $q^* \in \mathbb{Z}[t]$ is the reciprocal polynomial of q, so $q^*(0)$ corresponds to the leading coefficient of q). Indeed, note that $q(C) = 0$ because C is the companion matrix of q. Suppose that b_ℓ is not in $\{0, \ldots, |q^*(0)| - 1\}$; then one can add or subtract $C^{\ell-m}q(C) = 0$ in order to obtain another expression on the right side of (2.6), without altering the value of v. This can be done the appropriate number of times, so we can assume w.l.o.g. that $b_\ell \in \{0, \ldots, |q^*(0)| - 1\}$. Repeating this for $\ell - 1, \ell - 2, \ldots, m$, one arrives at $b_m, \ldots, b_\ell \in \{0, \ldots, |q^*(0)| - 1\}$, and the representation (2.6) with this property and ℓ minimal is unique.

By [25, Lemma 4.1] each polynomial $r \in \mathbb{Z}[t]/(q)$ can be expressed uniquely as $r = r' + \sum_{j=m}^{\ell} r_j t^j \mod q$, with $r' \in \mathbb{Z}[x]$, $\deg(r') < m$, $\ell \in \mathbb{N}$ and $r_m, \ldots, r_\ell \in \{0, \ldots, |q^*(0)| - 1\}$. Using this, one easily checks that

$$h : \mathbb{Z}^m[C] \to \mathbb{Z}[t]/(q); \quad \sum_{i=0}^{\ell} b_i C^i \mathbf{e}_1 \mapsto \sum_{i=0}^{\ell} b_i t^i$$

is an isomorphism and the claim in (2.5) is proved.

Because $t\,h(G) = h(CG)$ for any $G \in \mathbb{Z}^m[C]$, this isomorphism implies that

$$\mathbb{Z}^m[C]/C\mathbb{Z}^m[C] \simeq \mathbb{Z}[t]/(q, t).$$

It is easy to check that $|\mathbb{Z}[t]/(q,t)| = |q(0)|$ (see [25, p. 1460]) and, hence, we have that

$$|\mathbb{Z}^m[C]/C\mathbb{Z}^m[C]| = |q(0)|. \tag{2.7}$$

Applying (2.7) for $q = q_i$ $(1 \leqslant i \leqslant k)$ in (2.4), the left equation of (2.3) follows. The right equation of (2.3) is proved in the same way by replacing A by A^{-1}. The assertion $\det A = \frac{a}{b}$ follows from (2.3) (recall the definition of q_i and the fact that each p_i is monic), because

$$|\det A| = \prod_{i=1}^{k} |\det C_i| = \prod_{i=1}^{k} |p_i(0)| = \prod_{i=1}^{k} \frac{|q_i(0)|}{|q_i^*(0)|} = \frac{a}{b}. \qquad \square$$

2.2. **The representation space.** Properties of digit systems and digit expansions can be reflected geometrically via self-affine sets and tilings. The space $\mathbb{K}_A$, defined in what follows for a given $A \in \mathbb{Q}^{n\times n}$, will turn out to be a natural space where these sets and tilings can be defined. Suppose that we wanted to define a set $\mathcal{F}(A,\mathcal{D}) \subset \mathbb{R}^n$ satisfying (1.1). In order to define a tiling, we would require the union on the right side of (1.1) to be essentially disjoint. We know that the action of A scales the Lebesgue measure of a set by $|\det A|$, so if A has a nonintegral determinant and $\mathcal{F}$ has positive measure, we would need $\mathcal{D}$ to have a nonintegral amount of digits, which is of course not doable. We will show that the action of A in $\mathbb{K}_A$ multiplies the Haar measure of a set by a, where $a \in \mathbb{N}$ is as in (2.2).

Standard digit systems $(A,\mathcal{D})$ where the characteristic polynomial χ_A of $A \in \mathbb{Q}^{n\times n}$ is irreducible are considered in [26]. Rational self-affine tiles are introduced as subsets of a representation space of the form $\mathbb{R}^n \times \prod_{\mathfrak{p}} K_{\mathfrak{p}}$, where each $K_{\mathfrak{p}}$ is a completion of a number field K (defined in terms of χ_A) with respect to a certain absolute value $|\cdot|_{\mathfrak{p}}$. The representation space $\mathbb{K}_A$ from the present paper is a generalization of this, and can be defined in a simpler way. In the irreducible case, both settings are isomorphic.

Let $A \in \mathbb{Q}^{n\times n}$ be expanding. For convenience of notation, from here onwards we set

$$B := A^{-1}.$$

Consider the ring

$$\mathbb{Z}^n(B) = \bigcup_{k\geqslant 1} (B^{-k}\mathbb{Z}^n + B^{-k+1}\mathbb{Z}^n + \cdots + B^k\mathbb{Z}^n)$$

(note that $\mathbb{Z}^n(A) = \mathbb{Z}^n(B)$). We define on $\mathbb{Z}^n(B)$ the *B-adic valuation* $\nu : \mathbb{Z}^n(B) \to \mathbb{Z}\cup\{\infty\}$ as

$$\nu(y) := \begin{cases} \min\{k \in \mathbb{Z} \mid y \in B^k\mathbb{Z}^n[B] \setminus B^{k+1}\mathbb{Z}^n[B]\}, & y \neq 0, \\ \infty, & y = 0. \end{cases} \tag{2.8}$$

On $\mathbb{Z}^n(B)$ the *B-adic metric* is defined by

$$\mathbf{d}_B(y,y') := b^{-\nu(y-y')}, \tag{2.9}$$

for b as in (2.2) and $y, y' \in \mathbb{Z}^n(B)$, with the convention that $b^{-\infty} = 0$.

Definition 2.3 (B-adic series)**.** We define the space $\mathbb{Z}^n((B))$ of *B-adic series* as the completion of $\mathbb{Z}^n(B)$ with respect to the metric $\mathbf{d}_B$.

We extend the metric $\mathbf{d}_B$ to the completion $\mathbb{Z}^n((B))$, and hence we extend the B-adic valuation ν to $\mathbb{Z}^n((B))$ so that it satisfies (2.9). Then, every nonzero $y \in \mathbb{Z}^n((B))$ can be expressed as Laurent series

$$y = \sum_{j=\nu(y)}^{\infty} B^j y_j, \quad y_j \in \mathbb{Z}^n, \tag{2.10}$$

of powers of B with coefficients in $\mathbb{Z}^n$, which converges w.r.t. the metric $\mathbf{d}_B$. Then $\nu(y)$ is the smallest index such that y has an expansion (2.10) with $y_{\nu(y)} \neq 0$. We denote by $\mathbb{Z}^n[[B]]$ the subring of $\mathbb{Z}^n((B))$ consisting of points $y \in \mathbb{Z}^n((B))$ with $\nu(y) \geqslant 0$, the ring of power series in B with coefficients in $\mathbb{Z}^n$. The B-adic metric satisfies the *ultrametric inequality*, namely

$$\mathbf{d}_B(y,y') \leqslant \max\{\mathbf{d}_B(y,y''), \mathbf{d}_B(y'',y')\}$$

for every $y, y', y'' \in \mathbb{Z}^n((B))$. This metric turns $\mathbb{Z}^n((B))$ into a complete separable space, which is also a locally compact topological group. Thus there is a Haar measure μ_B on $\mathbb{Z}^n((B))$ which is normalized in a way that $\mu_B(\mathbb{Z}^n[[B]]) = 1$, and we call it the *$B$-adic measure*. If $M \subset \mathbb{Z}^n((B))$ is a measurable set, then

$$\mu_B(A^k M) = b^k \mu_B(M). \tag{2.11}$$

Remark 2.4. Let $n = 1$. In this case, $A = \frac{a}{b}$ where a and b are coprime integers. Suppose $b \geqslant 2$. We obtain that $\mathbb{Z}^n((B)) \simeq \mathbb{Q}_b$, where $\mathbb{Q}_b$ is the ring of b-adic numbers, and $\mathbb{Z}^n[[B]] \simeq \mathbb{Z}_b$, where $\mathbb{Z}_b$ is the ring of b-adic integers.

Definition 2.5 (The representation space)**.** Given an expanding matrix $A \in \mathbb{Q}^{n\times n}$, define the *representation space* $\mathbb{K}_A$ as

$$\mathbb{K}_A := \mathbb{R}^n \times \mathbb{Z}^n((B)).$$

We endow the space $\mathbb{K}_A$ with the following structures:

(1) It inherits the structure of an additive group from its cartesian factors.
(2) Consider the group of matrices given by

$$\mathbb{Z}[A] := \bigcup_{k\geqslant 1} (\mathbb{Z}A^{-k} + \mathbb{Z}A^{-k+1} + \cdots + \mathbb{Z}A^k).$$

Then $\mathbb{Z}[A]$ acts on $\mathbb{K}_A$ by multiplication, *i.e.*, $G\,(x, y) = (Gx, Gy)$ if $G \in \mathbb{Z}[A]$ and $(x, y) \in \mathbb{K}_A$.
(3) We define the metric

$$\mathbf{d}((x, y), (x', y')) := \max\{\|x - x'\|, \mathbf{d}_B(y, y')\},$$

for $(x, y), (x', y') \in \mathbb{K}_A$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\mathbf{d}_B$ is the B-adic metric in $\mathbb{Z}^n((B))$. This turns $\mathbb{K}_A$ into a locally compact topological group. It is easy to check that, for every closed ball $\mathbf{B}_r(x, y)$ of radius $r > 0$ and center $(x, y) \in \mathbb{K}_A$, there is a decomposition

$$\mathbf{B}_r(x, y) = \mathbf{B}_r(x) \times \mathbf{B}_r(y),$$

into closed balls on each respective space. This characterizes the topology of $\mathbb{K}_A$.
(4) We define a measure μ in $\mathbb{K}_A$ as the product measure

$$\mu := \lambda \times \mu_B,$$

with λ being the Lebesgue measure in $\mathbb{R}^n$ and μ_B the B-adic measure in $\mathbb{Z}^n((B))$. Then μ is the Haar measure on $\mathbb{K}_A$ satisfying $\mu([0, 1] \times \mathbb{Z}^n[[B]]) = 1$.

Remark 2.6. Note that, when $A \in \mathbb{Z}^{n\times n}$ is an integer matrix, the space $\mathbb{Z}^n((B))$ is trivial and plays no role, and hence $\mathbb{K}_A = \mathbb{R}^n$. However, $\mathbb{K}_A = \mathbb{R}^n$ may also happen in the noninteger case: for example, if $A = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 3 \end{pmatrix}$ we have $b = 1$. Suppose $b = 1$, *i.e.*, that $\mathbb{Z}^n[B]/B\mathbb{Z}^n[B]$ is trivial. Then by Lemma 2.2 $\det A = a$ is an integer. The results presented in Section 3 are proven by Lagarias and Wang in [17] for real expanding matrices with integer determinant. For this reason, in all that follows we assume $b \geqslant 2$.

Lemma 2.7. *If $M \subset \mathbb{K}_A$ is a measurable set, then $\mu(AM) = a\,\mu(M)$.*

Proof. Consider a measurable subset of $\mathbb{K}_A$ of the form $M_1 \times M_2$, where $M_1 \subset \mathbb{R}^n$ and $M_2 \subset \mathbb{Z}^n((B))$ are both measurable sets. We have seen in Proposition 2.2 that $\det A = \frac{a}{b}$. Then

$$\mu(A(M_1 \times M_2)) = \lambda(AM_1)\,\mu_B(AM_2) = \frac{a}{b}\,\lambda(M_1)\,b\,\mu_B(M_2) = a\,\mu(M_1 \times M_2).$$

Since $\mu = \lambda \times \mu_B$ is a product measure, the σ-algebra of μ-measurable sets is generated by sets of the form $M_1 \times M_2$. Therefore, if $M \subset \mathbb{K}_A$ is measurable we have $\mu(AM) = a\,\mu(M)$. ☐

The previous lemma implies that, in some sense, the base A has "enough space" for a digits when the digit system is embedded in $\mathbb{K}_A$. In fact, when considering a rational matrix A with integer determinant, it suffices to take $|\mathcal{D}| = |\det A|$, because A acts in some sense like an integer matrix: the action of A scales the measure of a set by an integer factor. When A has nonintegral determinant, it turns turns out that A acts in $\mathbb{K}_A$ like an integer matrix, and that is why this space is appropriate for our purposes. This relation between the measure and the cardinality of the digit set will be important in all that follows.

2.3. **Rational self-affine tiles.** We proceed to introduce a set $\mathcal{F}$ associated to the digit system $(A,\mathcal{D})$ that reflects features of its structure. It can be regarded as the set of "fractional parts" of the digit system, in the sense that it plays the same role as the interval $[0,1]$ does for the decimal digit system. In Section 3 we will study topological properties of $\mathcal{F}$.

We need the following lemma, which is in the spirit of Lind [20].

Lemma 2.8. *Let $A \in \mathbb{Q}^{n\times n}$ be expanding and assume $b \geqslant 2$, with b as in* (2.2)*. Then there exists a metric $\boldsymbol{\ell}$ on $\mathbb{K}_A$ w.r.t. which the action of $B = A^{-1}$ is a contraction. In particular, there exists $0 \leqslant \kappa < 1$ such that*

$$\boldsymbol{\ell}(B\,(x,y), B\,(x',y')) \leqslant \kappa\,\boldsymbol{\ell}((x,y),(x',y')) \quad ((x,y),(x',y') \in \mathbb{K}_A). \tag{2.12}$$

Proof. Let $Spec(A)$ denote the set of eigenvalues of A. Since A is expanding, there exists $\rho \in \mathbb{R}$ such that $1 < \rho < \min\{|\eta| \mid \eta \in Spec(A)\}$. For $x \in \mathbb{R}^n$, define

$$\|x\|' := \sum_{k=0}^{\infty} \rho^k\,\|B^k x\|. \tag{2.13}$$

Since all the eigenvalues of $\rho\,B = \rho A^{-1}$ are strictly smaller than 1 in modulus, the series on the right hand side of (2.13) converges and $\|\cdot\|'$ becomes a norm in $\mathbb{R}^n$ that satisfies

$$\|B\,x\|' = \frac{1}{\rho}\sum_{k=1}^{\infty} \rho^k\,\|B^k\,x\| \leqslant \frac{1}{\rho}\,\|x\|'.$$

Also, for all $y,y' \in \mathbb{Z}^n((B))$, it follows from the definition of the B-adic metric that $\mathbf{d}_B(By,By') = \frac{1}{b}\,\mathbf{d}_B(y,y')$. Let $(x,y),(x',y') \in \mathbb{K}_A$. Define on $\mathbb{K}_A$ the metric $\boldsymbol{\ell}$ given by

$$\boldsymbol{\ell}((x,y),(x',y')) := \max\{\|x-x'\|', \mathbf{d}_B(y,y')\}.$$

Then (2.12) follows with $\kappa := \max\{\frac{1}{\rho},\frac{1}{b}\} < 1$. □

Note that $\mathbf{d}$ and $\boldsymbol{\ell}$ are equivalent, because $\|\cdot\|'$ is equivalent to $\|\cdot\|$.

We now introduce a suitable way to embed our digit system into the representation space $\mathbb{K}_A$. Define the *diagonal embedding* φ as

$$\varphi : \mathbb{Z}^n(B) \to \mathbb{K}_A; \quad x \mapsto (x,x).$$

Definition 2.9 (Rational self-affine tile)**.** Let $(A,\mathcal{D})$ be a digit system. Define $\mathcal{F} = \mathcal{F}(A,\mathcal{D}) \subset \mathbb{K}_A$ as the unique nonempty compact set satisfying the set equation

$$A\mathcal{F} = \bigcup_{d\in\mathcal{D}} (\mathcal{F} + \varphi(d)). \tag{2.14}$$

If $\mu(\mathcal{F}) > 0$, then $\mathcal{F}$ is called a *rational self-affine tile.*

Because A is expanding, Lemma 2.8 implies that the mapping

$$\mathbb{K}_A \to \mathbb{K}_A; \quad (x,y) \mapsto A^{-1}((x,y)+\varphi(d))$$

is a contraction for each $d \in \mathcal{D}$. Let $\mathcal{H}(K)$ be the family of nonempty compact subsets of $\mathbb{K}_A$, and consider the map

$$\Psi : \mathcal{H}(K) \to \mathcal{H}(K); \quad X \mapsto \bigcup_{d\in\mathcal{D}} A^{-1}(X + \varphi(d)). \tag{2.15}$$

By Hutchinson [9], there is a unique nonempty compact set which is a fixed point of Ψ, hence $\mathcal{F}$ is well defined. This set is the attractor of an iterated function system, meaning that it is the Hausdorff limit of the sequence of compact sets $\{\Psi^k(X)\}_{k\geq 1}$, for any compact set X.

The set $\mathcal{F}$ can be interpreted as the set of "fractional parts" of the digit system $(A,\mathcal{D})$ embedded in $\mathbb{K}_A$; that is, every point of $\mathcal{F}$ can be expressed in base A with digits in $\varphi(\mathcal{D})$ using only negative powers of the base. In fact, $\mathcal{F}$ is given explicitly by

$$\mathcal{F} = \Big\{ \sum_{j=1}^{\infty} A^{-j}\varphi(d_j) \mid d_j \in \mathcal{D} \Big\}. \tag{2.16}$$

Indeed, it is easy to see that $\mathcal{F}$ is nonempty, bounded, and satisfies (2.14). The fact that $\mathcal{F}$ is closed follows by a Cantor diagonal argument.

Note that $\mathcal{F}$ is *self-affine* in the sense that it can be written as the union of $a = |\mathcal{D}|$ contracted affine copies of itself, because (2.14) is equivalent to $\mathcal{F} = A^{-1}(\mathcal{F} + \varphi(\mathcal{D}))$. When $\mathcal{F}$ has zero measure, it is a (generalization of a) Cantor set.

Suppose $\mathcal{F}$ has positive measure. In order to define a tiling, we want the union $\bigcup_{d\in\mathcal{D}}(\mathcal{F}+\varphi(d))$ to be essentially disjoint (that is, disjoint up to a μ-measure zero set); since multiplication by A on $\mathbb{K}_A$ enlarges the measure by a factor of a, then it is necessary for $\mathcal{D}$ to have exactly a elements. We will show in the next section that, if $(A,\mathcal{D})$ is a standard digit system, then $\mathcal{F}(A,\mathcal{D})$ has positive measure.

Remark 2.10. W.l.o.g. we will always assume that $0 \in \mathcal{D}$. This can be done because replacing $\mathcal{D}$ by $\mathcal{D} - v$, where $v \in \mathbb{Z}^n[A]$ is a constant vector, means that $\mathcal{F}(A, \mathcal{D} - v)$ is a translation of $\mathcal{F}(A,\mathcal{D})$, hence it is equivalent when we study the existence of tilings.

3. Results on rational self-affine tiles

Let $(A,\mathcal{D})$ be a digit system and let $\mathbb{K}_A$ be the representation space with metric $\mathbf{d}$ and Haar measure μ as before. In this section, we give some equivalent topological and combinatorial conditions for the set $\mathcal{F}(A,\mathcal{D})$ to have positive measure and we prove that, whenever $\mathcal{F}(A,\mathcal{D})$ is a rational self-affine tile, it induces a tiling. We also study some topological properties and present an example.

3.1. Properties of rational self-affine tiles and a tiling theorem.

We introduce some definitions before stating the results. To denote blocks of digits, let

$$\mathcal{D}_k := \big\{ d_0 + Ad_1 + \cdots + A^{k-1}d_{k-1} \mid d_0,\ldots,d_{k-1} \in \mathcal{D} \big\} \quad \text{and} \quad \mathcal{D}_\infty := \bigcup_{k\geqslant 1} \mathcal{D}_k. \tag{3.1}$$

From the set equation (2.14), we deduce the *iterated set equation*

$$A^k\mathcal{F} = \bigcup_{d\in\mathcal{D}_k} (\mathcal{F} + \varphi(d)). \tag{3.2}$$

Definition 3.1 (Uniform discreteness)**.** We say that a set $M \subset \mathbb{K}_A$ is *uniformly discrete* if there exists $r > 0$ such that every open ball of radius r in $\mathbb{K}_A$ contains at most one point of M.

Our first result is a criterion for $\mathcal{F}(A,\mathcal{D})$ to have positive measure formulated in terms of $\mathcal{D}$. It is an extension of [17, Theorem 1.1] and [12, Theorem 10] to the case of rational self-affine tiles.

Theorem 3.2. *Let $(A,\mathcal{D})$ be a digit system, and let $\mathcal{F} = \mathcal{F}(A,\mathcal{D}) \subset \mathbb{K}_A$. Then $\mathcal{F}$ has positive measure if and only if for every $k \geqslant 1$, all a^k expansions in $\mathcal{D}_k$ are distinct, and $\varphi(\mathcal{D}_\infty)$ is a uniformly discrete subset of $\mathbb{K}_A$.*

Proof. We omit some details of the proof, since the one in [17, p. 32 – 34] is similar, although it is provided in the setting of self-affine tiles in $\mathbb{R}^n$.

Assume first that $\varphi(\mathcal{D}_\infty)$ is a uniformly discrete set and that all the elements in $\mathcal{D}_k$ are distinct for every $k \geqslant 1$. Recall the metric $\boldsymbol{\ell}$ and the constant $0 \leqslant \kappa < 1$ defined in Lemma 2.8. Then $\boldsymbol{\ell}(B(x,y),0) \leqslant \kappa\,\boldsymbol{\ell}((x,y),0)$ for every $(x,y) \in \mathbb{K}_A$. Consider the closed ball $\mathbf{B}'_r(0) := \{(x,y) \in \mathbb{K}_A \mid \boldsymbol{\ell}((x,y),0) \leqslant r\}$, and let $(x,y) \in \mathbf{B}'_r(0)$. Let Ψ be the map defined in (2.15);

then by Hutchinson [9], $\mathcal{F}$ is the Hausdorff limit of the sequence $\{\Psi^k(\mathbf{B}'_r(0))\}_{k\geqslant 1}$. Suppose that $r \geqslant \frac{\kappa}{1-\kappa}\max_{d\in\mathcal{D}}\{\boldsymbol{\ell}(\varphi(d),0)\}$; then $\Psi(\mathbf{B}'_r(0)) \subset \mathbf{B}'_r(0)$. Consequently, by Lebesgue's dominated convergence theorem, we get

$$\mu(\mathcal{F}) = \lim_{k\to\infty}\mu(\Psi^k(\mathbf{B}'_r(0))).$$

It suffices to find a set of positive measure contained in every $\Psi^k(\mathbf{B}'_r(0))$. Since $\varphi(\mathcal{D}_\infty)$ is uniformly discrete, there exists $\delta > 0$ such that for every $(x,y) \neq (x',y')$ in $\varphi(\mathcal{D}_\infty)$ it holds that $\boldsymbol{\ell}((x,y),(x',y')) > \delta$. For $0 < \varepsilon < \min\{\frac{\delta}{2}, r\}$, consider the closed ball $\mathbf{B}'_\varepsilon(0)$. By hypothesis, $\mathcal{D}_k$ has a^k distinct elements for all k, and all sets of the form $\mathbf{B}'_\varepsilon(0) + \varphi(d)$ for $d \in \mathcal{D}_k$ are pairwise disjoint. Therefore $\mu(\Psi^k(\mathbf{B}'_\varepsilon(0))) = \mu(\mathbf{B}'_\varepsilon(0))$ for every k, and hence

$$\mu(\mathcal{F}) \geqslant \lim_{k\to\infty}\mu(\Psi^k(\mathbf{B}'_\varepsilon(0))) = \mu(\mathbf{B}'_\varepsilon(0)) > 0.$$

For the converse, assume $\mu(\mathcal{F}) > 0$. Then

$$a^k\mu(\mathcal{F}) = \mu(A^k\mathcal{F}) = \mu\Big(\bigcup_{d\in\mathcal{D}_k}(\mathcal{F}+\varphi(d))\Big) \leqslant \sum_{d\in\mathcal{D}_k}\mu(\mathcal{F}+\varphi(d)) \leqslant a^k\mu(\mathcal{F}),$$

so all the terms are equal. Hence, $|\mathcal{D}_k| = a^k$ and the union is essentially disjoint, meaning that for $d \neq d'$ in $\mathcal{D}_k$, we get

$$\mu((\mathcal{F}+\varphi(d)) \cap (\mathcal{F}+\varphi(d'))) = 0. \tag{3.3}$$

It remains to show that $\varphi(\mathcal{D}_\infty)$ is uniformly discrete. Suppose this was not the case, then we can find a sequence $\{(d_l, d'_l)\}_{l\geqslant 1}$ where, d_l and d'_l are distinct elements of some $\mathcal{D}_{k_l}$ for each $l \geqslant 1$, and such that

$$\lim_{l\to\infty}\mathbf{d}(\varphi(d_l),\varphi(d'_l)) = 0.$$

If $\mu(\mathcal{F}) > 0$, then by Federer [7, page 156, Corollary 2.9.9], there exists a Lebesgue point $(x^*,y^*) \in \mathcal{F}$. Given $\varepsilon > 0$, this implies the existence of a sufficiently small r for which

$$\mu(\mathbf{B}_r(x^*,y^*) \cap \mathcal{F}) \geqslant (1-\varepsilon)\,\mu(\mathbf{B}_r(x^*,y^*)). \tag{3.4}$$

Here, $\mathbf{B}_r(x^*,y^*)$ denotes the closed ball of center (x^*,y^*) and radius r w.r.t. the metric $\mathbf{d}$. Let $0 < \varepsilon' < r$, and consider $(x,y) \in \mathbb{K}_A$ such that $\mathbf{d}((x,y),0) < \varepsilon' < r$. Then

$$\begin{aligned}\mu(\mathbf{B}_r(x^*,y^*) \cap (\mathcal{F}+(x,y))) &\geqslant \mu(\mathbf{B}_{r-\varepsilon'}((x^*,y^*)+(x,y)) \cap (\mathcal{F}+(x,y)))\\ &\geqslant (1-\varepsilon)\mu(\mathbf{B}_{r-\varepsilon'}(x^*,y^*)).\end{aligned} \tag{3.5}$$

Recall that b satisfies (2.11). Note that $\mathbf{B}_r(y^*) = \mathbf{B}_{b^{\lfloor\log_b r\rfloor}}(y^*)$, and define $K := \lfloor\log_b(r)\rfloor - \lfloor\log_b(r-\varepsilon')\rfloor > 0$. Then $y \in \mathbf{B}_r(y^*)$ if and only if $B^K y \in \mathbf{B}_{r-\varepsilon'}(B^K y^*)$. Hence,

$$\mu(\mathbf{B}_{r-\varepsilon'}(x^*,y^*)) = \lambda\Big(\frac{r-\varepsilon'}{r}\mathbf{B}_r(x^*)\Big)\,\mu_B(B^K\mathbf{B}_r(y^*)) = \Big(\frac{r-\varepsilon'}{r}\Big)^n b^{-K}\mu(\mathbf{B}_r(x^*,y^*)),$$

and thus for the appropriate value of $\varepsilon'' > 0$ it follows from (3.5) that

$$\mu(\mathbf{B}_r(x^*,y^*) \cap (\mathcal{F}+(x,y))) > (1-\varepsilon'')\mu(\mathbf{B}_r(x^*,y^*)). \tag{3.6}$$

By inclusion-exclusion and combining (3.4) and (3.6), we get

$$\mu((\mathcal{F}+(x,y)) \cap \mathcal{F}) > (1-\varepsilon-\varepsilon'')\mu(\mathbf{B}_r(x^*,y^*) > 0$$

for (x,y) sufficiently close to 0. This implies that, for large enough l,

$$\mu((\mathcal{F}+\varphi(d_l)) \cap (\mathcal{F}+\varphi(d'_l)) = \mu(\mathcal{F}) \cap (\mathcal{F}+\varphi(d'_l-d_l)) > 0,$$

which is a contradiction. □

Corollary 3.3. *If $(A,\mathcal{D})$ is a standard digit system, then $\mathcal{F}(A,\mathcal{D})$ has positive measure.*

The second result of this section gives some topological equivalences for $\mathcal{F}$ being a rational self-affine tile. It is in the spirit of [17, Theorem 1.1].

Theorem 3.4. *Let $(A,\mathcal{D})$ be a digit system and let $\mathcal{F} = \mathcal{F}(A,\mathcal{D}) \subset \mathbb{K}_A$. The following assertions are equivalent:*

(*i*) $\mathcal{F}$ *has positive measure.*
(*ii*) $\mathcal{F}$ *has nonempty interior.*
(*iii*) $\mathcal{F}$ *is the closure of its interior, and its boundary* $\partial\mathcal{F}$ *has measure zero.*

Proof. $(iii) \Rightarrow (ii) \Rightarrow (i)$ is trivial.

$(ii) \Rightarrow (iii)$ is analogous to the proof found in [17, p. 30 – 32].

$(i) \Rightarrow (ii)$: Because $\mathcal{F}$ has positive measure, it has a Lebesgue point (x^*, y^*) satisfying (3.4). This implies that we can consider a sequence $\varepsilon_k \searrow 0$ together with a sequence of radii $r_k \searrow 0$ such that, for every $l > 0$,

$$\mu(A^l(\mathbf{B}_{r_k}(x^*, y^*) \cap \mathcal{F})) \geqslant (1 - \varepsilon_k)\, \mu(A^l \mathbf{B}_{r_k}(x^*, y^*)). \tag{3.7}$$

Claim 1: *For every index k there exists a large enough $l_k > 0$ and $(u^{(k)}, v^{(k)}) \in \mathbb{K}_A$ such that $\mathbf{B}_1(u^{(k)}, v^{(k)}) \subset A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)$ with*

$$\mu(\mathbf{B}_1(u^{(k)}, v^{(k)}) \cap A^{l_k}\mathcal{F}) \geqslant (1 - C\,\varepsilon_k)\, \mu(\mathbf{B}_1(u^{(k)}, v^{(k)})),$$

where (x^*, y^*) is a Lebesgue point of $\mathcal{F}$ and $C > 0$ is a constant depending only on the space $\mathbb{K}_A$.

To prove the claim, we draw on ideas from [17, p. 35]. With a slight abuse of notation, we will use $\mathbf{B}_r(\cdot)$ to denote closed balls of radius r in each respective space. Fix k, and note that, since A is expanding, $A^l\mathbf{B}_{r_k}(x^*) \subset \mathbb{R}^n$ is an ellipsoid whose shortest axis' length goes to infinity as l goes to infinity. Consider $l_k > 0$ large enough so that $b^{-l_k} \leqslant r_k$, and such that the ellipsoid $A^{l_k}\mathbf{B}_{r_k}(x^*)$ has a shortest axis greater than 4. Define $E_k := \{x \in A^{l_k}\mathbf{B}_{r_k}(x^*) \mid d(x, \partial(A^{l_k}\mathbf{B}_{r_k}(x^*))) \geqslant 1\}$ as the set of points of $A^{l_k}\mathbf{B}_{r_k}(x^*)$ whose (Euclidean) distance from the boundary is at least 1. Consider the set $2E_k - x^*$, obtained by doubling E_k and centering around x^*. Then $E_k \subsetneq A^{l_k}\mathbf{B}_{r_k}(x^*) \subsetneq 2E_k - x^*$ and $\lambda(A^{l_k}\mathbf{B}_{r_k}(x^*)) \leqslant \lambda(2E_k - x^*) = 2^n\lambda(E_k)$.

Consider the compact subset of $\mathbb{K}_A$ given by $\mathcal{U}_k := E_k \times A^{l_k}\mathbf{B}_{r_k}(y^*)$, which is trivially covered by the collection of unit balls $\mathcal{G} := \{\mathbf{B}_1(u, v) \mid (u, v) \in \mathcal{U}_k\}$. Let $\{\mathbf{B}_1(u_1, v_1), \ldots, \mathbf{B}_1(u_s, v_s)\}$ be a maximal disjoint subcollection of $\mathcal{G}$. Then $\mathcal{U}_k \subset \bigcup_{j=1}^s \mathbf{B}_2(u_j, v_j)$ because of the following reason: let $(x, y) \in \mathcal{U}_k$. If $\mathbf{B}_1(x, y) \notin \mathcal{G}$, then by maximality there exists $j \in \{1, \ldots, s\}$ with $\mathbf{B}_1(x, y) \cap \mathbf{B}_1(u_j, v_j) \neq \varnothing$. Take $(x', y') \in \mathbf{B}_1(x, y) \cap \mathbf{B}_1(u_j, v_j)$. Then

$$\mathbf{d}((x, y), (u_j, v_j)) \leqslant \mathbf{d}((x, y), (x', y')) + \mathbf{d}((x', y'), (u_j, v_j)) \leqslant 2.$$

This yields

$$\begin{aligned} \mu(A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)) &= \lambda(A^{l_k}\mathbf{B}_{r_k}(x^*))\, \mu_B(A^{l_k}\mathbf{B}_{r_k}(y^*)) \\ &\leqslant 2^n\lambda(E_k)\, \mu_B(A^{l_k}\mathbf{B}_{r_k}(y^*)) \\ &= 2^n\mu(\mathcal{U}_k) \leqslant 2^n \sum_{j=1}^s \mu(\mathbf{B}_2(u_j, v_j)). \end{aligned} \tag{3.8}$$

Note that $\lambda(\mathbf{B}_2(u_j)) = 2^n\lambda(\mathbf{B}_1(u_j))$ while $\mu_B(\mathbf{B}_2(v_j)) = b^{\lfloor \log_b 2 \rfloor}\mu_B(\mathbf{B}_1(v_j))$; hence

$$\sum_{j=1}^s \mu(\mathbf{B}_2(u_j, v_j)) = 2^n b^{\lfloor \log_b 2 \rfloor} \mu\Big(\bigsqcup_{j=1}^s \mathbf{B}_1(u_j, v_j)\Big)$$

because the unit balls are disjoint. Combining this with (3.8), we have

$$\mu(A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)) \leqslant C\,\mu\Big(\bigsqcup_{j=1}^s \mathbf{B}_1(u_j, v_j)\Big) \tag{3.9}$$

for $C := 4^n b^{\lfloor \log_b 2 \rfloor}$. We show next that all the balls $\mathbf{B}_1(u_j, v_j) = \mathbf{B}_1(u_j) \times \mathbf{B}_1(v_j)$ are contained in $A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)$. Fix $j \in \{1, \ldots, s\}$. For the real part, $u_j \in E_k$, meaning it is a point of $A^{l_k}\mathbf{B}_{r_k}(x^*)$ which is at distance at least one from its boundary, hence $\mathbf{B}_1(u_j) \subset A^{l_k}\mathbf{B}_{r_k}(x^*)$. For the B-adic part, consider $y \in \mathbf{B}_1(v_j)$ and recall that $A^{-l_k} v_j \in \mathbf{B}_{r_k}(y^*)$. From the ultrametric inequality it follows

$$\begin{aligned}\mathbf{d}_B(A^{-l_k} y, y^*) &\leqslant \max\{\mathbf{d}_B(A^{-l_k} y, A^{-l_k} v_j), \mathbf{d}_B(A^{-l_k} v_j, y^*)\} \\ &= \max\{b^{-l_k}\mathbf{d}_B(y, v_j), r_k\} \\ &\leqslant \max\{b^{-l_k}, r_k\} = r_k,\end{aligned} \tag{3.10}$$

since we assumed $b^{-l_k} \leqslant r_k$. Therefore, $A^{-l_k} y \in \mathbf{B}_{r_k}(y^*)$ for every $y \in \mathbf{B}_1(v_j)$, and so $\mathbf{B}_1(v_j) \subset A^{l_k}\mathbf{B}_{r_k}(y^*)$. From (3.7) it follows that

$$\mu(A^{l_k}\mathbf{B}_{r_k}(x^*, y^*) \setminus (A^{l_k}\mathbf{B}_{r_k}(x^*, y^*) \cap A^{l_k}\mathcal{F})) \leqslant \varepsilon_k \mu(A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)). \tag{3.11}$$

Equations (3.9) and (3.11) and the fact that $\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j) \subset A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)$ imply

$$\mu\Big(\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j) \setminus \Big(\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j) \cap A^{l_k}\mathcal{F}\Big)\Big) \leqslant \varepsilon_k\, C\, \mu\Big(\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j)\Big).$$

Since the balls $\mathbf{B}_1(u_j, v_j)$ are pairwise disjoint and contained in $A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)$, then for at least one $j_k \in \{1, \ldots, s\}$ it holds that

$$\mu(\mathbf{B}_1(u_{j_k}, v_{j_k}) \cap A^{l_k}\mathcal{F}) \leqslant (1 - \varepsilon_k\, C)\mu(\mathbf{B}_1(u_{j_k}, v_{j_k})),$$

which yields Claim 1 with $(u^{(k)}, v^{(k)}) = (u_{j_k}, v_{j_k})$.

Back to the main proof, Claim 1 together with the iterated set equation (3.2) implies that, for every k, there exists $l_k > 0$ and $(u^{(k)}, v^{(k)}) \in \mathbb{K}_A$ such that

$$\mu\Big(\mathbf{B}_1(u^{(k)}, v^{(k)}) \cap \Big(\bigcup_{d \in \mathcal{D}_{l_k}} \mathcal{F} + \varphi(d)\Big)\Big) \leqslant (1 - \varepsilon_k\, C)\mu(\mathbf{B}_1(u^{(k)}, v^{(k)})). \tag{3.12}$$

Define the finite sets

$$\mathcal{V}_k := \{\varphi(d) - (u^{(k)}, v^{(k)}) \mid d \in \mathcal{D}_{l_k},\ (\mathcal{F} + \varphi(d) - (u^{(k)}, v^{(k)})) \cap \mathcal{F} \neq \varnothing\}.$$

Then shifting the arguments inside the measures in (3.12) by $-(u^{(k)}, v^{(k)})$ and restricting to translates contained in $\mathcal{V}_k$ yields

$$\mu(\mathbf{B}_1(0) \cap (\mathcal{F} + \mathcal{V}_k)) \leqslant (1 - \varepsilon_k\, C)\mu(\mathbf{B}_1(0)).$$

Note that $\mathcal{F} \cap \mathbf{B}_1(e) \neq \varnothing$ for every $e \in \mathcal{V}_k$. Thus, because $\mathcal{F}$ is bounded, all $\mathcal{V}_k \subset \mathbf{B}_R(0)$ for a sufficiently large constant R. Recall that $\varphi(\mathcal{D}_\infty)$ is a uniformly discrete set by Theorem 3.2, and $\mathcal{D}_{l_k} \subset \mathcal{D}_\infty$, hence there exists $\delta > 0$ such that $\mathbf{d}(e, e') \geqslant \delta$ for every $e, e' \in \mathcal{V}_k$ for every k. This implies that the sequence of cardinalities $\{|\mathcal{V}_k|\}_{k \geqslant 1}$ is bounded. Therefore, $\{\mathcal{V}_k\}_{k \geqslant 1}$ has a convergent subsequence $\{\mathcal{V}_{k_j}\}_{j \geqslant 1}$ whose limit, denoted by $\mathcal{V}$, is a finite set. Then

$$\begin{aligned}\mu(\mathbf{B}_1(0) \cap (\mathcal{F} + \mathcal{V})) &\geqslant \liminf_{j \to \infty} \mu(\mathbf{B}_1(0) \cap (\mathcal{F} + \mathcal{V}_{k_j}) \\ &\geqslant \liminf_{j \to \infty} (1 - C\, \varepsilon_{k_j})\mu(\mathbf{B}_1(0)) = \mu(\mathbf{B}_1(0)).\end{aligned} \tag{3.13}$$

Because T is closed this implies that $(\mathcal{F} + \mathcal{V}) \cap \mathbf{B}_1(0) = \mathbf{B}_1(0)$. Thus $\mathcal{F} + \mathcal{V}$ is a finite union of translates of the compact set $\mathcal{F}$ containing inner points. Baire's theorem implies that $\mathcal{F}$ has nonempty interior. □

In what follows, we will restrict ourselves to the case where $\mathcal{F}$ has positive measure. We have referred to $\mathcal{F}$ in this case as a tile, because we will show that there exists a *tiling* of the space $\mathbb{K}_A$ by translates of $\mathcal{F}$.

Definition 3.5 (Tiling, self-replicating tiling, multiple tiling)**.** Assume $\mu(\mathcal{F}) > 0$. Let $\mathcal{S} \subset \mathbb{K}_A$ and consider the collection $\{\mathcal{F} + s \mid s \in \mathcal{S}\}$, which we denote as $\mathcal{F} + \mathcal{S}$ with a slight abuse of notation.

(1) $\mathcal{F} + \mathcal{S}$ is said to be a *tiling* of $\mathbb{K}_A$ if it is a covering of $\mathbb{K}_A$ such that, for any $s \neq s'$ in $\mathcal{S}$, it holds that $\mu((\mathcal{F} + s) \cap (\mathcal{F} + s')) = 0$, or, equivalently, if $\mathcal{F} + s$ and $\mathcal{F} + s'$ have disjoint interiors. $\mathcal{S}$ is called a tiling set for $\mathcal{F}$. We say that $\mathcal{F} + \mathcal{S}$ tiles $\mathbb{K}_A$.

(2) $\mathcal{F}+\mathcal{S}$ is said to be a *self-replicating tiling* if there exists an expanding linear map T on $\mathbb{K}_A$ such that, for each $s\in\mathcal{S}$, there exists a finite subset $J(s)\subset\mathcal{S}$ with

$$T(\mathcal{F}+s)=\bigcup_{s'\in J(s)}(\mathcal{F}+s').$$

(3) $\mathcal{F}+\mathcal{S}$ is said to be a *multiple tiling* of $\mathbb{K}_A$, if there exists $k\in\mathbb{N}$ such that μ-almost every point of $\mathbb{K}_A$ is contained in exactly k distinct sets of the form $\mathcal{F}+s$ with $s\in\mathcal{S}$.

It follows that a self-replicating tiling is completely determined by the set of tiles that touch the origin 0. We call a self-replicating tiling atomic if the origin touches exactly one tile.

We want to study the nature of the tilings of $\mathbb{K}_A$ obtained using self-affine tiles. For any $k\geqslant 1$, consider the difference sets

$$\mathcal{D}_k-\mathcal{D}_k=\{d-d'\mid d,d'\in\mathcal{D}_k\},$$

and define

$$\Delta:=\bigcup_{k=1}^{\infty}\varphi(\mathcal{D}_k-\mathcal{D}_k).$$

Theorem 3.6. *Suppose that $\mathcal{F}$ contains an open set. Then:*

(i) *There exists a set of translations $\mathcal{S}\subset\Delta$ such that $\mathcal{F}+\mathcal{S}$ tiles $\mathbb{K}_A$. Furthermore, there exists a translate $\mathcal{S}'$ of $\mathcal{S}$ such that $\mathcal{F}+\mathcal{S}'$ is an atomic self-replicating tiling of $\mathbb{K}_A$, and the expanding linear map associated to it is of the form $T=A^k$ for some sufficiently large k.*

(ii) *If Δ is a group, then $\mathcal{F}+\Delta$ is a tiling.*

Proof. (i) The proof is analogous to the one in [17, Theorem 1.2].

(ii) Assume that Δ is a group. Since by (i) there is a subset $\mathcal{S}\subset\Delta$ for which $\mathcal{F}+\mathcal{S}$ tiles $\mathbb{K}_A$, $\mathcal{F}+\Delta$ is a covering of $\mathbb{K}_A$. Given any $s\neq s'$ in Δ, then it suffices to show

$$0=\mu((\mathcal{F}+s)\cap(\mathcal{F}+s'))=\mu(\mathcal{F}\cap(\mathcal{F}+s'-s)).$$

Because Δ is a group, then $s'-s\in\Delta$. That means that there exists $k\geqslant 1$ and $d\neq d'$ in $\mathcal{D}_k$ such that $s'-s=\varphi(d-d')$, so the assertion is equivalent to

$$\mu((\mathcal{F}+\varphi(d))\cap(\mathcal{F}+\varphi(d')))=0,$$

which holds by (3.3). □

3.2. **Example.** We give now an example of a standard and a nonstandard digit system, together with an illustration of a rational self-affine tile.

Example 3.7. Let

$$A=\begin{pmatrix}2&1\\0&\frac{5}{3}\end{pmatrix}\in\mathbb{Q}^{2\times 2}.$$

Since its characteristic polynomial is $\chi_A(x)=(x-2)(x-\frac{5}{3})$, A is expanding. We proceed to find the representation space $\mathbb{K}_A$. We have

$$B=A^{-1}=\begin{pmatrix}\frac{1}{2}&-\frac{3}{10}\\0&\frac{3}{5}\end{pmatrix},$$

and we get

$$\mathbb{Z}^2[B]=\left\{\begin{pmatrix}\frac{s}{2^n5^m}\\\frac{t}{5^l}\end{pmatrix}\mid n,m,l\in\mathbb{N},\ s,t\in\mathbb{Z}\right\}$$

and

$$B\mathbb{Z}^2[B]=\left\{\begin{pmatrix}\frac{s}{2^n5^m}\\\frac{3t}{5^l}\end{pmatrix}\mid l,m,n\in\mathbb{N},\ s,t\in\mathbb{Z}\right\}.$$

Then, a residue set $\mathcal{E}$ for the quotient $\mathbb{Z}^n[B]/B\mathbb{Z}^n[B]$ is given by

$$\mathcal{E}=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix}\right\},$$

hence $b = 3$. It is not hard to check that, for this example, it is possible to establish an isomorphism $\mathbb{Z}^n((B)) \simeq \mathbb{Q}_3$, and thus we regard the elements of $\mathbb{K}_A$ as points in $\mathbb{R}^2 \times \mathbb{Q}_3$. We will consider a digit set $\widetilde{\mathcal{D}}$ and in Figure 1, we will illustrate a rational self-affine tile associated to $(A, \widetilde{\mathcal{D}})$, and represent it in $\mathbb{R}^3$ by embedding $\mathbb{Q}_3$ in $\mathbb{R}$.

We will first find a standard digit system $(A, \mathcal{D})$, and for that we compute the quotient $\mathbb{Z}^2[A]/A\mathbb{Z}^2[A]$. Powers of A are of the form

$$A^k = \begin{pmatrix} 2^k & c_k \\ 0 & (\frac{5}{3})^k \end{pmatrix} \in \mathbb{Q}^{2\times 2},$$

where $c_k = \sum_{i,j:\, i+j=k-1} 2^i \left(\frac{5}{3}\right)^j$. We have

$$\mathbb{Z}^2[A] = \left\{ \begin{pmatrix} \frac{s}{3^n} \\ \frac{t}{3^m} \end{pmatrix} \mid n, m \in \mathbb{N},\ s, t \in \mathbb{Z} \right\},$$

hence, an element of $A\mathbb{Z}^2[A]$ is of the form

$$\begin{pmatrix} 2 & 1 \\ 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} \frac{s}{3^n} \\ \frac{t}{3^m} \end{pmatrix} = \begin{pmatrix} \frac{2s}{3^n} + \frac{t}{3^m} \\ \frac{5t}{3^{m+1}} \end{pmatrix}.$$

Note that $\frac{2s}{3^n} + \frac{t}{3^m} \equiv \frac{5t}{3^{m+1}} \mod 2$ in $\mathbb{Q}$, because if we multiply both sides by 3^{n+m+1} we get $3^{m+1}2s + 3^{n+1}t \equiv t \mod 2$, and $3^n 5t \equiv t \mod 2$. This yields

$$A\mathbb{Z}^2[A] = \left\{ \begin{pmatrix} \frac{s}{3^n} \\ \frac{5t}{3^m} \end{pmatrix} \mid n, m \in \mathbb{N},\ s, t \in \mathbb{Z},\ s \equiv t \mod 2 \right\}.$$

A complete set of residue class representatives of $\mathbb{Z}^2[A]/A\mathbb{Z}^2[A]$ is given by

$$\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 9 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \end{pmatrix} \right\},$$

and so $(A, \mathcal{D})$ is a standard digit system. Note that $a = 10$ and so $|\det A| = \frac{10}{3} = \frac{a}{b}$, as expected.

Next, we want to find a nonstandard digit system. Note that we can write $\mathcal{D} = \mathcal{R}_1 + \mathcal{R}_2$, where

$$\mathcal{R}_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 9 \end{pmatrix} \right\}, \quad \mathcal{R}_2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

and the decomposition of the digits as a sum is unique. Consider

$$\widetilde{\mathcal{D}} := \mathcal{R}_1 + A\mathcal{R}_2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \end{pmatrix} \right\},$$

which is not a residue set for $\mathbb{Z}^n[A] \mod A$. We show that $\mathcal{F}(A, \widetilde{\mathcal{D}})$ has positive measure. Let

$$E_1 := \Big\{ \sum_{j=1}^{\infty} A^{-j} \varphi(e_j) \mid e_j \in \mathcal{R}_1 \Big\} \quad \text{and} \quad E_2 := \Big\{ \sum_{j=1}^{\infty} A^{-j} \varphi(e'_j) \mid e'_j \in \mathcal{R}_2 \Big\}.$$

Now, the unique decomposition of the elements of $\widetilde{\mathcal{D}}$ yields that

$$\begin{aligned} \mathcal{F}(A, \widetilde{\mathcal{D}}) &= E_1 + A\,E_2 = E_1 + \Big\{ \sum_{j=0}^{\infty} A^{-j} \varphi(e'_{j+1}) \mid e'_j \in \mathcal{R}_2 \Big\} \\ &= E_1 + E_2 + \varphi(\mathcal{R}_2) = \mathcal{F}(A, \mathcal{D}) + \varphi(\mathcal{R}_2). \end{aligned}$$

Since $(A, \mathcal{D})$ is a standard digit system, the set $\mathcal{F}(A, \mathcal{D})$ has positive measure (see Corollary 3.3), and hence so does $\mathcal{F}(A, \widetilde{\mathcal{D}})$, meaning it is rational self-affine tile associated to a nonstandard digit system. We illustrate it in Figure 1. Recall that the representation space is not Euclidean, so we embedded the points of $\mathbb{K}_A$ into $\mathbb{R}^3$ in order to draw the picture. We did that so the figure still reflects some of the properties of the set, but it is not a completely faithful representation, since this is not possible due to the B-adic factor.

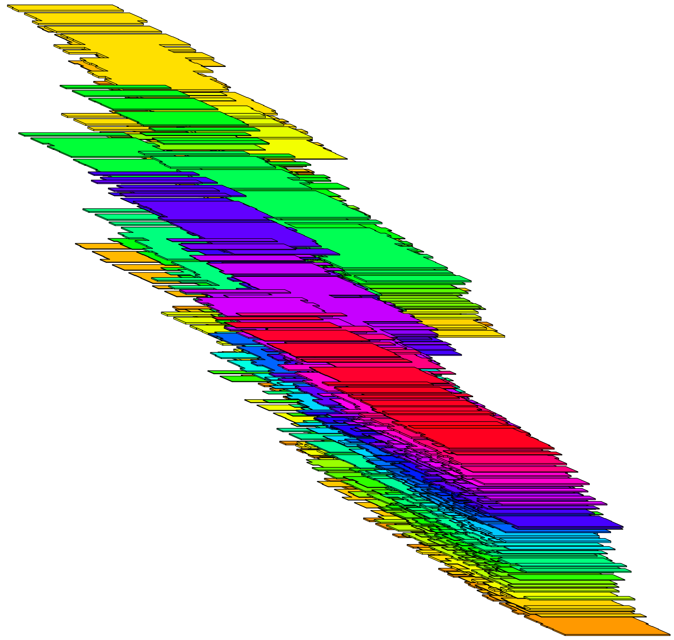

FIGURE 1. The tile $\mathcal{F}(A, \widetilde{\mathcal{D}})$.

4. Characters and multiple tiling

The main result of this section states that, whenever $\mathcal{F}(A, \mathcal{D})$ is a tile, it gives a multiple tiling of $\mathbb{K}_A$. Before arriving to the proof, we introduce some definitions in order to find the Pontryagin dual of $\mathbb{K}_A$, and we give a complete description of its characters. We make use of the character theory of locally compact abelian groups to show that the multiplication by A is ergodic on a certain torus in $\mathbb{K}_A$, and use this to prove the existence of the multiple tiling.

4.1. **Some basic results and definitions.** The module $\mathbb{Z}^n[A]$ plays a principal role in the study of tilings by rational self-affine tiles. We will prove first that, embedded into the representation space $\mathbb{K}_A$, this module becomes a lattice, and we show later that it is a translation set for a multiple tiling given by copies of $\mathcal{F}$. First, we formalize the notion of lattice in our setting.

Definition 4.1 (Lattice)**.** A subset Λ of $\mathbb{K}_A$ is a *lattice* if it satisfies the three following conditions:

(1) Λ is a group.
(2) Λ is uniformly discrete, meaning there exists $r > 0$ such that every open ball of radius r in $\mathbb{K}_A$ contains at most one point of Λ.
(3) Λ is relatively dense, meaning there exists $R > 0$ such that every closed ball of radius R in $\mathbb{K}_A$ contains at least one point of Λ.

We show next that $\varphi(\mathbb{Z}^n[A])$ satisfies these properties. We state a lemma first.

Lemma 4.2. *There exists an integer $K \geqslant 1$ such that*

$$\mathbb{Z}^n \cap B\mathbb{Z}^n[B] = \mathbb{Z}^n \cap \left(B\mathbb{Z}^n + B^2\mathbb{Z}^n + \cdots + B^K\mathbb{Z}^n\right). \tag{4.1}$$

Proof. For $k \geqslant 1$, define the lattices

$$\mathcal{L}_k[B] := \sum_{j=1}^{k} B^j\mathbb{Z}^n.$$

Since $\mathcal{L}_k[B]$ contains $B\mathbb{Z}^n$ and B is invertible, the lattice $\mathcal{L}_k[B]$ has full rank. Consider a nonzero integer m_k such that $m_k\mathcal{L}_k[B] \subset \mathbb{Z}^n$. Then $m_k\mathcal{L}_k[B]$ has finite index in $\mathbb{Z}^n$. From this fact, one deduces that the intersection $\mathbb{Z}^n \cap \mathcal{L}_k[B]$ has finite index in $\mathbb{Z}^n$ for every $k \geqslant 1$. Therefore, the chain of nested lattices

$$(\mathbb{Z}^n \cap \mathcal{L}_1[B]) \subset (\mathbb{Z}^n \cap \mathcal{L}_2[B]) \subset \cdots \subset (\mathbb{Z}^n \cap B\mathbb{Z}^n[B]) \subset \mathbb{Z}^n$$

must eventually stabilize after some $K \geqslant 1$. $\square$

Proposition 4.3. *The set $\varphi(\mathbb{Z}^n[A])$ is a lattice in $\mathbb{K}_A$.*

Proof. The fact that $\varphi(\mathbb{Z}^n[A])$ is a group follows from the additive group structure of $\mathbb{Z}^n[A]$ because φ is a group homomorphism.

To prove the uniform discreteness of $\varphi(\mathbb{Z}^n[A])$, we claim that there exists $0 < r \leqslant 1$ such that $\mathbf{d}(\varphi(z), 0) \geqslant r$ for every nonzero $z \in \mathbb{Z}^n[A]$. If $\mathbf{d}(\varphi(z), 0) \geqslant 1$ then we are done. Suppose on the contrary that $\mathbf{d}(\varphi(z), 0) < 1$. Since $z \in \mathbb{Z}^n[A]$, we can write it as

$$z = \sum_{j=0}^{k} A^j z_j, \quad z_j \in \mathbb{Z}^n, \tag{4.2}$$

with $z_k \neq 0$, and there is a minimal index k with this property. If $z_k \notin B\mathbb{Z}^n[B]$, then $B^k z \in \mathbb{Z}^n[B] \setminus B\mathbb{Z}^n[B]$, so one has $\mathbf{d}_B(z, 0) = b^k \mathbf{d}_B(B^k z, 0) \geqslant 1$, in contradiction to $\mathbf{d}(\varphi(z), 0) < 1$. Thus, $z_k \in \mathbb{Z}^n \cap B\mathbb{Z}^n[B]$. By Lemma 4.2, there are vectors $w_1, w_2, \ldots, w_K \in \mathbb{Z}^n$ such that

$$z_k = Bw_1 + B^2 w_2 + \cdots + B^K w_K.$$

We claim that, in such case, one must have $k \leq K$. Suppose that $k > K$. Then

$$z = A^k z_k + \sum_{j=0}^{k-1} A^j z_j = \sum_{j=0}^{k-K-1} A^j z_j + \sum_{j=k-K}^{k-1} (z_j + w_{k-j}) A^j = \sum_{j=0}^{k-1} A^j z_j', \quad z_j' \in \mathbb{Z}^n.$$

However, that would contradict the minimality of k. Therefore, $k \leq K$. Now, let m denote the least common multiple of the denominators of the entries of A. Since z is a sum of integer vectors multiplied by A^k, $0 \leqslant k \leqslant K$, the non-zero entries of z are at least $1/m^K$ in absolute value. Hence, $\mathbf{d}(\varphi(z), 0) \geqslant \|z\| \geqslant 1/m^K = r$.

We now turn to the proof of relative denseness. Let $(x, y) \in \mathbb{K}_A$ be arbitrary. Choose $x' \in \mathbb{Z}^n$ to be the closest integer vector to x, so that $\|x - x'\| < 1$, and choose $y' \in \mathbb{Z}^n[A]$ such that $\mathbf{d}_B(y, y') \leqslant 1$ (this holds by taking $y' = \{y\}_B$, see Definition 4.6 below). Hence, $\mathbf{d}((x, y), (x', y')) \leqslant 1$. Choose $y'' \in \mathbb{Z}^n$ to be the closest integer vector to $x' - y' \in \mathbb{R}^n$. Let $z := y' + y'' \in \mathbb{Z}^n[A]$. Then $\|x' - z\| = \|x' - y' - y''\| < 1$. Moreover, $\mathbf{d}_B(z, y') = \mathbf{d}_B(y'', 0) \leqslant 1$ because $y'' \in \mathbb{Z}^n$. Therefore, $\mathbf{d}((x', y'), \varphi(z)) \leqslant 1$. This yields $\mathbf{d}((x, y), \varphi(z)) \leqslant 2$, which proves the Lemma. $\square$

The next step is to define a space $\mathbb{Z}^n((B^*))$ that will be crucial later when we study the characters of $\mathbb{K}_A$. Prior to that, we prove the following Lemma.

Lemma 4.4. *The group $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B]$ is a lattice in $\mathbb{R}^n$.*

Proof. We show that $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B] \subset \mathbb{Z}^n + A\mathbb{Z}^n + \cdots + A^K\mathbb{Z}^n$ for some $K \geqslant 1$. Let $z \in \mathbb{Z}^n[A] \cap \mathbb{Z}^n[B]$. Since $z \in \mathbb{Z}^n[A]$, write $z = \sum_{j=0}^{k} A^j z_j$, with $z_j \in \mathbb{Z}^n$, $z_k \neq 0$ and k minimal. If $k = 0$, then $z \in \mathbb{Z}^n$. Assume $k \geqslant 1$. Since $z \in \mathbb{Z}^n[B]$, from solving for z_k it follows that $z_k \in \mathbb{Z}^n \cap B\mathbb{Z}^n[B]$. By Lemma 4.2, one can find vectors $w_1, w_2, \ldots, w_K \in \mathbb{Z}^n$ such that

$$z_k = Bw_1 + B^2 w_2 + \cdots + B^K w_K, \tag{4.3}$$

and by proceeding like in the proof of Proposition 4.3, one shows $k \leqslant K$. Therefore the inclusion follows. This implies that $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B]$ is contained in a lattice, and also it trivially contains the lattice $\mathbb{Z}^n$. Since it is a group, it is itself a lattice. $\square$

For an arbitrary lattice $\Lambda \subset \mathbb{R}^n$, define its *dual lattice* by

$$\Lambda^* := \{x \in \mathbb{Z}^n \mid \langle x, z \rangle \in \mathbb{Z} \text{ for every } z \in \Lambda\},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^n$. Denote by A^* the transpose of A, and let Λ and Γ be full rank lattices such that

$$\mathbb{Z}^n[A]\cap\mathbb{Z}^n[B]\subset\Lambda \quad\text{ and }\quad \Gamma[A^*]\cap\Gamma[B^*]\subset\Lambda^*\subset\mathbb{Z}^n. \tag{4.4}$$

This is possible because $\mathbb{Z}^n[A]\cap\mathbb{Z}^n[B]$ is a lattice by Lemma 4.4, and the proof that $\Gamma[A^*]\cap\Gamma[B^*]$ is also a lattice is analogous.

Definition 4.5 (B-adic and B^*-adic expansions)**.** Let $\mathcal{E}\subset\mathbb{Z}^n$ be a complete set of residue classes of $\mathbb{Z}^n[B]/B\mathbb{Z}^n[B]$ with $0\in\mathcal{E}$. Then every $y\in\mathbb{Z}^n((B))$ has a unique expansion of the form

$$y=\sum_{j=\nu(y)}^{\infty}B^jy_j,\quad y_j\in\mathcal{E}, \tag{4.5}$$

which we call the *B-adic expansion* of y with coefficients in $\mathcal{E}$. Recall that $\nu(0)=\infty$, so the B-adic expansion of 0 is the empty sum.

Let B^* denote the transpose of B. Consider the full rank integer lattice Γ satisfying (4.4) and let $\mathcal{E}^*\subset\Gamma\subset\mathbb{Z}^n$ be a complete set of coset representatives of $\Gamma[B^*]/B^*\Gamma[B^*]$ with $0\in\mathcal{E}^*$. Consider the space $\mathbb{Z}^n((B^*))$ defined analogously to $\mathbb{Z}^n((B))$. Then every $s\in\mathbb{Z}^n((B^*))$ has a unique expansion of the form

$$s=\sum_{j=\nu^*(s)}^{\infty}{B^*}^js_j,\quad s_j\in\mathcal{E}^*, \tag{4.6}$$

where ν^* is the valuation in $\mathbb{Z}^n((B^*))$ defined in the same way as ν. We call this the *B^*-adic expansion* of s with coefficients in $\mathcal{E}^*$.

Definition 4.6 (B-adic and B^*-adic fractional and integer part)**.** Given $y\in\mathbb{Z}^n((B))$ with B-adic expansion 4.5, we define the *B-adic fractional part* and the *B-adic integer part* of y, respectively, as

$$\{y\}_B:=\sum_{j=\nu(y)}^{-1}B^jy_j,\quad \lfloor y\rfloor_B:=\sum_{j=0}^{\infty}B^jy_j.$$

Given $s\in\mathbb{Z}^n((B^*))$ with B^*-adic expansion 4.6, we define the *B^*-adic fractional part* and the *B^*-adic integer part* of s, respectively, as

$$\{s\}_B^*:=\sum_{j=\nu^*(s)}^{-1}{B^*}^js_j,\quad \lfloor s\rfloor_B^*:=\sum_{j=0}^{\infty}{B^*}^js_j.$$

From here onwards, whenever we have a B-adic series (resp. B^*-adic series), we assume the coefficients to lie in $\mathcal{E}$ (resp. $\mathcal{E}^*$).

Remark 4.7. Recall that $b=|\mathcal{E}|$. We claim that, if $b=1$, then the multiple tiling theorem holds. Note that, in this case, $\det A=a$ is an integer. Indeed, an analogous version of Theorem 4.16 is proven by Lagarias and Wang in [16] for integer matrices. However, they show in [16, Lemma 2.1] that this results also holds for self-affine tiles associated to expanding real matrices $A\in\mathbb{R}^{n\times n}$ with integer determinant, as long as there exists an A-invariant lattice in $\mathbb{R}^n$ containing the difference set $\mathcal{D}-\mathcal{D}$. If $\mathcal{F}(A,\mathcal{D})$ is a rational self-affine tile and $b=1$, then $\mathbb{Z}^n[A]\cap\mathbb{Z}^n[B]$ is a lattice by Lemma 4.4, and it is A-invariant because $\mathbb{Z}^n[B]/B\mathbb{Z}^n[B]$ is trivial and hence $A\mathbb{Z}^n[B]\subset\mathbb{Z}^n[B]$. Consider $c\in\mathbb{Z}\setminus\{0\}$ such that $c\mathcal{D}\subset\mathbb{Z}^n$; then $\mathcal{D}-\mathcal{D}\subset\frac{1}{c}(\mathbb{Z}^n[A]\cap\mathbb{Z}^n[B])$, which is an A-invariant lattice. Therefore, the assumption that $b\geqslant 2$ made in Remark 2.6 also applies to this section.

Remark 4.8. We can assume without loss of generality that $\mathcal{E}$ has a subset $\{c_1,\dots,c_n\}$ such that the lattice $\Theta:=\langle c_1,\dots,c_n\rangle_{\mathbb{Z}}$ has full rank in $\mathbb{R}^n$. To show this, suppose first that $b\geqslant n$. Take a matrix $R\in\mathbb{Z}^{n\times n}$ whose columns are distinct elements $\{c_1,\dots,c_n\}$ of $\mathcal{E}$. Consider the integer matrix $N(t):=tB-R$, where $t\in\mathbb{N}$ is chosen so that $tB\in\mathbb{Z}^{n\times n}$. Its determinant is $\det(N(t))=\det(B)\det(t\cdot Id-AR)=\det(B)\,\chi_{AR}(t)$, where $\chi_{AR}(t)\in\mathbb{Q}[t]$ is the characteristic polynomial of AR. For all but finitely many $t\in\mathbb{N}$, it holds that $\chi_{AR}(t)\neq 0$. Hence, we can choose t in a way that the column vectors $\{\widetilde{c_1},\dots,\widetilde{c_n}\}$ of $N(t)$ are linearly independent, and hence they span a full rank integer lattice. Note that $c_j - \widetilde{c_j} \in B\mathbb{Z}^n$ for $j = 1, \dots, n$, so we can replace each c_j by $\widetilde{c_j}$, and this produces a new residue set $\widetilde{\mathcal{E}}$ with the required property.

Suppose now that $1 < b < n$. Choose $k \geqslant 1$ so that $b^k \geqslant n$. Note that a complete set of residues for $\mathbb{Z}^n[B^k]/B^k\mathbb{Z}^n[B^k]$ is given by $\mathcal{E} + B\mathcal{E} + \cdots + B^{k-1}\mathcal{E}$, so $|\mathbb{Z}^n[B^k]/B^k\mathbb{Z}^n[B^k]| = b^k \geqslant n$. Suppose that we consider the digit system $(A^k, \mathcal{D}_k)$ with $\mathcal{D}_k$ as in (3.1); then $\mathcal{E}_k$ satisfies the assumption of the previous paragraph. Note that from the iterated set equation (3.2) it follows that $\mathcal{F}(A^k, \mathcal{D}_k) = \mathcal{F}(A, \mathcal{D})$. Hence, whenever $|\mathcal{E}| < n$ we can work with $(A^k, \mathcal{D}_k)$ instead of $(A, \mathcal{D})$ and with $\mathcal{E} + B\mathcal{E} + \cdots + B^{k-1}\mathcal{E}$ instead of $\mathcal{E}$.

4.2. **Character theory.** In order to prove the multiple tiling theorem, we use some results on the characters of $\mathbb{K}_A$. For more on the topic of character theory on locally compact abelian groups, we refer the reader to [21, Chapter 4].

Definition 4.9 (Character)**.** A character χ on an abelian group G is a continuous function $\chi : G \to \mathbb{S}^1$ such that $\chi(x + y) = \chi(x)\chi(y)$ for all $x, y \in G$.

Lemma 4.10. *The set of all characters of G constitutes a group, called the Pontryagin dual of G, denoted by $\widehat{G}$. It satisfies the following properties:*

(1) *The Pontryagin dual of $\widehat{G}$ is isomorphic to G.*
(2) *The Pontryagin dual of the product $G_1 \times G_2$ is isomorphic to $\widehat{G}_1 \times \widehat{G}_2$ and the characters are of the form $\chi = \chi_1 \cdot \chi_2$ with $\chi_1 \in \widehat{G_1}$ and $\chi_2 \in \widehat{G_2}$.*
(3) *Given a subgroup $H \subset G$, define the annihilator of H on G as*

$$Ann(H) := \{\chi \in \widehat{G} \mid \chi(H) = 1\}.$$

Then $\widehat{(G/H)} \simeq Ann(H)$ and $\widehat{H} \simeq \widehat{G}/Ann(H)$.

Proof. See [21, Chapter 4]. □

It is well known that the characters on $\mathbb{R}^n$ are given by

$$\chi_r : \mathbb{R}^n \to S^1; \quad x \mapsto \exp(2\pi i \langle x, r\rangle),$$

where $r \in \mathbb{R}^n$, and $\widehat{\mathbb{R}}^n \simeq \mathbb{R}^n$ via the isomorphism $r \mapsto \chi_r$.

For any $s = \sum_{j=\nu^*(s)}^{\infty} B^{*j} s_j \in \mathbb{Z}^n((B^*))$ with $s_j \in \mathcal{E}^*$, define the map

$$\chi_s : \mathbb{Z}^n((B)) \to S^1; \quad y \mapsto \exp(2\pi i S_s(y)), \tag{4.7}$$

where

$$S_s(y) := \sum_{j=\nu^*(s)}^{\infty} \langle \{B^j y\}_B, s_j\rangle. \tag{4.8}$$

The map is well defined because $\{B^j y\}_B = 0$ for all but finitely many indices j. We show next that this map is indeed a character.

Proposition 4.11. *For every $s \in \mathbb{Z}^n((B^*))$, the map χ_s defined in (4.7) is continuous and multiplicative, that is, $\chi_s(y + y') = \chi_s(y)\chi_s(y')$.*

Proof. Fix $s = \sum_{j=\nu^*(s)}^{\infty} B^{*j} s_j \in \mathbb{Z}^n((B^*))$ with $s_j \in \mathcal{E}^*$. For the multiplicativity, if suffices to show that $S_s(y + y') = S_s(y) + S_s(y') \mod \mathbb{Z}$. We prove first the following claim: given $\omega, \omega' \in \mathbb{Z}^n((B))$, it holds that

$$\{\omega\}_B + \{\omega'\}_B - \{\omega + \omega'\}_B \in \Lambda, \tag{4.9}$$

where Λ is the lattice satisfying (4.4). By definition of the B-adic fractional part, $\{\omega\}_B + \{\omega'\}_B - \{\omega + \omega'\}_B \in \mathbb{Z}^n[A]$. Also,

$$\begin{aligned} \{\omega\}_B + \{\omega'\}_B - \{\omega + \omega'\}_B &= (\{\omega\}_B - \omega) + (\{\omega'\}_B - \omega') + (\omega + \omega' - \{\omega + \omega'\}_B) \\ &= -\lfloor \omega \rfloor_B - \lfloor \omega' \rfloor_B + \lfloor \omega + \omega' \rfloor_B \in \mathbb{Z}^n[A^{-1}]. \end{aligned}$$

Since $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[A^{-1}] \subset \Lambda$ by definition of Λ, this yields the claim. Now, for any y, y' we have

$$S_s(y) + S_s(y') - S_s(y+y') = \sum_{j=\nu^*(s)}^{\infty} \langle \{B^j y\}_B + \{B^j y'\}_B - \{B^j(y+y')\}_B, s_j \rangle \tag{4.10}$$

where $\{B^j y\}_B + \{B^j y'\}_B - \{B^j(y+y')\}_B \in \Lambda$ by (4.9). The summands in (4.10) are nonzero only for a finite number of j's. For every index j, we have $s_j \in \mathcal{E}^* \subset \Gamma \subset \Lambda^*$, thus by definition of dual lattice,

$$\langle \{B^j y\}_B + \{B^j y'\}_B - \{B^j(y+y')\}_B, s_j \rangle \in \mathbb{Z}$$

and the multiplicativity of χ_s is established.

For the continuity, let $y, y' \in \mathbb{Z}^n((B))$ such that $\mathbf{d}_B(y, y') \leqslant b^{\nu^*(s)}$. Then, for every $j \geqslant \nu^*(s)$, it holds that $\mathbf{d}_B(B^j y, B^j y') \leqslant 1$, and so $B^j(y - y') \in \mathbb{Z}^n[[B]]$ which implies $\{B^j(y-y')\}_B = 0$. Then $S_s(y - y') = 0$ and hence, by multiplicativity, $\chi_s(y) = \chi_s(y')$. Thus χ_s is locally constant and, hence, continuous. $\square$

We will show that the Pontryagin dual of $\mathbb{K}_A$ is isomorphic to $\mathbb{R}^n \times \mathbb{Z}^n((B^*))$. To do so, we prove some lemmas first.

Lemma 4.12. *Let $y = \sum_{k=\nu(y)}^{\infty} B^k y_k \in \mathbb{Z}^n((B))$ with $y_k \in \mathcal{E}$ and $s = \sum_{j=\nu^*(s)}^{\infty} B^{*j} s_j \in \mathbb{Z}^n((B^*))$ with $s_j \in \mathcal{E}^*$ be given. Then*

$$S_s(y) = \sum_{j=\nu^*(s)}^{\infty} \langle \{B^j y\}_B, s_j \rangle = \sum_{k=\nu(y)}^{\infty} \langle y_k, \{B^{*k} s\}_B^* \rangle.$$

Proof. From direct calculation, we obtain

$$\begin{aligned} S_s(y) &= \sum_{j=\nu^*(s)}^{\infty} \langle \{B^j y\}_B, s_j \rangle = \sum_{j=\nu^*(s)}^{\infty} \langle \sum_{k=\nu(y)}^{-j-1} B^{j+k} y_k, s_j \rangle \\ &= \sum_{j=\nu^*(s)}^{\infty} \sum_{k=\nu(y)}^{-j-1} \langle y_k, B^{*j+k} s_j \rangle = \sum_{k=\nu(y)}^{\infty} \sum_{j=\nu^*(s)}^{-k-1} \langle y_k, B^{*j+k} s_j \rangle \\ &= \sum_{k=\nu(y)}^{\infty} \langle y_k, \sum_{j=\nu^*(s)}^{-k-1} B^{*j+k} s_j \rangle = \sum_{k=\nu(y)}^{\infty} \langle y_k, \{B^{*k} s\}_B^* \rangle. \end{aligned}$$ $\square$

Our next step is to establish a Pontryagin duality between $\mathbb{Z}^n((B))$ and $\mathbb{Z}^n((B^*))$. For that purpose, we express both sets in terms of projective limits. For more on the topic we refer the reader to [23]. For each $k \in \mathbb{N}$, consider the quotients

$$\mathcal{E}_k := \mathbb{Z}^n[B]/B^k \mathbb{Z}^n[B].$$

Clearly, $\mathcal{E}_k \subset \mathcal{E}_{k+1}$ for every k, so we can define the canonical projections

$$\pi_k : \mathcal{E}_{k+1} \to \mathcal{E}_k, \quad x \mapsto x \mod B^k.$$

Therefore, we have a projective system

$$\cdots \longrightarrow \mathcal{E}_{k+1} \xrightarrow{\pi_k} \mathcal{E}_k \xrightarrow{\pi_{k-1}} \mathcal{E}_{k-1} \longrightarrow \dots \xrightarrow{\pi_1} \mathcal{E}_1,$$

which entitles the existence of the projective limit

$$\varprojlim_{k\in\mathbb{N}} \mathcal{E}_k = \{(M_k)_{k\in\mathbb{N}} \mid M_k \in \mathcal{E}_k \text{ and } \pi_k(M_{k+1}) = M_k \text{ for every } k\},$$

and it holds that

$$\mathbb{Z}^n((B)) \simeq \varinjlim_{j\in\mathbb{N}} \varprojlim_{k\in\mathbb{N}} B^{-j} \mathcal{E}_k. \tag{4.11}$$

Analogously, for $k \in \mathbb{N}$ consider

$$\mathcal{E}_k^* := \Gamma[B^*]/B^{*k}\Gamma[B^*].$$

Then
$$\mathbb{Z}^n((B^*)) \simeq \varinjlim_{j\in\mathbb{N}} \varprojlim_{k\in\mathbb{N}} B^{*-j}\mathcal{E}_k^*.$$

Proposition 4.13. *The characters of* $\mathbb{K}_A$ *are of the form*
$$\chi_{r,s} : \mathbb{K}_A \to S^1, \quad \chi_{r,s}(x,y) = \chi_r(x)\chi_s(y) = \exp(2\pi i\langle x, r\rangle)\exp(2\pi i S_s(y)),$$
for $r \in \mathbb{R}^n$, $s \in \mathbb{Z}^n((B^*))$, *with* $S_s(y)$ *as in (4.8). Moreover, there is a group morphism given by* $(r,s) \mapsto \chi_{r,s}$.

Proof. In view of Lemma 4.10 we have the isomorphism $\widehat{\mathbb{K}_A} \simeq \widehat{\mathbb{R}^n} \times \widehat{\mathbb{Z}^n((B))}$. It is known that there is an isomorphism $\mathbb{R}^n \simeq \widehat{\mathbb{R}^n}$ given by $r \mapsto \chi_r$. Consider the map $\mathbb{Z}^n((B^*)) \to \widehat{\mathbb{Z}^n((B))}$, $s \mapsto \chi_s$; we show that it is an isomorphism. We first prove that the group operations are compatible on both sets, that is, $\chi_{s+s'}(y) = \chi_s(y)\chi_{s'}(y)$ for every $y = \sum_{j=\nu(y)}^{\infty} B^j y_j \in \mathbb{Z}^n((B))$ with $y_j \in \mathcal{E}$. It is enough to show that $S_{s+s'}(y) = S_s(y) + S_{s'}(y) \mod \mathbb{Z}$. Applying Lemma 4.12, we get
$$S_s(y) + S_{s'}(y) - S_{s+s'}(y) = \sum_{j=\nu(y)}^{\infty} \langle y_j, \{B^{*j}s\}_B^* + \{B^{*j}s'\}_B^* - \{B^{*j}(s+s')\}_B^*\rangle.$$
Proceeding in analogy to the proof of Proposition 4.11 and using the definition of B^*-adic fractional part, we can see that, for every index $j \geqslant \nu(y)$,
$$\{B^{*j}s\}_B^* + \{B^{*j}s'\}_B^* - \{B^{*j}(s+s')\}_B^* \subset \Gamma[B^*] \cap \Gamma[B^{*-1}] \subset \Lambda^* \subset \mathbb{Z}^n,$$
with Λ^* as in (4.4). Since $y_j \in \mathcal{E} \subset \mathbb{Z}^n$ for every $j \geqslant \nu(y)$ and is finite only for a finite number of indices, this yields the first part of the proof.

Next, we show the injectivity. In view of the first part of the proof, it suffices to show that $\chi_s \neq 1$ for $s \neq 0$. Let $s = \sum_{j=\nu^*(s)}^{\infty} B^{*j}s_j \in \mathbb{Z}^n((B^*)) \setminus \{0\}$ with $s_j \in \mathcal{E}^*$, and consider a point of the form $B^{-l}c \in \mathbb{Z}^n((B))$ for $0 \neq c \in \mathcal{E}$ and $l \in \mathbb{N}$. Note that $\{B^{j-l}c\}_B = 0$ whenever $j \geqslant l$. Therefore, applying again Lemma 4.12 we get
$$S_s(B^{-l}c) = \sum_{j=\nu^*(s)}^{l-1} \langle B^{j-l}c, s_j\rangle = \sum_{j=\nu^*(s)}^{l-1} \langle c, B^{*j-l}s_j\rangle = \Big\langle c, \sum_{j=\nu^*(s)}^{l-1} B^{*j-l}s_j \Big\rangle.$$
Suppose $\chi_s = 1$, then $S_s(B^{-l}c) \in \mathbb{Z}$ for every $l \in \mathbb{N}$ and every $c \in \mathcal{E}$. Recall that in Remark 4.8 we assumed w.l.o.g. that $\mathcal{E}$ has a subset $\{c_1, \dots, c_n\}$ such that $\Theta := \langle c_1, \dots, c_n\rangle_{\mathbb{Z}}$ is a full rank lattice in $\mathbb{R}^n$. Thus $S_s(B^{-l}\Theta) \subset \mathbb{Z}$ and hence, by the definition of dual lattice, $\sum_{j=\nu^*(s)}^{l-1} B^{*l-j}s_j \in \Theta^*$ holds for all $l \in \mathbb{N}$. Since $s \neq 0$, we have that
$$\sum_{j=\nu^*(s)}^{l-1} B^{*l-j}s_j \in (B^{*l-\nu^*(s)}\Gamma + \cdots + B^*\Gamma) \setminus (B^{*l-\nu^*(s)-1}\Gamma + \cdots + B^*\Gamma).$$
Then $(B^{*l-\nu^*(s)}\Gamma + \cdots + B^*\Gamma)_{l\geqslant 1}$ is a strictly nested sequence of lattices in $\mathbb{Q}^n$. Given l, consider the entries of all the vectors of $B^{*l-\nu^*(s)}\Gamma + \cdots + B^*\Gamma$ expressed as irreducible fractions, and define m_l to be the maximum of the denominators of these fractions. It is clear that $m_l \leqslant m_{l+1}$. However, note that $(m_l)_{l\geqslant 1}$ does not stabilize: this would imply that, for some lattice Λ_l, there are infinitely many steps in which we can add a point and form a strictly larger lattice, while not increasing the bound on the denominators, which is not possible. This means that Θ^* contains points with entries having arbitrarily large denominators, which is a contradiction.

For the surjectivity, consider a character $\chi \in \widehat{\mathbb{Z}^n((B))}$. By classical arguments following [21, p. 139], we obtain from (4.11) that
$$\widehat{\mathbb{Z}^n((B))} \simeq \varprojlim_{j\in\mathbb{N}} \varinjlim_{k\in\mathbb{N}} \widehat{B^{-j}\mathcal{E}_k}. \tag{4.12}$$

Since $B^{-j}\mathcal{E}_k = B^{-j}\mathbb{Z}^n[B]/B^{k-j}\mathbb{Z}^n[B]$ is a finite group of cardinality b^k, so is its dual. Consider the group
$$G_{j,k} := B^{*j-k}\Gamma[B^*]/B^{*j}\Gamma[B^*],$$

and note that $|G_{j,k}| = b^k$ because Γ is a full rank lattice (and hence, isomorphic to $\mathbb{Z}^n$). Given $s \in G_{j,k}$, we can regard it as an element in $B^{*j-k}\mathcal{E}^* + \cdots + B^{*j-1}\mathcal{E}^*$ and consider the character χ_s as in (4.7). Note that $\chi_s \in \mathrm{Ann}(B^{k-j}\mathbb{Z}^n[B])$ (see Lemma 4.10); hence, χ_s is a character of $B^{-j}\mathcal{E}_k$. Also, for $s \neq 0$ in $G_{j,k}$, there is $y \in B^{-j}\mathcal{E}_k$ such that $S_s(y) \neq 0$: in fact, there exists l with $j-k \leqslant l \leqslant j-1$ with $s_l \neq 0$; since $\mathcal{E}$ spans a full rank lattice, find $c \in \mathcal{E}$ such that $\langle c, s_l\rangle \neq 0$ and take $y = B^{-l}c$. Hence all the characters χ_s for $s \in G_{j,k}$ are distinct over $B^{-j}\mathcal{E}_k$, and since $|B^{-j}\mathcal{E}_k| = b^k = |G_{j,k}|$, this implies that $\widehat{B^{-j}\mathcal{E}_k} \simeq G_{j,k}$, so (4.12) yields

$$\widehat{\mathbb{Z}^n((B))} \simeq \varprojlim_{j\in\mathbb{N}} \varinjlim_{k\in\mathbb{N}} G_{j,k}$$

It is not hard to establish an isomorphism $G_{j,k} \simeq B^{*-j}\mathcal{E}_k^*$, and so

$$\chi \in \varprojlim_{j\in\mathbb{N}} \varinjlim_{k\in\mathbb{N}} \{\chi_s \mid s \in G_{j,k}\} \simeq \varprojlim_{j\in\mathbb{N}} \varinjlim_{k\in\mathbb{N}} B^{*-j}\mathcal{E}_k^* \simeq \mathbb{Z}^n((B^*)),$$

therefore every character χ is of the form $\chi = \chi_s$ for some $s \in \mathbb{Z}^n((B^*))$. □

4.3. **Multiple tiling theorem.** In this final section, we will prove that rational self-affine tiles give a multiple tiling of the representation space. We will make use of the character theory we have developed in the previous section.

Recall that the set $\varphi(\mathbb{Z}^n[A])$ is a lattice by Lemma 4.3, hence the torus $\mathbb{T} := \mathbb{K}_A/\varphi(\mathbb{Z}^n[A])$ is well defined and compact. We endow it with the normalized quotient measure $\bar{\mu}$, which is the Haar measure on $\mathbb{T}$. Denote the multiplication by A on $\mathbb{T}$ as

$$\tau_A : \mathbb{T} \to \mathbb{T}; \quad (x,y) \mapsto A(x,y) \mod \varphi(\mathbb{Z}^n[A]),$$

which is well defined because $\varphi(\mathbb{Z}^n[A])$ is A-invariant. We will prove that this map is ergodic by using the following lemma.

Lemma 4.14. *If G is a compact abelian group with normalized Haar measure and $\tau : G \to G$ is a surjective continuous endomorphism of G, then τ is ergodic if and only if the trivial character $\chi = 1$ is the only character of G that satisfies $\chi \circ \tau^k = \chi$ for some $k \geqslant 1$.*

Proof. See [28, Theorem 1.10.1]. □

We are in position to prove the following result.

Lemma 4.15. *The map τ_A is ergodic.*

Proof. Because A is an invertible matrix, τ_A is a continuous surjective homomorphism. We first prove that it is measure preserving. Note that $\varphi(\mathbb{Z}^n[A])$ is a sublattice of index a of $A^{-1}\varphi(\mathbb{Z}^n[A])$. This implies that, for any measurable set $E \subset \mathbb{T}$,

$$\begin{aligned}
\bar{\mu}(\tau_A^{-1}(E)) &= \bar{\mu}(\{(x,y) \in \mathbb{T} \mid A(x,y) \mod \varphi(\mathbb{Z}^n[A]) \in E\}) \\
&= \bar{\mu}(\{(x,y) \in \mathbb{T} \mid (x,y) \mod A^{-1}\varphi(\mathbb{Z}^n[A]) \in A^{-1}E\}) \\
&= a\,\bar{\mu}(\{(x,y) \in \mathbb{T} \mid (x,y) \mod \varphi(\mathbb{Z}^n[A]) \in A^{-1}E\}) \\
&= a\,\bar{\mu}(A^{-1}E) = \bar{\mu}(E).
\end{aligned}$$

By Proposition 4.13, the characters of $\mathbb{K}_A$ are of the form $\chi_{r,s}$ for $r \in \mathbb{R}^n$, $s \in \mathbb{Z}^n((B^*))$. Since $\mathbb{T} = \mathbb{K}_A/\varphi(\mathbb{Z}^n[A])$, by Lemma 4.10 there is an isomorphism $\widehat{\mathbb{T}} \simeq \mathrm{Ann}(\varphi(\mathbb{Z}^n[A]))$; this means that a character of $\mathbb{T}$ is of the form $\chi_{r,s}$ and satisfies $\chi_{r,s}(\varphi(z)) = 1$ for every $z \in \mathbb{Z}^n[A]$. Suppose there exists a character $\chi_{r,s} \in \widehat{\mathbb{T}}$ satisfying $\chi_{r,s} \circ \tau_A^k = \chi_{r,s}$ for some $k \geqslant 1$. In view of Lemma 4.14, it suffices to show that $\chi_{r,s}$ is constantly equal to 1. We have that, for every $(x,y) \in \mathbb{T}$,

$$\chi_{r,s} \circ \tau_A^k(x,y) = \chi_{r,s}(x,y),$$

meaning

$$\exp(2\pi i(\langle r, A^k x\rangle + S_s(A^k y))) = \exp(2\pi i(\langle r, x\rangle + S_s(y)))$$

with S_s defined in (4.8). Letting $y = 0$ implies

$$\langle r, A^k x\rangle = \langle r, x\rangle \mod \mathbb{Z}$$

and so, for every $x \in [0,1]^n$,

$$\langle {A^*}^k r - r, x\rangle \in \mathbb{Z},$$

which can only be true if ${A^*}^k r - r = 0$. Suppose $r \neq 0$; this implies that ${A^*}^k$ has 1 as an eigenvalue, and so therefore A has 1 as an eigenvalue, which contradicts the fact that A is expanding. Hence, $r = 0$.

Next, we prove that $\chi_{(0,s)} \in \widehat{\mathbb{T}}$ implies $s = 0$. For every $z \in \mathbb{Z}^n[A]$ we have $1 = \chi_{(0,s)}(\varphi(z)) = \chi_0(z)\chi_s(z) = \chi_s(z)$. Consider points of the form $z = A^l c = B^{-l}c$ with $l \geqslant 1$, $c \in \mathcal{E}$. Then $S_s(B^{-l}c) = \langle c, \{{B^*}^{-l}s\}_B^*\rangle \in \mathbb{Z}$. Recall that $\mathcal{E}$ contains a subset that spans a full rank lattice Θ. By the definition of dual lattice, this implies that $\{{B^*}^{-l}s\}_B^* \in \Theta^*$ for every $l \geqslant 1$. Suppose $s \neq 0$, then the leading B^*-adic coefficient satisfies $s_{\nu^*(s)} \in \Gamma \setminus B^*\Gamma$, because the coefficients live in $\mathcal{E}^* \subset \Gamma$ which is defined to be a residue set for $\Gamma[B^*]/B^*\Gamma[B^*]$. Moreover, we have that

$$\{{B^*}^{-l}s\}_B^* \in ({B^*}^{-l}\Gamma + \cdots + {B^*}^{-1}\Gamma + \Gamma) \setminus ({B^*}^{-l+1}\Gamma + \cdots + {B^*}^{-1}\Gamma + \Gamma).$$

Define inductively $\Lambda_0 := \Gamma$ and Λ_{l+1} the lattice spanned by Λ_l and $\{{B^*}^{-l}s\}_B^*$. Then $(\Lambda_l)_{l\geqslant 1}$ is a strictly nested sequence of lattices in $\mathbb{Q}^n$, all of which are contained in Θ^*. Given l, consider the entries of all the vectors of Λ_l expressed as irreducible fractions, and define m_l to be the maximum of the denominators of these fractions. It is clear that $m_l \leqslant m_{l+1}$. However, note that $(m_l)_{l\geqslant 1}$ does not stabilize: this would imply that, for some lattice Λ_l, there are infinitely many steps in which we can add a point and form a strictly larger lattice, while not increasing the bound on the denominators, which is not possible. Hence the sequence of denominators $(m_l)_{l\geqslant 1}$ tends to infinity. This contradicts the uniform discreteness of the lattice Θ^*. $\square$

We arrive at our final result. The proof is based on the one of [16, Theorem 1.1]. For completeness, we include it here. We recall the reader of the definition of multiple tiling given in Definition 3.5.

Theorem 4.16. *Let $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ be a rational self-affine tile. Then $\mathcal{F} + \varphi(\mathbb{Z}^n[A])$ is a multiple tiling of $\mathbb{K}_A$.*

Proof. Let $\bar{\mu}$ be the normalized Haar measure on the torus $\mathbb{T} = \mathbb{K}_A/\varphi(\mathbb{Z}^n[A])$. Consider the canonical projection $\pi : \mathbb{K}_A \to \mathbb{T}$, and define the function $\Phi : \mathbb{T} \to \mathbb{Z}_{\geqslant 0}$ as

$$\Phi(x,y) := |\pi^{-1}(x,y) \cap \mathcal{F}|,$$

meaning Φ counts the points on $\mathcal{F}$ that are congruent to (x,y) modulo $\varphi(\mathbb{Z}^n[A])$. Since $\mathcal{F}$ is compact, Φ is finite everywhere, hence it is well defined. Also, Φ is positive in a set of positive measure since $\mu(\mathcal{F}) > 0$. If we prove that $\Phi(x,y)$ is equal almost everywhere to some $k \in \mathbb{N}$, this implies the statement of the theorem: it means that almost every point of $\mathbb{K}_A$ gets covered by exactly k translates of $\mathcal{F}$ when translating via the set $\varphi(\mathbb{Z}^n[A])$. Note that Φ is constant almost everywhere if and only if there exists k such that every $S \subset \mathbb{T}$ satisfies

$$\int_S \Phi(x,y)\, d\bar{\mu}(x,y) = k\,\bar{\mu}(S). \tag{4.13}$$

To obtain this, we show first that Φ satisfies

$$\Phi(x,y) = \frac{1}{a} \sum_{(x',y') \in \tau_A^{-1}(x,y)} \Phi(x',y'), \tag{4.14}$$

where τ_A is the multiplication by A on the torus.

Using the set equation (2.14) of $\mathcal{F}$ and the fact that $\varphi(\mathcal{D}) \subset \varphi(\mathbb{Z}^n[A])$, we have

$$\begin{aligned} |\pi^{-1}(x,y) \cap A\mathcal{F}| &= \Big|\pi^{-1}(x,y) \bigcap \Big(\bigcup_{d\in\mathcal{D}} (\mathcal{F} + \varphi(d))\Big)\Big| \\ &= \Big|\bigcup_{d\in\mathcal{D}} (\pi^{-1}(x,y) - \varphi(d)) \cap \mathcal{F}\Big| \\ &= \sum_{d\in\mathcal{D}} |\pi^{-1}(x,y) \cap \mathcal{F}| = a\,\Phi(x,y). \end{aligned} \tag{4.15}$$

Recall that the index of the sublattice $A\varphi(\mathbb{Z}^n[A])$ of $\varphi(\mathbb{Z}^n[A])$ is a, which implies that $|\tau_A^{-1}(x,y)| = a$, and so

$$|\pi^{-1}(x,y) \cap A\mathcal{F}| = \frac{1}{a} \sum_{(x',y') \in \tau_A^{-1}(x,y)} |\pi^{-1}(\tau_A(x',y')) \cap A\mathcal{F}|. \tag{4.16}$$

Also, for any $(u,v) \in \mathbb{K}_A$, it holds that $|((u,v)+\varphi(\mathbb{Z}^n[A]))\cap A\mathcal{F}| = a|((u,v)+A\varphi(\mathbb{Z}^n[A]))\cap A\mathcal{F}|$, and hence, for each $(x',y') \in \tau_A^{-1}(x,y)$,

$$|\pi^{-1}(\tau_A(x',y')) \cap A\mathcal{F}| = a\,|A\pi^{-1}(x',y') \cap A\mathcal{F}| = a\,|\pi^{-1}(x',y') \cap \mathcal{F}| = a\,\Phi(x',y'). \tag{4.17}$$

Thus from (4.16) and (4.17) follows that

$$|\pi^{-1}(x,y) \cap A\mathcal{F}| = \sum_{(x',y') \in \tau_A^{-1}(x,y)} \Phi(x',y'). \tag{4.18}$$

Combining (4.15) and (4.18) yields (4.14). Applying (4.14) and doing the change of variables $(x,y) \mapsto \tau_A(x,y)$ we get

$$\begin{aligned}\int_S \Phi(x,y)d\bar{\mu}(x,y) &= \int_S \frac{1}{a} \sum_{(x',y') \in \tau_A^{-1}(x,y)} \Phi(x',y')\,d\bar{\mu}(x,y)\\ &= \int_{\tau_A^{-1}(S)} \frac{1}{a} \sum_{\tau_A(x',y')=\tau_A(x,y)} \Phi(x',y')\,a\,d\bar{\mu}(x,y)\\ &= \int_{\tau_A^{-1}(S)} \Phi(x,y)\,d\bar{\mu}(x,y).\end{aligned} \tag{4.19}$$

Because the map τ_A is ergodic by Lemma 4.15, iterating (4.19) $j \geqslant 1$ times and afterwards applying the ergodic theorem (see [6, Theorem 3.20]), yields

$$\begin{aligned}\int_S \Phi(x,y)d\bar{\mu}(x,y) &= \int_{\tau_A^{-j}(S)} \Phi(x,y)d\bar{\mu}(x,y)\\ &= \int_{\mathbb{T}} 1_s(\tau_A^j(x,y))\Phi(x,y)d\bar{\mu}(x,y)\\ &= \int_{\mathbb{T}} \Big(\frac{1}{N}\sum_{j=1}^{N} 1_s(\tau_A^j(x,y))\Big)\Phi(x,y)d\bar{\mu}(x,y)\\ &\xrightarrow[N\to\infty]{} \bar{\mu}(S)\int_{\mathbb{T}} \Phi(x,y)d\bar{\mu}(x,y) = k\,\bar{\mu}(S),\end{aligned} \tag{4.20}$$

and this concludes the proof. □

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