D I P L O M A R B E I T

Digital Expansions and the Distribution of Related Functions

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Chapter 1

Introduction and Notations

The distribution of the sum of digits function is very well known, especially for q-ary digital expansions (e.g. decimal, binary expansions). This work will present other types of digital expansions as well as generalizations of the sum of digits function and show that the distribution of these functions satisfies a central limit theorem.

More precisely, a problem posed by Drmota [8] is partially solved. Drmota studied the distribution of patterns in digital expansions related to specific finite recurrences and obtained a global and a local limit law. He wondered if corresponding laws hold for infinite linear recurrences related to the Parry expansion of a real number $\alpha > 1$. Now, the global limit law can be proved for (infinite) linear recurrences related to α-numbers (Section 5.5). In addition, the local limit law can be proved for a larger set of finite recurrences (those related to simple α-numbers, Section 5.6).

The Parry expansion (including α-numbers and simple α-numbers) and other digital expansions of non-negative integers and real numbers will be presented in Chapter 2. Chapter 3 deals with number systems in general integral domains, e.g. in the Gaussian integers. Chapter 4 recapitulates what is known about the distribution of the sum of digits function and related functions, whereas Chapter 5 contains the new results on this domain.

As in Drmota's work [8], adjacency matrices of generalized De Bruijn graphs are used. Drmota's conjecture that the characteristic polynomial of these graphs is in principle the characteristic polynomial of the underlying linear recurrence will be proved in Section 5.1.

Throughout the work the following notations will be used: $\mathbf{N}$ will denote the set of non-negative integers, $\mathbf{Z}$ the set of integers, $\mathbf{Q}$ the set of rational numbers, $\mathbf{R}$ the set of real numbers, and $\mathbf{C}$ the set of complex numbers.

$R[x]$ will denote the polynomial ring over a ring R, $\mathbf{Q}(\alpha)$ the extension field of $\mathbf{Q}$ generated by α and $N_{K/\mathbf{Q}}(\beta)$ the norm of $\beta \in K$ over $\mathbf{Q}$. The relation "$<$" will denote the lexicographic order for sequences.

Chapter 2

Digital Expansions

In this chapter some of the most important digital expansions of integers and real numbers will be presented, with the focus on Parry's α-expansion which will be needed in Chapter 5.

2.1 Definition

Let $G = (G_j)_{j\geq 0}$ be a strictly increasing sequence of integers with $G_0 = 1$. Then every non-negative integer n has a (unique) proper G-ary digital expansion

$$n = \sum_{j\geq 0} \epsilon_j(n) G_j$$

with integer digits $\epsilon_j(n) \geq 0$ provided that

$$\sum_{j=0}^{k} \epsilon_j(n) G_j < G_{k+1} \quad \forall k \in \mathbf{N} \tag{2.1}$$

A sequence $(\epsilon_j)_{j\geq 0}$ shall be called realizable if there exists an n with $\epsilon_j = \epsilon_j(n)\ \forall j \in \mathbf{N}$.

For real numbers let $G = (G_j)_{j\in\mathbf{Z}}$ be a strictly increasing sequence of real numbers with $G_0 = 1$, $\lim\limits_{j\to-\infty} G_j = 0$. Then every non-negative real number x has a (unique) proper G-ary digital expansion

$$x = \sum_{j\in\mathbf{Z}} \epsilon_j(x) G_j$$

with integer digits $\epsilon_j(x) \geq 0$ provided that

$$\sum_{j\leq k} \epsilon_j(x) G_j < G_{k+1} \quad \forall k \in \mathbf{Z} \tag{2.2}$$

A sequence $(\epsilon_j)_{j\in\mathbf{Z}}$ shall be called realizable if there exists an x with $\epsilon_j = \epsilon_j(x)\ \forall j \in \mathbf{Z}$.

2.2 q-ary Expansions

q-ary expansions are the classical cases $G_j = q^j$ (with $q > 1$ an integer). The digits ϵ_j are in $\{0, 1, \ldots, q-1\}$. For integers j runs in $\mathbf{N}$, for real numbers in $\mathbf{Z}$.

A sequence is realizable iff $\epsilon_j \neq 0$ only for a finite number of $j \geq 0$ and (for real numbers) its "negative tail" is not $(q-1, q-1, \ldots)$, i.e. it exists no $k < 0$ such that $\epsilon_j = q-1\ \forall j \leq k$.

2.3 Cantor's Expansion

In a Cantor's expansion the sequence G is defined as

$$G_j = \begin{cases} q_1 q_2 \ldots q_j & \text{for } j > 0 \\ 1 & \text{for } j = 0 \\ \frac{1}{q_{-1}q_{-2}\cdots q_{-j}} & \text{for } j < 0 \end{cases}$$

where $q_j \in \mathbf{N}, q_j \geq 2$. Then

$$\epsilon_j \in \begin{cases} \{0, 1, \ldots, q_{j+1} - 1\} & \text{for } j \geq 0 \\ \{0, 1, \ldots, q_j - 1\} & \text{for } j < 0 \end{cases}.$$

Like in q-ary expansions j runs in $\mathbf{N}$ for integers and in $\mathbf{Z}$ for real numbers and a sequence is realizable iff $\epsilon_j \neq 0$ only for a finite number of $j \geq 0$ and (for real numbers) it exists no $k < 0$ so that $\epsilon_j = q_j - 1\ \forall j \leq k$.

In q-ary expansions and Cantor's expansions the digits are independent.

2.4 Parry's α-Expansion

Instead of choosing $q \in \mathbf{N}$ like in q-ary expansions we choose $\alpha \in \mathbf{R},\ \alpha > 1$.

Rényi [36] proved that every non-negative real number x has an α-expansion

$$x = \epsilon_0(x) + \frac{\epsilon_{-1}(x)}{\alpha} + \frac{\epsilon_{-2}(x)}{\alpha^2} + \cdots \tag{2.3}$$

where $\epsilon_0(x) = [x],\ \epsilon_{-1}(x) = [\alpha\langle x\rangle],\ \epsilon_{-2}(x) = [\alpha\langle\alpha\langle x\rangle\rangle]$ etc. and $[x]$ denotes the integral part, $\langle x\rangle$ the fractional part of x. In the sequel the sequence $(\epsilon_0(x), \epsilon_{-1}(x), \ldots)$ will be called α-expansion as well.

Let the α-expansion of α be

$$\alpha = a_1 + \frac{a_2}{\alpha} + \frac{a_3}{\alpha^2} + \cdots . \tag{2.4}$$

Parry [34] showed the following relation between the α-expansions of a real number x and of α:

$$(\epsilon_k(x), \epsilon_{k-1}(x), \ldots) < (a_1, a_2, \ldots) \ \forall k < 0 \tag{2.5}$$

and, in particular,

$$(a_k, a_{k+1}, \ldots) < (a_1, a_2, \ldots) \ \forall k > 1. \tag{2.6}$$

("$<$" denotes the lexicographic order.)

Conversely, if a sequence $(a_1, a_2, \ldots)$ satisfies the relation (2.6), we have a real number α with α-expansion $(a_1, a_2, \ldots)$.

Those α which have recurrent "tails" in their α-expansions, i.e. $a_{j+m} = a_j \ \forall j > n$ for some integers n and m, are called α-numbers. The α-numbers which have a finite α-expansion are called simple α-numbers.

With $G_j := \alpha^j \ \forall j \leq 0$, $(\epsilon_j(x))_{j \leq 0}$ constitutes the G-ary digital expansion of $x \in [0, 1)$.

If we set $G_j := \alpha^j \ \forall j \in \mathbf{Z}$, the relation (2.5) is valid for all real numbers x and for all integers k. (To see this, it suffices to look at the digital expansion of x/α^M with M so that $x < \alpha^M$.)

A sequence is realizable iff $\epsilon_j > 0$ only for a finite number of $j \geq 0$, (2.5) holds and, if α is a simple α-number with α-expansion $(a_1, \ldots, a_q)$, the sequence $(\epsilon_k, \epsilon_{k-1}, \ldots)$ does not coincide with $(c_j)_{j \geq 1}$ for a $k \in \mathbf{Z}$:

$$c_j = \begin{cases} a_k & \text{if } j \equiv k \not\equiv 0 \pmod{q}, \ 0 < k < q \\ a_q - 1 & \text{if } j \equiv 0 \pmod{q} \end{cases} .$$

Clearly α lies between a_1 and $a_1 + 1$.

If α is a simple α-number with α-expansion

$$\alpha = a_1 + \frac{a_2}{\alpha} + \cdots + \frac{a_r}{\alpha^{r-1}},$$

it is a root of the polynomial

$$x^r - a_1 x^{r-1} - \ldots - a_{r-1} x - a_r$$

which is called characteristic polynomial of α.

If α is a non-simple α-number with n and m as above, it is a root of the polynomial

$$(x^{n+m}-a_1x^{n+m-1}-\cdots-a_{n+m-1}x-a_{n+m})-(x^n-a_1x^{n-1}-\cdots-a_{n-1}x-a_n).$$

This polynomial is called characteristic polynomial, if n and m are minimal with this property.

The digits in these expansions are dependent.

Example 2.1. Let α be the (only) positive root of $x = 1 + \frac{1}{x}$ ($\alpha \approx 1.618$). Clearly α is a simple α-number with

$$(a_1, a_2, \ldots) = (1, 1, 0, 0, 0, \ldots).$$

Hence the realizable sequences are exactly those where $\epsilon_j \in \{0, 1\}$ and no two subsequent digits are 1.

Example 2.2. Let α be the positive root of $x^3 - 2x^2 - 1$ ($\alpha \approx 2.206$). Then

$$(a_1, a_2, \ldots) = (2, 0, 1, 0, 0, 0, \ldots).$$

The possible subblocks of length 3 of a digital expansion are therefore

$$(0,0,0),(0,0,1),(0,0,2),(0,1,0),(0,1,1),(0,1,2),(0,2,0),(1,0,0),$$

$$(1,0,1),(1,0,2),(1,1,0),(1,1,1),(1,1,2),(1,2,0),(2,0,0).$$

Example 2.3. Let α be the root satisfying $\alpha > 1$ of

$$x^3 - 2x^2 - 3x + 1 = (x^3 - 2x^2 - 2x - 1) - (x - 2) \qquad (\alpha \approx 2.912).$$

Then α is a non-simple α-number and

$$(a_1, a_2, \ldots) = (2, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots).$$

2.5 Expansions Related to Parry's α-Expansion

The above sequence G cannot be used for digital expansions of integers since the G_j are in general no integers, but we can use the α-expansion of α to build a sequence G with a recurrence. The recurrence can be infinite or finite:

If infinitely many $a_j \neq 0$,

$$G_j = \sum_{i=1}^{j} a_i G_{j-i} + 1 \;\forall j > 0 \tag{2.7}$$

if α is a simple α-number, i.e. $a_r \neq 0,\ a_j = 0\ \forall j > r$

$$G_j = \begin{cases} \sum\limits_{i=1}^{j} a_i G_{i-j} + 1 & \text{for } j < r \\ \sum\limits_{i=1}^{r} a_i G_{j-i} & \text{for } j \geq r \end{cases} \tag{2.8}$$

Proposition 2.1 shows that the realizable sequences are exactly those which appear in Parry's α-expansions.

Proposition 2.1. *Let G be defined as in (2.7) or (2.8) and $(a_j)_{j\geq 1}$ satisfy (2.6). A sequence $(\epsilon_j)_{j\in\mathbf{Z}}$ with $\epsilon_j = 0\ \forall j < 0$ is realizable iff*

1. $(\epsilon_{j-1}, \epsilon_{j-2}, \ldots) < (a_1, a_2, \ldots)\ \forall j \geq 1$
2. $\epsilon_j \neq 0$ *only for a finite number of* $j \in \mathbf{Z}$.

Proof. If 1. and 2. hold, we have to show $\sum\limits_{j=0}^{m} \epsilon_j G_j < G_{m+1}\ \forall m \in \mathbf{N}$.

To do this, we define $M := \max\{j \in \mathbf{N} : \epsilon_j \neq 0\}$ and use induction on M. The hypothesis is valid for $M = 0$ and we conclude from $M - 1$ to M:

If we set $\epsilon_M = 0$, we have

$$\sum_{j=0}^{m} \epsilon_j G_j < G_{m+1}\ \forall m \leq M - 1$$

because of the hypothesis. Hence it suffices to show

$$\sum_{j=0}^{M} \epsilon_j G_j < G_{M+1}.$$

If $\epsilon_M < a_1$, then

$$\sum_{j=0}^{M} \epsilon_j G_j \leq (a_1 - 1)G_M + \sum_{j=0}^{M-1} \epsilon_j G_j < a_1 G_M \leq G_{M+1}.$$

If $\epsilon_k < a_{M-k+1}$ and $(\epsilon_M, \epsilon_{M-1}, \ldots, \epsilon_{k+1}) = (a_1, a_2, \ldots, a_{M-k})$, then

$$\sum_{j=0}^{M} \epsilon_j G_j = \sum_{i=0}^{M} \epsilon_{M-i} G_{M-i} = \sum_{i=0}^{M-k-1} a_i G_{M-i} + \sum_{j=0}^{k} \epsilon_j G_j < \sum_{i=0}^{M-k} a_i G_{M-i} \leq G_{M+1}.$$

To show the other direction, we use the following lemma:

Lemma 2.1. *Let G be defined as in (2.7) or (2.8) and $(a_j)_{j\geq 1}$ satisfy (2.6). Then*

$$G_{j-k} > \sum_{i=k+1}^{j} a_i G_{j-i} \ \forall j \geq 1, \ 1 \leq k < j. \tag{2.9}$$

Proof. With the part of the Theorem 2.1 which is already shown and (2.6), $\sum_{i=k+1}^{j} a_i G_{j-i}$ is a digital expansion and therefore

$$\sum_{i=k+1}^{j} a_i G_{j-i} < G_{j-k}.$$

□

If $(\epsilon_j)_{j\in\mathbf{Z}}$ is realizable, then 2. is obvious. To prove 1., we assume ">" in 1. Then we have some $k \geq 1$ so that

$$\epsilon_{j-k} > a_k, \ (\epsilon_{j-1}, \epsilon_{j-2}, \ldots, \epsilon_{j-k+1}) = (a_1, a_2, \ldots, a_{k-1})$$

and

$$\sum_{i=1}^{k} \epsilon_{j-i} G_{j-i} \geq \sum_{i=1}^{k} a_i G_{j-i} + G_{j-k} > \sum_{i=1}^{j} a_i G_{j-i} = G_j.$$

Hence (2.1) is violated and the sequence $(\epsilon_j)_{j\in\mathbf{Z}}$ is not realizable.

If "=" holds in 1., i.e. $(\epsilon_{j-1}, \epsilon_{j-2}, \ldots) = (a_1, a_2, \ldots)$, we are in the finite case. Then (2.1) is violated because of

$$\sum_{i=1}^{j} \epsilon_{j-i} G_{j-i} = \sum_{i=1}^{r} a_i G_{j-i} = G_j$$

and the sequence $(\epsilon_j)_{j\in\mathbf{Z}}$ is not realizable. □

Proposition 2.2 shows together with Lemma 2.1 that (2.9) is almost equivalent to (2.6).

Proposition 2.2. *Let G be defined as in (2.7) or (2.8). If (2.9) holds, then*

$$(a_k, a_{k+1}, \ldots) \leq (a_1, a_2, \ldots) \ \forall k > 1 \tag{2.10}$$

If G is defined by a finite recurrence, (2.10) is equivalent to (2.6). If G is defined by an infinite recurrence, it can always be built with a sequence $(a'_j)_{j\geq 1}$ which satisfies (2.6).

Proof. We assume that ">" in (2.10) for some k. Then

$$a_j > a_{j-k},\ (a_{k+1}, a_{k+2}, \dots, a_{j-1}) = (a_1, a_2, \dots, a_{j-k-1})$$

for some $j > k$. For all $l \in \mathbf{N}$ with $l \geq j$

$$G_{l-k} = \underbrace{a_1 G_{l-k-1} + \cdots + a_{j-k-1} G_{l-j+1}}_{C} + a_{j-k} G_{l-j} + \underbrace{a_{j-k+1} G_{l-j-1} + \cdots + a_r G_{l-k-r}}_{A}$$

and

$$G_{l-k} > \underbrace{a_{k+1} G_{l-k-1} + \cdots + a_{j-1} G_{l-j+1}}_{C} + a_j G_{l-j} + \underbrace{a_{j+1} G_{l-j-1} + \cdots + a_r G_{l-r}}_{B}.$$

This implies

$$0 \leq A < G_{l-j},\ 0 \leq B < G_{l-j},\ A - B > (a_j - a_{j-k}) G_{l-j} \geq G_{l-j}$$

which is a contradiction.

In the finite case, "=" is impossible in (2.10) since $a_{r+j} = 0$ and $a_r \neq 0$. Therefore (2.10) and (2.6) are equivalent.

If $(a_j)_{j\geq 1}$ is periodic, i.e. $(a_k, a_{k+1}, \dots) = (a_1, a_2, \dots)$, then the sequence G' built with $(a'_j)_{j\geq 1}$:

$$(a'_1, a'_2, \dots, a'_{k-1}) = (a_1, a_2, \dots, a_{k-2}, a_{k-1} + 1),\ a'_j = 0\ \forall j \geq k$$

is identic with G, because the realizable sequences are identic. If k is minimal with this property, $(a'_j)_{j\geq 1}$ satisfies (2.6). To show this, we assume

$$(a'_j, a'_{j+1}, \dots, a'_{k-1}, 0, \dots, 0) > (a'_1, a'_2, \dots, a'_{k-1})$$

("=" is impossible). With (2.10) we have

$$(a_j, a_{j+1}, \dots, a_{k-1}) = (a_1, a_2, \dots, a_{k-j})$$

and

$$(a_j, a_{j+1}, \dots) = (a_1, a_2, \dots, a_{k-j}, a_1, a_2, \dots) \geq (a_1, a_2, \dots)$$

which is a contradiction to (2.10) or the minimality of k. □

Example 2.4. For the finite recurrence with $(a_1, a_2) = (1, 1)$, the elements of the resulting sequence G are the Fibonacci numbers $1, 2, 3, 5, 8, \dots$

Example 2.5. For the finite recurrence with $(a_1, a_2) = (2, 0, 1)$, the resulting sequence G is $(1, 3, 7, 15, 33, 73, 161, \ldots)$. The digital expansion of 160 is

$$(\epsilon_5, \ldots, \epsilon_0) = (2, 0, 0, 2, 0, 0).$$

Example 2.6. For the infinite recurrence related to the non-simple α-number of Example 2.3, the resulting sequence G is $(1, 3, 7, 18, 56, 163, \ldots)$. The digital expansion of 162 is

$$(\epsilon_4, \ldots, \epsilon_0) = (2, 2, 1, 2, 1).$$

Chapter 3

Number Systems in Integral Domains

The concept of digital expansions of integers and real numbers can be generalized to the concept of number systems in integral domains. In this chapter we will give conditions for the existence of number systems in an integral domain and focus on quadratic fields over **Q** where all number systems are known. For the sake of shortness the proofs of the theorems will be omitted. They can be found in the cited papers.

3.1 Definition

Let R be an integral domain, $\alpha \in R$, $\mathcal{N} = \{n_1, n_2, \ldots, n_m\} \subset \mathbf{Z}$. $\{\alpha, \mathcal{N}\}$ is called a number system in R if any $\gamma \in R$ has a unique representation

$$\gamma = c_0 + c_1\alpha + \cdots + c_h\alpha^h : \; c_j \in \mathcal{N} \; \forall j \in \{0, 1, \ldots, h\}, \; c_h \neq 0 \text{ if } h \neq 0 \quad (3.1)$$

If $\mathcal{N} = \mathcal{N}_0 = \{0, 1, \ldots, m\}$ for some $m \geq 1$, then $\{\alpha, \mathcal{N}\}$ is called canonical number system (CNS).

α is called base and $\mathcal{N}$ is called set of digits of $\{\alpha, \mathcal{N}\}$.

3.2 Existence and Determination

The question of determining all the CNS in some special algebraic number fields has been raised by Kátai and Szabó [26] and completely solved for Gaussian integers:

Theorem 3.1. *$\{\alpha, \mathcal{N}_0\}$ is a CNS in the ring of Gaussian integers $\mathbf{Z}[i]$ iff $Re(\alpha) < 0,\ Im(\alpha) = \pm 1,\ \mathcal{N}_0 = \{0, 1, \ldots, |\alpha|^2 - 1\}$.*

For the other imaginary quadratic fields and real quadratic fields this has be done by Kátai and Kovács [24, 25]:

Theorem 3.2. *Let $N \geq 2,\ -N \not\equiv 1 \pmod 4$. $\{\alpha, \mathcal{N}_0\}$ is a CNS in $\mathbf{Q}(i\sqrt{N})$ iff*

$$\alpha = A \pm i\sqrt{N},\ 0 \leq -2A \leq A^2 + N \geq 2,\ A \text{ is integer.}$$

Let $N \geq 2,\ -N \equiv 1 \pmod 4$. $\{\alpha, \mathcal{N}_0\}$ is a CNS in $\mathbf{Q}(i\sqrt{N})$ iff

$$\alpha = \frac{1}{2}(B \pm i\sqrt{N}),\ -1 \leq -B \leq \frac{1}{4}(B^2 + N) \geq 2,\ B \text{ is an odd integer.}$$

Theorem 3.3. *Let $N \not\equiv 1 \pmod 4$. $\{\alpha, \mathcal{N}_0\}$ is a CNS in $\mathbf{Q}(\sqrt{N})$ iff*

$$\alpha = A \pm \sqrt{N},\ 0 \leq -2A \leq A^2 - N \geq 2,\ A \text{ is integer.}$$

Let $N \equiv 1 \pmod 4$. $\{\alpha, \mathcal{N}_0\}$ is a CNS in $\mathbf{Q}(\sqrt{N})$ iff

$$\alpha = \frac{1}{2}(B \pm \sqrt{N}),\ 0 < -B \leq \frac{1}{4}(B^2 - N) \geq 2,\ B \text{ is an odd integer.}$$

These CNS can be used to represent all complex numbers and real numbers respectively:

Theorem 3.4. *If $\{\alpha, \mathcal{N}_0\}$ is a CNS in $\mathbf{Q}(i\sqrt{N})$, then every complex number z can be written as*

$$z = \sum_{i=k}^{-\infty} a_i \alpha^i \qquad (a_i \in \mathcal{N}_0\ \forall i \in \{k, k-1, \ldots\}).$$

If $\{\alpha, \mathcal{N}_0\}$ is a CNS in $\mathbf{Q}(\sqrt{N})$, then every real number x can be written as

$$x = \sum_{i=k}^{-\infty} a_i \alpha^i \qquad (a_i \in \mathcal{N}_0\ \forall i \in \{k, k-1, \ldots\}).$$

Kovács [30] solved the problem of the existence of CNS for algebraic number fields of higher degree:

Theorem 3.5. *Let $\mathbf{Q}(\vartheta)$ be an n^{th} degree extension of $\mathbf{Q}$, $n \geq 3$. In $\mathbf{Q}[\vartheta]$ there exists CNS iff there exists $\alpha \in \mathbf{Q}[\vartheta]$, such that $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is an integer-basis in $\mathbf{Q}(\vartheta)$.*

Finally Kovács and Pethő [31] gave a necessary and sufficient condition for the existence of number systems in an integral domain R:

Theorem 3.6. *There exists a number system in R iff*

1. $R = \mathbf{Z}[\alpha]$ *for an* α*, algebraic over* $\mathbf{Q}$*, if* $charR = 0$
2. $R = F_p[x]$*, where* F_p *denotes the finite field with* p *elements and* x *is transcendental over* F_p*, if* $charR = p$*,* p *is a prime*

They described the number systems in $R = F_p[x]$:

Theorem 3.7. $\{\alpha, \mathcal{N}\}$ *is a number system in* $F_p[x]$ *iff* $\alpha = a_0 + a_1 x$*, where* $a_0,\ a_1 \in F_p,\ a_1 \neq 0$ *and* $\mathcal{N} = \mathcal{N}_0 = \{0, 1, \ldots, p-1\}$*.*

and in $R = \mathbf{Z}[\alpha]$, where $K = \mathbf{Q}(\alpha)$ is of degree n and $\gamma = \gamma^{(1)}, \ldots, \gamma^{(n)}$ denote the conjugates of $\gamma \in K$:

Theorem 3.8. *Let* α *be an algebraic integer over* $\mathbf{Q}$*. Let* $\beta \in \mathbf{Z}[\alpha]$*,* $\mathcal{N} \subset \mathbf{Z}$ *and put* $A := \max_{a \in \mathcal{N}} |a|$*.* $\{\beta, \mathcal{N}\}$ *is a number system in* $\mathbf{Z}[\alpha]$ *iff*

1. $|\beta^{(j)}| > 1\ \forall j \in \{1, 2, \ldots, n\}$
2. $\mathcal{N}$ *is a complete residue system* $\bmod |N_{K/\mathbf{Q}}(\beta)|$ *containing* 0
3. $\alpha \in \mathbf{Z}[\beta]$
4. *all* $\gamma \in \mathbf{Z}[\alpha]$ *with*
$$|\gamma^{(j)}| \leq \frac{A}{|\beta^{(j)}| - 1}\ \forall j \in \{1, \ldots, n\}$$
have a representation (3.1) in $\{\beta, \mathcal{N}\}$

Kovács and Pethő [31] also gave a computational algorithm to determine all CNS of orders of algebraic number fields which is based on the following theorem, where α is an algebraic integer and $\mathcal{N}_0(\alpha) = \{0, 1, \ldots, |N_{K/\mathbf{Q}}(\alpha)| - 1\}$:

Theorem 3.9. *Let* $\mathcal{O}$ *be an order in the algebraic number field* K*. There exist* $\alpha_1, \ldots, \alpha_t \in \mathcal{O}$*;* $n_1, \ldots, n_t \in \mathbf{Z}$*,* $N_1, \ldots, N_t$ *finite subsets of* $\mathbf{Z}$*, which are all effectively computable, such that* $\{\alpha, \mathcal{N}_0(\alpha)\}$ *is a CNS in* $\mathcal{O}$*, iff* $\alpha = \alpha_i - h$ *for some integers* $i,\ h$ *with* $1 \leq i \leq t$ *and either* $h \geq n_i$ *or* $h \in N_i$*.*

Chapter 4

The Distribution of the Sum of Digits Function and Related Functions

The intention of this chapter is to present some of the known facts about the sum of digits function and other functions which depend on the digital expansion like q-additive functions. As in Chapter 3, the proofs are omitted for the sake of shortness and can be found in the cited papers.

4.1 q-Additive Functions on Integers

Let $q > 1$ be a given integer. A real-valued fucnction f, defined on the non-negative integers, is called q-additive if $f(0) = 0$ and

$$f(n) = \sum_{j \geq 0} f(\epsilon_j(n) q^j)$$

where $(\epsilon_j(n))_{j \geq 0}$ is the q-ary digital expansion of n. A special q-additive function is the sum of digits function

$$s_q(n) = \sum_{j \geq 0} \epsilon_j(n).$$

The statistical behaviour of the sum of digits function and q-additive functions has been well studied by several authors.

One of the first significant results was obtained by Delange [7] in 1975 who computed the average of $s_q(n)$:

$$\frac{1}{N}\sum_{n<N} s_q(n) = \frac{q-1}{2}\log_q N + \gamma(\log_q N),$$

where γ is a continuous, nowhere differentiable and periodic function with period 1. Other asymptotic and exact formulas are due to Bush [4], Bellman and Shapiro [3], Tenenbaum [37] and Trollope [39]. Formulas for digital expansions related to Parry's α-expansion are due to Pethő and Tichy [35] and Grabner and Tichy [22, 23].

Kirschenhofer [28] and Kennedy and Cooper [27] obtained a formula for the variance

$$\frac{1}{N}\sum_{n<N} s_q^2(n) - \frac{1}{N^2}\left(\sum_{n<N} s_q(n)\right)^2 = \frac{q^2-1}{12}\log_q N + \gamma(\log_q N)$$

with a continuous fluctuation γ of period 1. Grabner, Kirschenhofer, Prodinger and Tichy [21] extended this result (dth moment for the case $q = 2$) and showed

$$\frac{1}{N}\sum_{n<N} s_q^d(n) = \frac{1}{2^d}(log_2 N)^d + \sum_{i=0}^{d-1}(log_2 N)^i\gamma_i(log_2 N),$$

where the γ_i are again continuous fluctuations of period 1. Other formulas for higher moments can be found in Coquet [6] and in Dumont and Thomas [13].

The most general result concerning the mean value of q-additive functions is due to Manstavičius [32]. Let $E := \{0, 1, \ldots, q-1\}$,

$$m_{k,q} := \frac{1}{q}\sum_{c\in E} f(cq^k), \qquad m_{2;k,q}^2 := \frac{1}{q}\sum_{c\in E} f^2(cq^k),$$

and

$$M_q(x) := \sum_{k=0}^{N} m_{k,q}, \qquad B_q^2(x) = \sum_{k=0}^{N} m_{2;k,q}^2$$

with $N = [\log_q x]$. Then

$$\frac{1}{x}\sum_{n<x}(f(n) - M_q(x))^2 \leq cB_q^2(x),$$

which implies

$$\frac{1}{x}\sum_{n<x} f(n) = M_q(x) + \mathcal{O}(B_q(x)).$$

There exist distributional results for q-additive functions which use the higher moments and Fréchet-Shohat's theorem. The most general theorem known concerning a central limit theorem is due to Manstavičius [32]: Suppose that, as $x \to \infty$,

$$\max_{cq^j < x} |f(cq^j)| = o(B_q(x))$$

and that $D_q(x) \to \infty$, where

$$D_q^2(x) = \sum_{k=0}^{N} \sigma_{k,q}^2 \text{ and } \sigma_{k,q}^2 := \frac{1}{q} \sum_{c \in E} f^2(cq^k) - m_{k,q}^2.$$

Then, as $x \to \infty$

$$\frac{1}{x} \# \left\{ n < x | \frac{f(n) - M_q(x^r)}{D_q(x^r)} < y \right\} \to \Phi(y),$$

where Φ is the normal distribution function.

Bassily and Kátai [2] extended this on polynomial sequences:

Theorem 4.1. *Let f be a q-additive funciton such that $f(cq^j) = \mathcal{O}(1)$ as $j \to \infty$ and $c \in E$. Assume that $\frac{D_q(x)}{(\log x)^{1/3}} \to \infty$ as $x \to \infty$ and let $P(x)$ be a polynomial with integer coefficients, degree r, and positive leading term. Then, as $x \to \infty$,*

$$\frac{1}{x} \# \left\{ n < x | \frac{f(P(n)) - M_q(x^r)}{D_q(x^r)} < y \right\} \to \Phi(y).$$

Drmota [9] studied the joint distribution of q_l-additive functions $f_l(n)$ (if $q_1, q_2, \ldots, q_d > 1$ are pairwisely coprime integers) and showed that the $1/3$ in the above theorem can be replaced by $\eta > 0$.

Similar distribution results for the sum of digits function of number systems related to substitution automata were considered by Dumont and Thomas [14].

Drmota and Gajdosik [11] used a generating function approach to show that the sum of digits function for digital expansions related to Parry's α-expansion satisfies a central limit theorem.

Finally several authors studied subblocks of digital expansions and functions depending on them, e.g. Kirschenhofer [29], Barat, Tichy, and Tijdeman [1] and Cateland [5]. Drmota [8] showed that these functions satisfy a central limit theorem for expansions related to certain finite recurrences. We will extend this result on expansions related to Parry's α-numbers in Chapter 5.

4.2 b-Additive Functions on Gaussian Integers

Grabner, Kirschenhofer and Prodinger [20] and Thuswaldner [38] generalized Delange's result to canonical number systems in the Gaussian integers and to arbitrary canonical number systems respectively. For the Gaussian integers we have

$$\frac{1}{N\pi + \mathcal{O}(\sqrt{N})} \sum_{|z|^2<N} (s_b(z))^d =$$

$$= \left(\frac{|b|^2-1}{2}\right)^d \log_{|b|^2}^d N + \sum_{j=0}^{d-1} \log_{|b|^2}^j N \Phi_j(\log_{|b|^2} N) + \mathcal{O}\left(\sqrt{N} \log_{|b|^2}^d N\right),$$

where $\Phi_0, \ldots, \Phi_{d-1}$ are continuous periodic fluctuations of period 1, b the base of a canonical number system in $\mathbf{Z}[i]$ and $s_b(z)$ the sum of digits function.

A treatment of the higher moments in the general case was done by Gittenberger and Thuswaldner [18].

Gittenberger and Thuswaldner [19] extended the result of Bassily and Kátai [2] to b-additive functions

$$\left(f(0) = 0,\ f(\gamma) = \sum_{j\geq 0} f(c_j(\gamma) b^j) \text{ for } \gamma = \sum_{j\geq 0} a_j(\gamma) b^j \ (a_j(\gamma) \in \mathcal{N}_0) \right).$$

Theorem 4.2. *Let f be a b-additive function such that $f(cb^j) = \mathcal{O}(1)$ as $j \to \infty$ and $c \in \mathcal{N}_0$. Furthermore let*

$$m_k := \frac{1}{|b|^2} \sum_{c\in\mathcal{N}_0} f(cb^k), \qquad \sigma_k^2 := \frac{1}{|b|^2} \sum_{c\in\mathcal{N}_0} f^2(cb^k) - m_k^2,$$

and

$$M(x) := \sum_{k=0}^{L} m_k, \qquad D^2(x) = \sum_{k=0}^{L} \sigma_k^2$$

with $L = [\log_{|b|} x]$. Assume that $\frac{D(x)}{(\log x)^{1/3}} \to \infty$ as $x \to \infty$ and let $P(x)$ be a polynomial with integer coefficients and degree r. Then, as $x \to \infty$,

$$\frac{1}{\#\{z||z|^2 < x\}} \# \left\{ |z|^2 < x | \frac{f(P(z)) - M(x^r)}{D(x^r)} < y \right\} \to \Phi(y),$$

where Φ is the normal distribution function.

Chapter 5

The Distribution of Patterns in Expansions Related to α-Numbers

The aim of this chapter is to present some new results on the distribution of functions F depending on subblocks of digital expansions related to α-numbers α (see Sections 2.4, 2.5). We will prove asymptotic normality of the distribution of X_N, which will be defined in the sequel, and derive a local limit law if F attains only integer values and α is a simple α-number. The methods are adapted from [8], [16], [11] and [12].

Let α be an α-number, G defined as in Section 2.5 and

$$\mathcal{B}_L = \{(\epsilon_{L-1}(n), \epsilon_{L-2}(n), \ldots, \epsilon_0(n)) : n < G_L\}$$

be the set of blocks $B \subseteq \{0, 1, \ldots, a_1\}^L$ of length L which actually occur in digital expansions. In the q-ary case we trivially have

$$\mathcal{B}_L = \{0, 1, \ldots, q-1\}^L.$$

Let $F : \mathcal{B}_{L+1} \to \mathbf{R}$ be any given function (for some $L \geq 0$) with $F(0, 0, \ldots, 0) = 0$.

Furthermore, set

$$s_F(n) = \sum_{j \geq 0} F\left(\epsilon_{j+L}(n), \epsilon_{j+L-1}(n), \ldots, \epsilon_j(n)\right).$$

This means that we consider a weighted sum over all subsequent digital patterns of length $L+1$ of the digital expansion of n. For example, for

$L = 0$ and $F(\epsilon) = \epsilon$ we just obtain the sum-of-digits function, or if $L = 1$ and $F(\epsilon, \eta) = 1 - \delta_{\epsilon,\eta}$ ($\delta_{x,y}$ denoting the Kronecker delta) then $s_F(n)$ is just counting the number of times that a digit is different from the preceding one etc.

In order to get an insight into the distribution of $s_F(n)$, it is convenient to consider a related sequence of random variables X_N, $N \geq 1$, defined by

$$\mathbf{Pr}[X_N \leq x] = \frac{1}{N} |\{n < N : s_F(n) \leq x\}|$$

Expected value and variance of X_N are given by

$$\mathbf{E}X_N = \frac{1}{N} \sum_{n<N} s_F(n) \quad \text{and by} \quad \mathbf{V}X_N = \frac{1}{N} \sum_{n<N} (s_F(n) - \mathbf{E}X_N)^2 \tag{5.1}$$

We introduce the function

$$c_N(z) = \sum_{n<N} z^{s_F(n)}$$

and consider for any block $B = (\eta_1, \ldots, \eta_L) \in \mathcal{B}_L$ the functions

$$a_j^B(z) := \sum_{n<G_j, (\epsilon_{j-1}(n), \ldots, \epsilon_{j-L}(n)) = B} z^{s_F(n)}.$$

Then

$$\sum_{B \in \mathcal{B}_L} a_j^B(z) = c_N(z).$$

In order to obtain recurrent relations for the functions a_j^B we need the following notation:

For $B = (\eta_1, \ldots, \eta_L) \in \mathcal{B}_L$ let $B' = (\eta_2, \ldots, \eta_L)$ denote the block consisting of the last $L-1$ elements of B and η_B the first element η_1, i.e. $B = (\eta_B, B')$. (Similarly $'B = (\eta_1, \ldots, \eta_{L-1})$.) Furthermore, for $(\epsilon, B) = (\epsilon, \eta_1, \ldots, \eta_L) \in \mathcal{B}_{L+1}$ set

$$\begin{aligned} \kappa(\epsilon, B) \quad &= \quad \sum_{i=0}^{L-1} (F(0, \ldots, 0, \epsilon, \eta_1, \ldots, \eta_{L-i}) - F(0, 0, \ldots, 0, \eta_1, \ldots, \eta_{L-i})) + \\ &\qquad F(0, 0, \ldots, 0, \epsilon). \end{aligned}$$

Note that $\kappa(0, B) = 0$.

5.1 Simple α-Numbers

In the case of simple α-numbers, the coefficients of the α-expansion of α are $(a_1, a_2, \ldots, a_r, 0, 0, \ldots)$ and the G_j are determined by the finite recurrence (2.8).

Without loss of generality we may assume that $L \ge r-1$. (If we are only interested in $L+1$ subsequent digits with $L < r-1$, then we consider a new function $\tilde{F} : \mathcal{B}_r \to \mathbf{R}$ that does not depend on the first $(r-L-1)$ digits.)

Lemma 5.1. *The functions $a_j^B(z)$, $j > L$, are recursively given by*

$$a_j^B(z) = \sum_{C \in \mathcal{B}_L:\ 'C = B',\ (\eta_B, C) \in \mathcal{B}_{L+1}} a_{j-1}^C(z) z^{\kappa(\eta_B, C)}.$$

Proof. The set

$$\{n < G_j : (\epsilon_{j-1}(n), \ldots, \epsilon_{j-L}(n)) = B\}$$

is divided into subsets of the form

$$\{n < G_j : \epsilon_{j-1}(n) = \eta_B,\ (\epsilon_{j-2}(n), \epsilon_{j-3}(n), \ldots, \epsilon_{j-L-1}(n)) = (B', \epsilon) = C\}$$
$$= \{n < G_{j-1} : (\epsilon_{j-2}(n), \epsilon_{j-3}(n), \ldots, \epsilon_{j-L-1}(n)) = C\} + \eta_B G_{j-1}.$$

Since $\{(\eta_B, C) : C \in \mathcal{B}_L, 'C = B'\} \supseteq \mathcal{B}_{L+1}$, we cover all possible cases.

Furthermore, for $n < G_{j-1}$ with $(\epsilon_{j-2}(n), \epsilon_{j-3}(n), \ldots, \epsilon_{j-L-1}(n)) = C$ we have

$$s_F(n + \eta_B G_{j-1}) = s_F(n) + \kappa(\eta_B, C).$$

□

Corollary 5.1. *The vector $\mathbf{a}_j(z) = (a_j^B(z))_{B \in \mathcal{B}_L}$ satisfies the matrix recursion*

$$\mathbf{a}_j(z) = \mathbf{A}_L(z) \mathbf{a}_{j-1}(z) \qquad (j > L)$$

where the $G_L \times G_L$-matrix $\mathbf{A}_L(z) = (a_{B,C}(z))_{B,C \in \mathcal{B}_L}$ is given by

$$a_{B,C}(z) = \begin{cases} z^{\kappa(\eta_B, C)} & \text{if } 'C = B' \text{ and } (\eta_B, C) \in \mathcal{B}_{L+1} \\ 0 & \text{otherwise} \end{cases}$$

Example 5.1. For $q = 2$ $(r = 1,\ a_1 = 2)$, the matrix $\mathbf{A}_2(1)$ has the form

$$\mathbf{A}_2(1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Example 5.2. For $r = 2$ and $a_1 = a_2 = 1$ (the G_j are the Fibonacci numbers) we have $\mathcal{B}_2 = \{00, 01, 10\}$ and

$$\mathbf{A}_2(1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Remark 5.1. In the q-ary case, $\mathbf{A}_L(1)$ is the adjacency matrix of the (directed) De Bruijn graph corresponding to $\mathcal{B}_L$.

Remark 5.2. De Bruijn graphs are Eulerian graphs since the indegree and the outdegree of all vertices are equal ($= q$).

Remark 5.3. For general simple α-numbers, a generalization of De Bruijn graphs can be defined by the adjacency matrix $\mathbf{A}_L(1)$.

Remark 5.4. The generalized De Bruijn graph corresponding to $\mathbf{A}_{L+1}(1)$ is the line graph of the generalized De Bruijn graph corresponding to $\mathbf{A}_L(1)$ ($L \geq r - 1$).

Remark 5.5. If D is a digraph, $\mathcal{L}(D)$ its linegraph and $\mathbf{A}(D)$, $\mathbf{A}(\mathcal{L}(D))$ the adjacency matrices, then we have for the characteristic polynomials

$$\chi(\mathbf{A}(D))(x) = x^{q-p}\chi(\mathbf{A}(\mathcal{L}(D)))(x)$$

where q denotes the number of edges and p the number of vertices.

Theorem 5.1. *The characteristic polynomial of* $\mathbf{A}_L(1)$ *is*

$$\chi(\mathbf{A}_L(1))(x) = x^{G_L - r}p(x),$$

where

$$p(x) = x^r - a_1 x^{r-1} - a_2 x^{r-2} - \cdots - a_{r-1}x - a_r$$

is the characteristic polynomial of α *(and of the finite recurrence).*

Proof. First we remark that $\#(\mathcal{B}_L) = G_L$ and for each $B \in \mathcal{B}_L$

$$B = (\epsilon_{L-1}(i-1), \epsilon_{L-2}(i-1), \ldots, \epsilon_0(i-1))$$

for some $i \in \{0, 1, \ldots, G_L - 1\}$.

The conditions $B' = {}'C$ and $(\eta_B, C) \in \mathcal{B}_{L+1}$ can thus be written for $i, j \in \{1, 2, \ldots, G_L\}$ as

$$(\epsilon_{L-2}(i-1), \ldots, \epsilon_0(i-1)) = (\epsilon_{L-1}(j-1), \ldots, \epsilon_1(j-1)) \qquad (5.2)$$

and

$$(\epsilon_{L-1}(i-1), \epsilon_{L-1}(j-1), \dots, \epsilon_0(j-1)) \in \mathcal{B}_{L+1} \tag{5.3}$$

respectively.

Therefore the coefficients of $\mathbf{A}_L(1) = (a_{ij}^{(L)})_{1 \le i,j \le G_L}$ are

$$a_{ij}^{(L)} = \begin{cases} 1 & \text{if (5.2) and (5.3)} \\ 0 & \text{else} \end{cases}$$

If (5.2) holds, (5.3) is violated only for

$$L = r-1,\ (\epsilon_{r-2}(i-1), \dots, \epsilon_0(i-1)) = (a_1, \dots, a_{r-1}),$$

$$(\epsilon_{r-2}(j-1), \dots, \epsilon_0(j-1)) = (a_2, \dots, a_{r-1}, x)$$

with $a_r \le x \le a_1$.

Hence we have for $L \ge r$

$$a_{ij}^{(L)} = \begin{cases} 1 & \text{if (5.2)} \\ 0 & \text{else} \end{cases}$$

and

$$a_{i+kG_{L-1},j}^{(L)} = a_{i,j}^{(L)},$$

because of

$$\begin{aligned} &(\epsilon_{L-1}(i+kG_{L-1}-1), \epsilon_{L-2}(i+kG_{L-1}-1), \dots, \epsilon_0(i+kG_{L-1}-1)) \\ &\quad = (\epsilon_{L-1}(i-1)+k, \epsilon_{L-2}(i-1), \dots, \epsilon_0(i-1)). \end{aligned}$$

We define a matrix $\mathbf{P}_L := (p_{ij}^{(L)})_{1 \le i,j \le G_L}$ with

$$p_{i,j}^{(L)} := \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } j \le G_{L-1},\ i = j + kG_{L-1},\ k > 0 \\ 0 & \text{else} \end{cases} .$$

Hence $\mathbf{P}_L^{-1} = (p_{ij}^{(-L)})_{1 \le i,j \le G_L}$ is

$$p_{i,j}^{(-L)} = \begin{cases} 1 & \text{if } i = j \\ 1 & \text{if } j \le G_{L-1},\ i = j + kG_{L-1},\ k > 0 \\ 0 & \text{else} \end{cases} .$$

With $\mathbf{P}_L$ we create a matrix $\mathbf{A}'_L = (a_{ij}^{\prime(L)})_{1 \le i,j \le G_L}$ which is similar to $\mathbf{A}_L(1)$.

$$\mathbf{A}'_L := \mathbf{P}_L \mathbf{A}_L \mathbf{P}_L^{-1}$$

In the construction of $\mathbf{A}'_L$ the rows i of $\mathbf{A}_L(1)$ are subtracted of the rows $i+kG_{L-1}$ and the columns $i+kG_{L-1}$ are added to the columns i. Therefore

$$a'^{(L)}_{ij} = 0 \quad \forall i \in \{G_{L-1}+1, G_{L-1}+2, \dots, G_L\}, j \in \{1,2,\dots,G_L\}.$$

Now we look at the matrix

$$\mathbf{A}_{L-1} = (a^{(L-1)}_{ij})_{1\le i,j\le G_{L-1}} := \mathbf{A}'_L \begin{pmatrix} 1 & 2 & \dots & G_{L-1} \\ 1 & 2 & \dots & G_{L-1} \end{pmatrix}$$

(this notation means the rows and columns $1,2,\dots,G_{L-1}$ of $\mathbf{A}'_L$).

We show that $\mathbf{A}_{L-1} = \mathbf{A}_{L-1}(1)$, i.e. the two definitions are equivalent.

$a^{(L-1)}_{ij} = 1$ holds not only when (5.2) holds, but also when

$$\begin{aligned}(\epsilon_{L-2}(i-1), \dots, \epsilon_0(i-1)) &= (\epsilon_{L-1}(j+kG_{L-1}), \dots, \epsilon_1(j+kG_{L-1}-1)) \\ &= (k, \epsilon_{L-2}(j-1), \dots, \epsilon_1(j-1))\end{aligned}$$

for a $k \in \{0,1,\dots,a_1\}$ with $j+kG_{L-1} \le G_L$. Therefore

$$a^{(L-1)}_{ij} = \begin{cases} 1 & \text{if } (\epsilon_{L-3}(i-1), \dots, \epsilon_0(i-1)) = (\epsilon_{L-2}(j-1), \dots, \epsilon_1(j-1)) \\ 0 & \text{else} \end{cases}$$

For $L = r$ we have to check $a^{(r-1)}_{ij} = 0$ for

$$(\epsilon_{r-1}(i-1), \dots, \epsilon_0(i-1)) = (0, a_1, a_2, \dots, a_{r-1}),$$

$$(\epsilon_{r-1}(j-1), \dots, \epsilon_0(j-1)) = (0, a_2, \dots, a_{r-1}, x)$$

with $a_r \le x \le a_1$, i.e.

$$a_{i,j+kG_{r-1}} = 0 \qquad (j+kG_{r-1} \le G_r).$$

This is true because we had otherwise

$$k = \epsilon_{r-1}(j+kG_{r-1}-1) = \epsilon_{r-2}(i-1) = a_1$$

and $k < a_1$ because of $j+kG_{r-1} \le G_r$.

We have

$$\chi(\mathbf{A}_L(1))(x) = \chi(\mathbf{A}'_L)(x) = x^{G_L - G_{L-1}} \chi(\mathbf{A}_{L-1}(1))(x)$$

and hence

$$\chi(\mathbf{A}_L(1))(x) = x^{G_L - G_{r-1}} \chi(\mathbf{A}_{r-1}(1))(x) \tag{5.4}$$

Now we look at $\mathbf{A}_{r-1}(1)$. For $i < G_{r-1}, j \leq G_{r-1}$ (but not for $i = G_{r-1}$)

$$a_{ij}^{(r-1)} = \begin{cases} 1 & \text{if } (\epsilon_{r-3}(i-1), \ldots, \epsilon_0(i-1)) = (\epsilon_{r-2}(j-1), \ldots, \epsilon_1(j-1)) \\ 0 & \text{else} \end{cases}$$

and

$$a_{i+kG_{r-2},j}^{(r-1)} = a_{i,j}^{(r-1)}.$$

We define the matrix $\mathbf{P}_{r-1} := (p_{ij}^{(r-1)})_{1 \leq i,j \leq G_{r-1}}$ with

$$p_{i,j}^{(r-1)} := \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } j \leq G_{r-2},\ i = j + kG_{r-2},\ k > 0 \\ 0 & \text{else} \end{cases} .$$

Like above $\mathbf{A}'_{r-1} := \mathbf{P}_{r-1}\mathbf{A}_{r-1}(1)\mathbf{P}_{r-1}^{-1}$ and

$$a_{ij}^{\prime(r-1)} = 0 \quad \forall i \in \{G_{r-2}+1, G_{r-2}+2, \ldots, G_{r-1}-1\}, j \in \{1, 2, \ldots, G_{r-1}\}.$$

We build the matrix $\mathbf{A}_{r-2} := \mathbf{A}' \left(\begin{smallmatrix} 1 & 2 & \ldots & G_{r-2} & G_{r-1} \\ 1 & 2 & \ldots & G_{r-2} & G_{r-1} \end{smallmatrix}\right)$, where the numeration of the rows and columns is kept, i.e. $\mathbf{A}_{r-2} = (a_{ij}^{(r-2)})_{i,j \in \{1,2,\ldots,G_{r-2},G_{r-1}\}}$. (Now $\mathbf{A}_{r-2}$ cannot be interpreted as adjacency matrix of a generalized De Bruijn graph.) We have

$$\chi(\mathbf{A}_{r-1}(1))(x) = \chi(\mathbf{A}'_{r-1})(x) = x^{G_{r-1}-G_{r-2}-1}\chi(\mathbf{A}_{r-2})(x),$$

and for $i < G_{r-2},\ j \leq G_{r-1}$

$$a_{ij}^{(r-2)} = \begin{cases} 1 & \text{if } (\epsilon_{r-4}(i-1), \ldots, \epsilon_0(i-1)) = (\epsilon_{r-3}(j-1), \ldots, \epsilon_1(j-1)) \\ 0 & \text{else} \end{cases}$$

Hence this procedure can be iterated. We define matrices $\mathbf{P}_{r-l}$ for $2 \leq l < r$ by replacing $r-1$ by $r-l$ in the above definition, and

$$\mathbf{A}_{r-l-1} := (\mathbf{P}_{r-l}\mathbf{A}_{r-l}\mathbf{P}_{r-l}^{-1}) \begin{pmatrix} 1 & 2 & \ldots & G_{r-l-1} & G_{r-l} & \ldots & G_{r-1} \\ 1 & 2 & \ldots & G_{r-l-1} & G_{r-l} & \ldots & G_{r-1} \end{pmatrix}.$$

Now we look at $\mathbf{A}_0$:

$$\mathbf{A}_0 = (a_{ij}^{(0)})_{i,j \in \{G_0, G_1, \ldots, G_{r-1}\}},\ \dim(\mathbf{A}_0) = r$$

$$a_{i1}^{(0)} = \sum_{j=1}^{G_1-1} a_{ij}^{(1)} = \sum_{\substack{j \in \{1,2,\ldots,G_2-1\} \\ j \neq k_1 G_1\ \forall k_1 \leq a_1}} a_{ij}^{(1)} = \sum_{\substack{j \in \{1,2,\ldots,G_3-1\} \\ j \neq k_1 G_2 + k_2 G_1\ \forall (k_1,k_2): \\ (k_1,k_2) \leq (a_1,a_2),\ k_2 \leq a_1}} a_{ij}^{(2)} = \cdots$$

$$= \sum_{\substack{j\in\{1,2,\ldots,G_{r-1}-1\} \\ j\neq k_1G_{r-1}+k_2G_{r-2}+\cdots\ +k_{r-1}G_1\ \ \forall(k_1,\ldots,k_{r-1}): \\ (k_n,k_{n+1},\ldots,k_{r-1})\leq(a_1,a_2,\ldots,a_{r-n})\ \ \forall n\in\{1,2,\ldots,r-1\}}} a_{ij}^{(r-1)} = \sum_{\substack{j\in\{1,2,\ldots,G_{r-1}\} \\ \epsilon_0(j)\neq 0}} a_{ij}^{(r-1)}$$

For $1 \leq l < r$ we have

$$(\epsilon_{r-2}(G_l-1),\ldots,\epsilon_0(G_l-1)) = (0,\ldots,0,a_1,a_2,\ldots,a_l).$$

Hence for $l < r-1$ we have $a_{G_l,j} = 1$ iff

$$(\epsilon_{r-2}(j-1),\ldots,\epsilon_0(j-1)) = (0,\ldots,0,a_1,a_2,\ldots,a_l,x) \qquad (0 \leq x \leq a_{l+1}).$$

$x = a_{l+1}$ implies $j = G_{l+1}$ and $\epsilon_0(j) = 0$, otherwise $\epsilon_0(j) = x+1$.

With (5.3) we have $a_{G_{r-1},j} = 1$ iff

$$(\epsilon_{r-2}(j-1),\ldots,\epsilon_0(j-1)) = (a_2,\ldots,a_{r-1},x) \qquad (0 \leq x < a_r).$$

Therefore we have

$$a_{G_l 1}^{(0)} = a_{l+1}$$

for $1 \leq l < r$.

For $1 \leq m \leq r-1$ we have

$$a_{G_l G_m}^{(0)} = \delta_{l+1,m}$$

To show this, we have to look at $a_{G_l j}$, where

$$(\epsilon_{r-2}(j-1),\ldots,\epsilon_0(j-1)) = (k_1,\ldots,k_{r-m},a_1,\ldots,a_m)$$

$$= (0,\ldots,0,a_1,\ldots,a_l,x) \qquad (0 \leq x \leq a_{l+1})$$

for $l < r-1$.

For $m > l+1$ all these $a_{G_l j} = 0$, for $m = l+1$ only $a_{G_l G_m} = 1$.

For $m < l+1$ we must have

$$(\epsilon_{r-2}(j-1),\ldots,\epsilon_0(j-1)) = (0,\ldots,0,a_1,\ldots,a_{l+1-m},a_1,\ldots,a_m)$$

$$= (0,\ldots,0,a_1,\ldots,a_l,x).$$

Then

$$(a_{l+2-m},\ldots,a_l) = (a_1,\ldots,a_{m-1})$$

and $a_{l+1} \leq a_m$ because of (2.6). With $x \leq a_{l+1}$ we have $a_m = x = a_{l+1}$ and $j = G_{l+1}$. $a_{G_l G_{l+1}} = 1$ has no influence on $a_{G_l G_m}$, because the column G_{l+1} is never added to another column in the construction of $\mathbf{A}_0$.

For $l = r-1,\ 1 \le m < r$ we have to look at $a_{G_{r-1}j}$ where

$$(\epsilon_{r-2}(j-1), \ldots, \epsilon_0(j-1)) = (k_1, \ldots, k_{r-m}, a_1, \ldots, a_m)$$
$$= (a_2, \ldots, a_{r-1}, x) \qquad (0 \le x \le a_1).$$

For $a_r \le x \le a_1$ we have $a_{G_{r-1}j} = 0$, because (5.3) does not hold. For $0 \le x < a_r$ we get a contradiction with

$$(a_1, \ldots, a_m) = (a_{r-m+1}, \ldots, a_{r-1}, x) < (a_{r-m+1}, \ldots, a_r) < (a_1, \ldots, a_m).$$

Therefore all $a_{G_{r-1}G_m} = 0$ for $1 \le m < r$.

Hence $\mathbf{A}_0$ has the form

$$\mathbf{A}_0 = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ a_r & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

and the characteristic polynomial of $\mathbf{A}_{r-1}(1)$ is

$$\chi(\mathbf{A}_{r-1}(1))(x) = x^{G_{r-1}-r}(x^r - a_1 x^{r-1} - a_2 x^{r-2} - \cdots - a_{r-1}x - a_r).$$

With (5.4) the theorem is proved. $\square$

In the special case of q-ary expansions $\chi(\mathbf{A}_L(1))(x) = x^{q^L-1}(x-q)$.

Example 5.3. $(a_1, a_2) = (1, 1)$ (Fibonacci numbers)

$$G_0 = 1,\ G_1 = 2,\ G_2 = 3$$
$$\mathcal{B}_2 = \{(0,0), (0,1), (1,0)\}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{\mathbf{P}_2} \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{\mathbf{A}_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{\mathbf{P}_2^{-1}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}_1 = \mathbf{A}_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\chi(\mathbf{A}_2)(x) = x\chi(\mathbf{A}_0)(x) = x(x^2 - x - 1)$$

Example 5.4. $(a_1, a_2, a_3) = (2, 0, 1)$

$$G_0 = 1,\ G_1 = 3,\ G_2 = 7,\ G_3 = 15$$

$$\mathcal{B}_3 = \{(0,0,0),(0,0,1),(0,0,2),(0,1,0),(0,1,1),(0,1,2),(0,2,0),(1,0,0),$$
$$(1,0,1),(1,0,2),(1,1,0),(1,1,1),(1,1,2),(1,2,0),(2,0,0)\}$$

$$\mathbf{A}_3 = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\mathbf{P}_3^{\pm 1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mp 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mp 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mp 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mp 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mp 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mp 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mp 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\mp 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$
\mathbf{P}_3\mathbf{A}_3\mathbf{P}_3^{-1} = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\mathbf{A}_2 = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\mathbf{P}_2^{\pm 1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\mp 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \mp 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \mp 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\mathbf{P}_2\mathbf{A}_2\mathbf{P}_2^{-1} = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{P}_1} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{A}_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{P}_1^{-1}} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}_0 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\chi(\mathbf{A}_3)(x) = x^{12}\chi(\mathbf{A}_0)(x) = x^{12}(x^3 - 2x^2 - 1)$$

5.2 Non-simple α-Numbers

Let α be a non-simple α-number, i.e.

$$(a_{n+m+1}, a_{n+m+2}, \ldots) = (a_{n+1}, a_{n+2}, \ldots)$$

for some integers $m,\ n$ and $m,\ n$ minimal with this property.

As above let $\mathcal{B}_l$ denote the set

$$\mathcal{B}_l = \{(\epsilon_{l-1}(i), \epsilon_{l-2}(i), \ldots, \epsilon_0(i)) : i < G_l\},$$

F be a function $F : \mathcal{B}_{l+1} \to \mathbf{R}$ (with $F(0, 0, \ldots, 0) = 0$) and

$$s_F(n) = \sum_{j \geq 0} F\left(\epsilon_{j+l}(n), \epsilon_{j+l-1}(n), \ldots, \epsilon_j(n)\right).$$

Now we may assume that $l = km \geq n + m - 1$ for some $k \in \mathbf{N}$. (If we are interested in $l+1$ subsequent digits with $l < km$ then we consider a new function $\tilde{F} : \mathcal{B}_{km+1} \to \mathbf{R}$ that does not depend on the first $(km - l)$ digits.)

Lemma 5.2. *For all $l \in \mathbf{N}$ we can find an integer $L \geq l$ such that*

$$(a_j, a_{j+1}, \ldots, a_{L+1}) < (a_1, a_2, \ldots, a_{L-j+2}) \ \forall j \in \{2, 3, \ldots, L+1\} \tag{5.5}$$

Proof. If (5.5) holds for $L := l$, we are finished. Otherwise we have a $j \leq l+1$ such that

$$(a_j, a_{j+1}, \ldots, a_{l+1}) = (a_1, a_2, \ldots, a_{l-j+2})$$

and an integer $g > l + 1$ such that

$$(a_j, a_{j+1}, \ldots, a_{g-1}) = (a_1, a_2, \ldots, a_{g-j})$$

and

$$(a_j, a_{j+1}, \ldots, a_g) < (a_1, a_2, \ldots, a_{g-j+1}).$$

If (5.5) holds for $L := g-1$, we are finished. Otherwise we have a $j' \le g$ such that

$$(a_{j'}, a_{j'+1}, \ldots, a_g) = (a_1, a_2, \ldots, a_{g-j'+1}).$$

For $j' \ge j$, we have

$$(a_{j'}, a_{j'+1}, \ldots, a_g) < (a_{j'-j+1}, a_{j'-j+2}, \ldots, a_{g-j+1}) \le (a_1, a_2, \ldots, a_{g-j'+1}).$$

Therefore $j' < j$. We can find $g' > g$ such that

$$(a_{j'}, a_{j'+1}, \ldots, a_{g-1}) = (a_1, a_2, \ldots, a_{g-j'}),$$

$$(a_{j'}, a_{j'+1}, \ldots, a_g) < (a_1, a_2, \ldots, a_{g-j'+1})$$

and repeat this procedure.

Since $j > j' > j'' > \ldots > 1$, we find a L that satisfies (5.5) after a finite number of steps. □

Remark 5.6. If (5.5) holds for L, it holds for $L + m$ since

$$(a_j, a_{j+1}, \ldots, a_{L+m+1}) = (a_{j-m}, a_{j-m+1}, \ldots, a_{L+1})\ \forall j \in \{L+2, L+3, \ldots, L+m+1\}$$

and, with induction, for $L + km$.

Now we consider the functions

$$a_j^B(z) := \sum_{n<G_j, (\epsilon_{j-1}(n), \ldots, \epsilon_{j-L}(n))=B} z^{s_F(n)} \qquad (B \in \mathcal{B}_L).$$

Lemma 5.3. *Let L satisfy (5.5) and $L \ge km \ge n + m - 1$. Then the functions a_j^B, $j > L$, $B \in \mathcal{B}_L$ are (recursively) given by*

$$a_j^B(z) = \sum_{\substack{C \in \mathcal{B}_L:\ 'C = B', \\ (\eta_B, C) \in \mathcal{B}_{L+1}}} a_{j-1}^C(z)\ z^{\kappa(\eta_B, C)}$$

if $B \ne (a_1, a_2, \ldots, a_L)$ and

$$a_j^{(a_1,\ldots,a_L)}(z) = \sum_{\substack{C=(a_2,\ldots,a_L,\eta_L), \\ \eta_L \le a_{L+1}}} a_{j-1}^C(z)\ z^{\kappa(a_1, C)} - \sum_{D \in \mathcal{C}_{km}} b_{j-L-1}^D(z)\ z^{\lambda(a_1,\ldots,a_{L+1}, D)}, \tag{5.6}$$

where

$$\mathcal{C}_{km} := \{D \in \mathcal{B}_{km} : D \geq (a_{L+2}, a_{L+3}, \ldots, a_{L+km+1})\},$$

$$b_j^D(z) := \sum_{\substack{i<G_j:\ (\epsilon_{j-1}(i),\ldots,\epsilon_{j-km}(i))=D,\\ (\epsilon_{j-1}(i),\ldots,\epsilon_0(i))>(a_{L+2},\ldots,a_{L+j+1})}} z^{s_F(n)}$$

and

$$\lambda(\theta_1, \theta_2, \ldots, \theta_{L+1}, D) := s_F(n_1) - s_F(n_2)$$

with

$$D = (\zeta_1, \zeta_2, \ldots, \zeta_{km}), \qquad \epsilon_i(n_1) = \epsilon_i(n_2) = 0\ \forall i \geq km + L + 1$$

$$(\epsilon_{km+L}(n_1), \epsilon_{km+L+1}(n_1), \ldots, \epsilon_0(n_1)) = (\theta_1, \theta_2, \ldots, \theta_{L+1}, \zeta_1, \zeta_2, \ldots, \zeta_{km}),$$

$$(\epsilon_{km+L}(n_2), \epsilon_{km+L+1}(n_2), \ldots, \epsilon_0(n_2)) = (0, \ldots, 0, \zeta_1, \zeta_2, \ldots, \zeta_{km}).$$

The functions $b_j^D(z)$, $j \geq km$, $D \in \mathcal{C}_{km}$ are recursively given by

$$b_j^D(z) = \begin{cases} \sum\limits_{B\in\mathcal{B}_L:\ (\eta_1,\ldots,\eta_{km})=D} a_j^B(z) & \text{if } D > (a_{L+2}, \ldots, a_{L+km+1}) \\ \sum\limits_{E\in\mathcal{C}_{km}} b_{j-km}^E(z) z^{\lambda(0,\ldots,0,a_{L+2},\ldots,a_{L+km+1},E)} & \text{if } D = (a_{L+2}, \ldots, a_{L+km+1}) \end{cases}$$

Proof. The proof of the first equation is the same as that of Lemma 5.1. We just have to check

$$(\eta_B, \epsilon_{j-2}(n), \ldots, \epsilon_0(n), 0, \ldots) < (a_1, a_2, \ldots) \tag{5.7}$$

for $C = (\epsilon_{j-2}(n), \ldots, \epsilon_{j-L-1}(n))$, $'C = B'$, $(\eta_B, C) \in \mathcal{B}_{L+1}$.

(5.7) can only be violated, if $(\eta_B, C) = (a_1, \ldots, a_{L+1})$ (which implies $B = (a_1, \ldots, a_L)$).

$$\begin{aligned} a_j^{(a_1,\ldots,a_L)}(z) \quad = \quad & \sum_{\substack{(\varphi_{j-1},\ldots,\varphi_0):\\ (\varphi_{j-1},\ldots,\varphi_{j-L})=(a_1,\ldots,a_L),\ \varphi_{j-L-1}\leq a_{L+1},\\ (\varphi_{j-2},\ldots,\varphi_0)\in\mathcal{B}_{j-1}}} z^{s_F(\varphi_{j-1},\ldots,\varphi_0)} - \\ & \sum_{\substack{(\varphi_{j-1},\ldots,\varphi_0):\\ (\varphi_{j-1},\ldots,\varphi_{j-L})=(a_1,\ldots,a_L),\ \varphi_{j-L-1}=a_{L+1},\\ (\varphi_{j-2},\ldots,\varphi_0)\in\mathcal{B}_{j-1},\\ (\varphi_{j-L-2},\ldots,\varphi_0)>(a_{L+2},\ldots,a_j)}} z^{s_F(\varphi_{j-1},\ldots,\varphi_0)} \end{aligned} \tag{5.8}$$

where, with $\varphi_i = 0\ \forall i \geq j$,

$$s_F(\varphi_{j-1}, \ldots, \varphi_0) := \sum_{i\geq 0} F(\varphi_{i+l}(n), \ldots, \varphi_i(n)).$$

The first sum of (5.8) is equal to the first sum of (5.6) (see lemma 5.1 and consider (5.5)). The patterns of the second sum of (5.8) are exactly those of the first sum which do not satisfy $(\varphi_{j-1},\ldots,\varphi_0)\in\mathcal{B}_j$. Because of (5.5) the patterns $(\varphi_{j-L-2},\ldots,\varphi_0)$ which appear in the second sum of (5.8) are not influenced by the digits in front of them. Therefore this sum is equal to the second sum of (5.6).

The equation for $b_j^D(z)$, $D > (a_{L+2},\ldots,a_{L+km+1})$ is clear. For $D = (a_{L+2},\ldots,a_{L+km+1})$ we have to consider Remark 5.6 and that $(a_{L+km+2},\ldots,a_{L+2km+1}) = (a_{L+2},\ldots,a_{L+km+1})$. $\square$

Corollary 5.2. *The vector*

$$\mathbf{a}_j(z) = (a_j^{B_1}(z),\ldots,a_j^{B_{G_L}}(z), b_{j-1}^{D_1}(z),\ldots,b_{j-1}^{D_M}(z),\ldots,b_{j-L}^{D_1}(z),\ldots,b_{j-L}^{D_M}(z))^T$$

with

$$M := \#(\mathcal{C}_{km}),\ B_i := (\epsilon_{L-1}(i-1),\ldots,\epsilon_0(i-1))$$

$$D_1 := (a_{L+2},\ldots,a_{L+km+1}),\ \ldots,\ D_M := (a_1,\ldots,a_{km})$$

satisfies the matrix recursion

$$\mathbf{a}_j(z) = \mathbf{A}_L(z)\mathbf{a}_{j-1}(z) \qquad (j > L)$$

where $\mathbf{A}_L(z) = (a_{i,j}(z))_{1\le i,j\le G_L+LM}$ *is given by*

$$a_{i,j}(z) = \begin{cases} z^{\kappa(\eta_{B_i},B_j)} & \textit{if } i,\ j\le G_L,\ 'B_j = B_i' \\ -z^{\lambda(a_1,\ldots,a_{L+1},D_{j-G_L-(L-1)M})} & \textit{if } i = G_L,\ j > G_L+(L-1)M \\ z^{\lambda(0,\ldots,0,a_{L+2},\ldots,a_{L+km+1},D_{j-G_L-(km-1)M})} & \textit{if } i = G_L+1,\ j\le G_L+kmM, \\ & j > G_L+(km-1)M \\ 1 & \textit{if } G_L+2\le i\le G_L+M, \\ & D_{i-G_L} = (\eta_1,\ldots,\eta_{km})(B_j) \\ & \textit{or } i > G_L+M,\ j = i-M \\ 0 & \textit{otherwise} \end{cases}$$

Hence, if $L > km$, $\mathbf{A}_L(1)$ has the form

$$\left(\begin{array}{c|c|c|c|c|c|c}
\tilde{\mathbf{A}}_L & 0 & \cdots & \cdots & \cdots & 0 & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -1 & \cdots & -1 \end{matrix} \\
\hline
\begin{matrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{matrix} & 0 & \cdots & \begin{matrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & \cdots & 0 & 0 \\
\hline
0 & E_M & 0 & \cdots & \cdots & \cdots & 0 \\
\hline
\vdots & \ddots & \ddots & \ddots & & & \vdots \\
\hline
\vdots & & \ddots & \ddots & \ddots & & \vdots \\
\hline
\vdots & & & \ddots & \ddots & \ddots & \vdots \\
\hline
0 & \cdots & \cdots & \cdots & 0 & E_M & 0
\end{array}\right)$$

where $\tilde{\mathbf{A}}_L$ is the matrix $\mathbf{A}_L(1)$ corresponding to the simple α-number with α-expansion $(a_1, a_2, \ldots, a_L, a_{L+1}+1, 0, \ldots, 0)$.

$\mathbf{A}_{km}(1)$ has the form

$$\left(\begin{array}{c|c|c|c|c|c}
\tilde{\mathbf{A}}_{km} & 0 & \cdots & \cdots & 0 & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -1 & \cdots & -1 \end{matrix} \\
\hline
\begin{matrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{matrix} & 0 & \cdots & \cdots & 0 & \begin{matrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} \\
\hline
0 & E_M & 0 & \cdots & \cdots & 0 \\
\hline
\vdots & \ddots & \ddots & \ddots & & \vdots \\
\hline
\vdots & & \ddots & \ddots & \ddots & \vdots \\
\hline
0 & \cdots & \cdots & 0 & E_M & 0
\end{array}\right)$$

Theorem 5.2. *The characteristic polynomial of* $\mathbf{A}_L(1)$ *is*

$$\chi(\mathbf{A}_L(1))(x) = p(x)\ (x^{(k-1)m} + x^{(k-2)m} + \cdots + 1)\ x^{G_L + LM - km - n} \qquad (5.9)$$

where

$$p(x) = (x^{n+m} - a_1 x^{n+m-1} - \cdots - a_{n+m}) - (x^n - a_1 x^{n-1} - \cdots - a_n)$$

is the characteristic polynomial of α.

Proof. First we construct a matrix $\mathbf{A}' = (a'_{ij})_{1\le i,j\le G_L+L}$ with

$$\chi(\mathbf{A}_L(1))(x) = \chi(\mathbf{A}')(x)\ x^{LM-L}.$$

To get $\mathbf{A}'$, we define $\mathbf{P}_h = (p_{ij}^{(h)})_{1\le i,j\le G_L+LM},\ 0 \le h < L$, with

$$p_{i,i}^{(h)} := 1\ \forall i \le G_L + LM,$$

$$p_{G_L+hM+1,j}^{(h)} := 1\ \forall j : G_L + hM + 1 \le j \le G_L + (h+1)M.$$

Then

$$\mathbf{A}' := \bar{\mathbf{A}} \begin{pmatrix} 1\ 2\ \dots\ G_L+1\ G_L+M+1\ \dots\ G_L+(L-1)M+1 \\ 1\ 2\ \dots\ G_L+1\ G_L+M+1\ \dots\ G_L+(L-1)M+1 \end{pmatrix}$$

where

$$\bar{\mathbf{A}} := \mathbf{P}_0\mathbf{P}_1\dots\mathbf{P}_{L-1}\mathbf{A}_L\mathbf{P}_{L-1}^{-1}\mathbf{P}_L^{-1}\dots\mathbf{P}_0^{-1}.$$

$\mathbf{A}'$ has the form

$$\left(\begin{array}{ccccccc|ccccccc|c} & & & & & & & & & & & & & & 0\\ & & & \tilde{\mathbf{A}}_L & & & & & & & 0 & & & & \vdots\\ & & & & & & & & & & & & & & 0\\ & & & & & & & & & & & & & & -1\\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0\\ \hline & & & & & & & & & & & & & & 0\\ & & & 0 & & & & & & & E_{L-1} & & & & \vdots\\ & & & & & & & & & & & & & & 0 \end{array}\right)$$

if $L > km$ and

$$\left(\begin{array}{cccccc|ccc|c} & & & & & & & & & 0\\ & & \tilde{\mathbf{A}}_{km} & & & & & 0 & & \vdots\\ & & & & & & & & & 0\\ & & & & & & & & & -1\\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1\\ \hline & & & & & & & & & 0\\ & & 0 & & & & & E_{L-1} & & \vdots\\ & & & & & & & & & 0 \end{array}\right)$$

if $L = km$.

Since $\tilde{\mathbf{A}}_L$ is the matrix $\mathbf{A}_L(1)$ corresponding to the simple α-number with α-expansion $(a_1, a_2, \ldots, a_L, a_{L+1} + 1, 0, \ldots, 0)$, it can be transformed to

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_L & \vdots & & \ddots & 1 \\ a_{L+1}+1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

(see proof of Theorem 5.1). In the transformation the last row is never added to or subtracted from another row. Because of this and the fact that $a'_{ij} = 0 \ \forall i < G_L,\ j > G_L$, we can apply this transformation to the whole matrix $\mathbf{A}'$ and get the $(2L+1) \times (2L+1)$-matrix $\mathbf{A}''$ which has the form

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & & & & & & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & & & & & \vdots \\ a_L & \vdots & & \ddots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_{L+1}+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 \\ y_0 & y_1 & \cdots & \cdots & y_{l-1} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & & & & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & & & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ a_L & \vdots & & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ a_{L+1}+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & -1 \\ y_0 & y_1 & \cdots & \cdots & y_{l-1} & 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}$$

respectively.

$$\chi(\mathbf{A}')(x) = \chi(\mathbf{A}'')(x)\ x^{G_L - L - 1}$$

The $y_j,\ 0 \leq j \leq L$, are given by $y_j := \#(\mathcal{D}_j)$ with

$$\begin{aligned}\mathcal{D}_j \quad = \quad & \{B = (\eta_1, \ldots, \eta_L) \in \mathcal{B}_L :\ (\eta_1, \ldots, \eta_{km}) > (a_{L+2}, \ldots, a_{L+km+1}), \\ & (\eta_{L-j+1}, \ldots, \eta_L) = (a_1, \ldots, a_j), (\eta_{L+1-h}, \ldots, \eta_L) \neq (a_1, \ldots, a_h)\ \forall h > j\}\end{aligned}$$

Lemma 5.4. *The $y_{L-j},\ 0 \leq j \leq L$, are recursively given by*

$$y_L = 1,\ y_{L-j} = \sum_{h=1}^{j} y_{L-j+h} a_h - \begin{cases} a_{L+j+1} & \text{if } 1 \leq j < km \\ a_{L+1+km} + 1 & \text{if } j = km \\ 0 & \text{if } j > km \end{cases}$$

Proof. $y_L = 1$ is obvious.

If $B = (\eta_1, \ldots, \eta_L) \in \mathcal{D}_{L-j}$, let $h \geq 1$ be maximal with the property

$$B = (\eta_1, \ldots, \eta_{j-h}, a_1, \ldots, a_{h-1}, \eta_j, a_1, \ldots, a_{L-j})$$

Then

$$(\eta_1, \ldots, \eta_{j-h}, a_1, \ldots, a_{L-j+h}) \in \mathcal{D}_{L-j+h}.$$

If we take $(\theta_1, \ldots, \theta_{j-h}, a_1, \ldots, a_{L-j+h}) \in \mathcal{D}_{L-j+h}$, $1 \leq h < j$, then

$$(\eta_1, \ldots, \eta_L) := (\theta_1, \ldots, \theta_{j-h}, a_1, \ldots, a_{h-1}, \eta_j, a_1, \ldots, a_{L-j}) \in \mathcal{D}_{l-j}$$

iff $\eta_j < a_h$ and $(\eta_1, \ldots, \eta_{km}) > (a_{L+2}, \ldots, a_{L+km+1})$.

If $j > km$, the definition of $(\eta_1, \ldots, \eta_L)$ guarantees that the last condition holds. If $j \leq km$, it guarantees

$$(\eta_1, \ldots, \eta_{j-1}) \geq (a_{L+2}, \ldots, a_{L+j}).$$

Therefore $(\eta_1, \ldots, \eta_{km}) > (a_{L+2}, \ldots, a_{L+km+1})$ is violated iff

$$j \leq km,\ (\eta_1, \ldots, \eta_{j-1}) = (a_{L+2}, \ldots, a_{L+j}),\ \eta_j < a_{L+j+1}$$

or

$$j = km,\ (\eta_1, \ldots, \eta_{j-1}) = (a_{L+2}, \ldots, a_{L+j}),\ \eta_{km} \leq a_{L+km+1}.$$

□

We calculate $\chi(\mathbf{A}'')$ by developping $\det(x\mathbf{I} - \mathbf{A}'')$ at the columns $(2L+1)$ and $(L + km + 1)$:

$$\begin{aligned}\chi(\mathbf{A}'')(x) \quad = \quad & x^{L-km}\Big(x^{km}(x^{L+1} - a_1 x^L - \cdots - a_L x - a_{L+1} - 1) + \\ & (-1)^{km}(-1)^{L+km+1+L+2}(x^{L+1} - a_1 x^L - \cdots - a_L x - a_{L+1} - 1)\Big) + \\ & (-1)^{L-1}(-1)^{2L+1+L+1} det(\hat{\mathbf{A}}_L) \\ = \quad & x^{L-km}(x^{km} - 1)(x^{L+1} - a_1 x^L - \cdots - a_L x - a_{L+1} - 1) - \det(\hat{\mathbf{A}}_L) \\ = \quad & x^{L-km}(x^{(k-1)m} + x^{(k-2)m} + \cdots + 1)(x^m - 1) \\ & (x^{L+1} - a_1 x^L - \cdots - a_L x - a_{L+1} - 1) - \det(\hat{\mathbf{A}}_L)\end{aligned}$$

where $\hat{\mathbf{A}}_j,\ 1 \le j \le L$, is the matrix

$$\hat{\mathbf{A}}_j = \begin{pmatrix} x-a_1 & -1 & 0 & \cdots & \cdots & 0 \\ -a_2 & x & -1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ -a_j & 0 & \cdots & 0 & x & -1 \\ -y_0 & -y_1 & \cdots & \cdots & \cdots & -y_j \end{pmatrix}$$

$$\begin{aligned} \det(\hat{\mathbf{A}}_L) &= -y_L(x^L - a_1x^{L-1} - \cdots - a_L) + \det(\hat{\mathbf{A}}_{L-1}) = \cdots \\ &= -y_Lx^L + (a_1y_L - y_{L-1})x^{L-1} + (a_2y_L + a_1y_{L-1} - y_{L-2})x^{L-2} \\ &\quad + \cdots + (a_Ly_L + \cdots + a_1y_1 - y_0) \\ &= -x^L + a_{L+2}x^{L-1} + \cdots + a_{L+km}x^{L-km+1} + (a_{L+km+1}+1)x^{L-km} \\ &= -x^{L-km}(x^{km} - a_{L+2}x^{km-1} - \cdots - a_{L+km}x - a_{L+km+1} - 1) \\ &= -x^{L-km}(x^{(k-1)m} + \cdots + 1)(x^m - a_{L+2}x^{m-1} - \cdots - a_{L+m+1} - 1) \end{aligned}$$

Hence

$$\begin{aligned} \chi(\mathbf{A}'')(x) &= x^{L-km}(x^{(k-1)m} + x^{(k-2)m} + \cdots + 1) \\ &\quad \Big((x^{L+m+1} - a_1x^{L+m} - \cdots - a_{L+m+1} - 1) - (x^{L+1} - a_1x^L - \cdots - a_{L+1} - 1)\Big) \\ &= x^{2L+1-km-n}(x^{(k-1)m} + x^{(k-2)m} + \cdots + 1)p(x) \end{aligned}$$

Therefore

$$\begin{aligned} \chi(\mathbf{A}(1))(x) &= x^{G_L-L-1+LM-L}\chi(\mathbf{A}'')(x) \\ &= x^{G_L+LM-km-n}(x^{(k-1)m} + x^{(k-2)m} + \cdots + 1)p(x) \end{aligned}$$

and the theorem is proved. □

Remark 5.7. The roots of $x^{(k-1)m} + x^{(k-2)m} + \cdots + 1$ are km-th roots of unity.

5.3 The Conjugates of α

We look at the conjugates of an α-number α with respect to its characteristic polynomial (see Section 2.4).

Using the notation employed in Section 2.4, we define the transformation $T(x) = \langle \alpha x \rangle$ which sends $[0, 1)$ onto itself. Then, with $T^0(x) = x$, inductively $T^n(1) = T^{n-1}(\langle \alpha \rangle)$ for $n \geq 1$, consequently $T^n(1) = \langle \alpha T^{n-1}(1) \rangle$ and $\epsilon_{-n}(\alpha) = [\alpha T^{n-1}(\langle \alpha \rangle)] = [\alpha T^n(1)]$. Therefore $a_n = \epsilon_{-n+1}(\alpha) = [\alpha T^{n-1}(1)]$.

Hence the characteristic polynomial is

$$p(x) = x^r - [\alpha]x^{r-1} - [\alpha T(1)]x^{r-2} - \cdots - [\alpha T^{r-1}(1)]$$

and

$$\begin{aligned} p(x) &= (x^{n+m} - [\alpha]x^{n+m-1} - [\alpha T(1)]x^{n+m-2} - \cdots - [\alpha T^{n+m-1}(1)]) - \\ &\quad (x^n - [\alpha]x^{n-1} - [\alpha T(1)]x^{n-2} - \cdots - [\alpha T^{n-1}(1)]) \end{aligned}$$

respectively.

Lemma 5.5. *The conjugates $\alpha_1, \alpha_2, \ldots, \alpha_{r-1}(\alpha_{n+m-1})$ of α with respect to its characteristic polynomial are roots of*

$$\bar{p}(x) := x^{r-1} + T(1)x^{r-2} + T^2(1)x^{r-3} + \cdots + T^{r-1}(1) \tag{5.10}$$

and

$$\begin{aligned} \bar{p}(x) &:= (x^{n+m-1} + T(1)x^{n+m-2} + T^2(1)x^{n+m-3} + \cdots + T^{n+m-1}(1)) - \\ &\quad (x^{n-1} + T(1)x^{n-2} + T^2(1)x^{n-3} + \cdots + T^{n-1}(1)) \end{aligned} \tag{5.11}$$

respectively.

Proof. We show $\bar{p}(x)(x - \alpha) = p(x)$. If α is a simple α-number, then the coefficient of $x^j,\ 1 \leq j \leq r-1$, of the left hand side polynomial is

$$T^{r-j}(1) - \alpha T^{r-j-1}(1) = \langle \alpha T^{r-j-1}(1) \rangle - \alpha T^{r-j-1} = -[\alpha T^{r-j-1}]$$

and the constant coefficient is

$$-\alpha T^{r-1}(1) = -[\alpha T^{r-1}(1)]$$

since

$$[\alpha T^j(1)] = a_{j+1} = 0 \ \forall j \geq r$$

and therefore $T^r(1) = 0$.

For non-simple α-numbers we can make a similar reasoning. The constant coefficient is then

$$-\alpha T^{n+m-1}(1) + \alpha T^{n-1}(1) = -[\alpha T^{n+m-1}(1)] + [\alpha T^{n-1}(1)]$$

since

$$\begin{aligned}
&(\alpha T^{n+m-1}(1) - [\alpha T^{n+m-1}(1)]) - (\alpha T^{n-1}(1) - [\alpha T^{n-1}(1)] \\
&= \langle \alpha T^{n+m-1}(1)\rangle - \langle \alpha T^{n-1}(1)\rangle = T^{n+m}(1) - T^{n}(1) = 0 \\
&\Big([\alpha T^{j}(1)] = a_{j+1} = a_{j+m+1} = [\alpha T^{j+m}(1)] \ \forall j \geq n\Big).
\end{aligned}$$

□

Proposition 5.1. *The conjugates of an α-number α with respect to the characteristic polynomial have absolute value less than α.*

Proof. If α is a simple α-number, set

$$g(x) := 1 - x^r p(x^{-1}) = \sum_{j=1}^{r} a_j x^j.$$

If $|x| > \alpha$, then

$$|g(x^{-1})| \leq g(|x^{-1}|) < g(\alpha^{-1}) = 1$$

and $p(x) \neq 0$. If $|x| = \alpha$, $x \neq \alpha$, this is assured by

$$|g(x^{-1})| < g(|x^{-1}|) = 1$$

since $a_1 \neq 0$.

If α is a non-simple α-number, set, for $|x| > 1$,

$$f(x) := 1 - \frac{a_1}{x} - \frac{a_2}{x^2} - \frac{a_3}{x^3} - \cdots$$

and, for $|x| < 1$,

$$g(x) := 1 - f(x^{-1}) = \sum_{j=1}^{\infty} a_j x^j.$$

Then, for the same reasons as above, the roots of $f(x)$ have absolute value less than α for $x \neq \alpha$.

If we set

$$p_k(x) = p(x)(1 + x^m + x^{2m} + \cdots + x^{(k-1)m}),$$

then

$$p_k(x) := (x^{n+km} - a_1 x^{n+km-1} - \cdots - a_{n+km}) - (x^n - a_1 x^{n-1} - \cdots - a_n).$$

With $q_k(x) := x^{-n-km}p_k(x)$ we have

$$q_k(x) = (1 - \frac{a_1}{x} - \cdots - \frac{a_{n+km}}{x^{n+km}}) - (\frac{1}{x^{km}} - \frac{a_1}{x^{km+1}} - \cdots - \frac{a_n}{x^{n+km}})$$

and, for $|x| > 1$, $q_k(x) \to f(x)$ as $k \to \infty$.

Therefore $f(x) = 0$ is a necessary condition for $p(x) = 0$, $|x| > 1$ and the roots of $p(x)$ have absolute value less than α for $x \neq \alpha$.

Since α is simple root of $p(x)$ ($\bar{p}(\alpha) > 0$), the proposition is proved. □

Proposition 5.2. *The conjugates of α with respect to the characteristic polynomial have absolute value less than 2.*

Proof. If $x \in \mathbf{C}$ is a conjugate with $|x| > 1$ and α is a non-simple α-number, then

$$|x^{n+m-1}+T(1)x^{n+m-2}+\cdots+T^{n+m-1}(1)| = |x^{n-1}+T(1)x^{n-2}+\cdots+T^{n-1}(1)|.$$

Hence

$$(|x|^m - 1)|x^{n-1} + T(1)x^{n-2} + \cdots + T^{n-1}(1)| < |x|^{m-1} + \cdots + 1,$$

$$|x^{n-1} + T(1)x^{n-2} + \cdots + T^{n-1}(1)| < \frac{1}{|x| - 1}$$

and

$$|x|^{n-1} < \frac{1}{|x|-1} + |T(1)x^{n-2}+\cdots+T^{n-1}(1)| \leq \frac{1}{|x|-1} + \frac{|x|^{n-1}-1}{|x|-1} = \frac{|x|^{n-1}}{|x|-1}.$$

If $x \in \mathbf{C}$ is a conjugate with $|x| > 1$ and α is a non-simple α-number, then

$$\begin{aligned} |x|^{r-1} &= |T(1)x^{r-2} + T^2(1)x^{r-3} + \cdots + T^{r-1}(1)| \\ &\leq |x|^{r-2} + |x|^{r-3} + \cdots + 1 = \frac{|x|^{r-1} - 1}{|x| - 1} < \frac{|x|^{r-1}}{|x| - 1}. \end{aligned}$$

Therefore $|x| < 2$ in both cases. □

α resembles thus a Pisot number which is an algebraic integer greater than 1, with conjugates of maximum absolute value less than 1.

The proofs of Lemma 5.5 and Proposition 5.2 are due to Parry [34].

Proposition 5.3. *If the α-expansion $(a_1, a_2, \ldots, a_r, 0, \ldots)$ satisfies $a_1 \geq a_2 \geq \ldots \geq a_r > 0$, then α is a Pisot number.*

Proof. We use the following two lemmas:

Lemma 5.6. *If* $(a_1, a_2, \ldots, a_r)$ *satisfies* $a_1 \geq a_2 \geq \ldots \geq a_r > 0$*, then*

$$1 > T(1) > T^2(1) > \cdots > T^{r-1}(1) > 0$$

Proof. We assume $T^{n-1}(1) \leq T^n(1)$. This implies

$$a_n = [\alpha T^{n-1}(1)] \leq [\alpha T^n(1)] = a_{n+1}$$

and consequently $a_n = a_{n+1}$. Hence

$$T^n(1) = \langle \alpha T^{n-1}(1) \rangle \leq \langle \alpha T^n(1) \rangle = T^{n+1}(1).$$

We can repeat this consideration and get $T^{r-1}(1) \leq T^r(1) = 0$. Then $a_r = [\alpha T^{r-1}(1)] = 0$ which is not allowed. □

Lemma 5.7. $q(z) := a_0 + a_1 x + \cdots + a_n x^n$*. If* $a_0 > a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$*, then all roots of* $q(x)$ *have absolute value bigger than 1.*

Proof. Let ξ be a root of $q(x)$. Clearly $\xi \neq 1$.

Assume $|\xi| \leq 1,\ \xi \neq 1$. We have

$$(1-x)q(x) = a_0 - \underbrace{(a_0 - a_1)}_{>0} x - \underbrace{(a_1 - a_2)}_{\geq 0} x^2 - \cdots - \underbrace{(a_{n-1} - a_n)}_{\geq 0} x^n - a_n x^{n+1},$$

hence

$$|(1-\xi)q(\xi)| > a_0 - (a_0 - a_1) - (a_1 - a_2) - \cdots - (a_{n-1} - a_n) - a_n = 0$$

and ξ cannot be a root of $q(x)$.

Therefore $|\xi| > 1$. □

We set $q(x) := x^{r-1}\bar{p}(x^{-1})$, i.e.

$$q(x) := 1 + T(1)x + T^2(1)x^2 + \cdots + T^{r-1}(1)x^{r-1}.$$

With Lemma 5.6, $q(x)$ satisfies the conditions of Lemma 5.7 and all roots of $q(x)$ have absolute value bigger than 1. Therefore all roots of $\bar{p}(x)$ have absolute value less than 1. □

Corollary 5.3. *If* $(a_1, a_2, \ldots, a_r)$ *satisfies* $a_1 \geq a_2 \geq \ldots \geq a_r > 0$*, the characteristic polynomial is irreducible in* $\mathbf{Z}[x]$*.*

Proof.

$$p(x) = (x-\alpha)(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{r-1})$$

If

$$(x-\alpha_{i_1})(x-\alpha_{i_2})\dots(x-\alpha_{i_k}) \in \mathbf{Z}[x] \qquad (1 \le i_j \le r-1),$$

then

$$\prod_{j=1}^{k} \alpha_{i_j} \in \mathbf{Z},\ |\prod_{j=1}^{k} \alpha_{i_j}| < 1.$$

Therefore $\alpha_{i_j} = 0$ for a $j \in \{1, 2, \dots, k\}$ and $a_r = \alpha \prod_{i=1}^{r-1} \alpha_i = 0$ which is not allowed. □

Remark 5.8. In general, the characteristic polynomial is not irreducible. E.g.

$$x^5 - 3x^4 - 2x^3 - 2x - 3 = (x^3 - 2x^2 - 5x - 3)(x^2 - x + 1).$$

5.4 Expected Value and Variance

Lemma 5.8. *Let $G(t, z) = \det(t\mathbf{I} - \mathbf{A}_L(z))$ be the characteristic polynomial of the matrix $\mathbf{A}_L(z)$. Then there exists a (complex) neighbourhood of $z = 1$ such that $G(t, z) = 0$ has a unique solution $t = \alpha(z)$ of maximal modulus. Furthermore, the function $\alpha(z)$ is analytic in this neighbourhood.*

Proof. Since the eigenvalues of $\mathbf{A}_L(1)$ are α (which is simple) and ($\dim\mathbf{A}_L$-1) complex numbers β_i with $\max|\beta_i| < \alpha$, there exists a neigbourhood of $z = 1$ such that $(\dim\mathbf{A}_L - 1)$ eigenvalues of $\mathbf{A}_L(z)$ are bounded by $\frac{\max|\beta_i|+\alpha}{2}$ and one eigenvalue is contained in the circle $|t - \alpha| < \frac{\alpha-\max|\beta_i|}{2}$. Hence, the equation $G(t, z) = 0$ has a unique solution $t = \alpha(z)$ of maximal modulus.

Since α is a simple root of $G(t, 1) = 0$ we also have

$$\frac{\partial}{\partial t} G(t, 1)|_{t=q} \neq 0.$$

Hence, by the implicit function theorem, there exists a neighbourhood of $z = 1$ such that $\alpha(z)$ is analytic, too. □

Corollary 5.4. *There exists a neighbourhood of $z = 1$ such that for every block $B \in \mathcal{B}_L$*

$$a_j^B(z) = a^B(z)\alpha(z)^j + \mathcal{O}(\alpha^{(1-\delta)j})$$

as $j \to \infty$, where $\delta > 0$ and $a^B(z)$ is a properly chosen analytic function.

Now we look at the function $c_N(z)$ $(= \sum_{n<N} z^{s_F(n)})$. By the preceding corollary we know the following:

Lemma 5.9. *The asymptotic behaviour of $c_{\eta G_j}(z)$, $j \geq 0$, $1 \leq \eta \leq a_1$, locally around $z = 1$ is given by*

$$c_{\eta G_j}(z) = C_\eta(z)\alpha(z)^j + \mathcal{O}(\alpha^{(1-\delta)j})$$

as $j \to \infty$, where $C_\eta(z)$ is a properly chosen analytic function with $C_\eta(1) = \eta$.

Proof. We just have to add up over all $a_{j+1}^B(z)$ with blocks $B \in \mathcal{B}_L$ with $\eta_B = \eta$ and observe that $c_{\eta G_j}(1) = \eta G_j$. □

The next recurrence will help to extend this property to general N.

Lemma 5.10. *Suppose that $N = \eta G_k + N'$, $k \geq L$, with $1 \leq \eta \leq a_1$ and $N' < G_k$ and let $B_N = (\epsilon_{k-1}(N), \ldots, \epsilon_{k-L}(N))$ be the block of digits of N preceding $\eta = \epsilon_k(N)$. Then*

$$c_N(z) = c_{\eta G_k}(z) + c_{N'}(z)\, z^{\kappa(\eta, B_N)}. \tag{5.12}$$

Proof. For $n < N'$ we have

$$s_F(n + \eta G_k) = s_F(n) + \kappa(\eta, B_N).$$

Hence, (5.12) follows immediately. □

Obviously, we have

$$\mathbf{E}X_N = \frac{c_N'(1)}{N}$$

and

$$\mathbf{V}X_N = \frac{c_N''(1)}{N} + \frac{c_N'(1)}{N} - \left(\frac{c_N'(1)}{N}\right)^2.$$

Therefore we need expressions for $c_N'(1)$ and $c_N''(1)$.

Lemma 5.11. *Suppose that the digital representation of N with respect to G_j is given by*

$$N = \sum_{m=1}^{M} \eta_m G_{j_m}$$

with $G_{j_1} > G_{j_2} > \cdots > G_{j_M}$ and $\eta_m > 0$. Then there exist κ_m depending on N such that

$$c_N(z) = \sum_{m=1}^{M} z^{\kappa_m} c_{\eta_m G_{j_m}}(z) \tag{5.13}$$

and $|\kappa_m| \leq Dm$ for some constant $D > 0$.

Proof. We just have to apply Lemma 5.10 recursively. Furthermore $|\kappa_m| \le Dm$ with

$$D = \max_{(\eta,B)\in\mathcal{B}_{L+1}} |\kappa(\eta,B)|.$$

□

Corollary 5.5. *With the same notation as in Lemma 5.11 we have*

$$c_N'(1) = \sum_{m=1}^{M} \left(\kappa_m\eta_m + C'_{\eta_m}(1) + j_m\frac{\alpha'(1)}{\alpha(1)}\eta_m\right) G_{j_m} + \mathcal{O}(N^{1-\delta})$$

and

$$\begin{aligned} c_N''(1) &= \sum_{m=1}^{M} \left(\kappa_m(\kappa_m-1)\eta_m + 2\kappa_m C'_{\eta_m}(1) + C''_{\eta_m}(1)\right) G_{j_m} \\ &+ \sum_{m=1}^{M} \left(2\kappa_m\eta_m\frac{\alpha'(1)}{\alpha(1)} + 2C'_{\eta_m}(1)\frac{\alpha'(1)}{\alpha(1)} + \frac{\alpha''(1)}{\alpha(1)}\eta_m\right) j_m G_{j_m} \\ &+ \sum_{m=1}^{M} \left(\eta_m j_m(j_m-1)\frac{\alpha'(1)^2}{\alpha(1)^2}\right) G_{j_m} + \mathcal{O}(N^{1-\delta}) \end{aligned}$$

for some $\delta > 0$.

Proof. We just apply that $c_{\eta G_j}(z)$ and its derivatives are given by

$$\begin{aligned} c_{\eta G_j}(z) &= C_\eta(z)\alpha(z)^j + \mathcal{O}(\alpha^{(1-\delta)j}), \\ c'_{\eta G_j}(z) &= C'_\eta(z)\alpha(z)^j + C_\eta(z)j\alpha'(z)\alpha(z)^{j-1} + \mathcal{O}(\alpha^{(1-\delta)j}), \\ c'_{\eta G_j}(z) &= C''_\eta(z)\alpha(z)^j + 2C'_\eta(z)j\alpha'(z)\alpha(z)^{j-1} \\ &\quad + C_\eta(z)j\alpha''(z)\alpha(z)^{j-1} + C_\eta(z)j(j-1)\alpha'(z)^2\alpha(z)^{j-2} + \mathcal{O}(\alpha^{(1-\delta)j}) \end{aligned}$$

in a sufficiently small neighbourhood of $z = 1$ and for some $\delta > 0$. Note that by Cauchy's formula, the absolute value of the derivative of an analytic function f in some circle $|z-1| \le R$ can be essentially bounded by $\max|f|$ in a slightly larger circle $|z-1| \le R + \varepsilon$. □

Theorem 5.3.

$$\mathbf{E}X_N = \frac{1}{N}\sum_{n<N} s_G(n) = \mu\frac{\log N}{\log\alpha} + \mathcal{O}(1) \tag{5.14}$$

and

$$\mathbf{V}X_N = \frac{1}{N}\sum_{n<N} (s_G(n) - \mathbf{E}X_N)^2 = \sigma^2\frac{\log N}{\log\alpha} + \mathcal{O}(1),$$

where

$$\mu = \frac{\alpha'(1)}{\alpha} \text{ and } \sigma^2 = \frac{\alpha''(1)}{\alpha} + \mu - \mu^2.$$

Proof. Let us start with $\mathbf{E}X_N$. Firstly, we have

$$\begin{aligned}
\mathbf{E}X_N &= \frac{c_N'(1)}{N} \\
&= \frac{1}{N}\sum_{m=1}^{M}\left(\kappa_m\eta_m + C_{\eta_m}'(1) + j_m\frac{\alpha'(1)}{\alpha(1)}\eta_m\right)G_{j_m} + \mathcal{O}(N^{-\delta}) \\
&= j_1\frac{\alpha'(1)}{\alpha(1)}\frac{1}{N}\sum_{m=1}^{M}\eta_m G_{j_m} - \frac{1}{N}\sum_{m=1}^{M}\frac{\alpha'(1)}{\alpha(1)}(j_1 - j_m)\eta_m G_{j_m} \\
&\quad + \frac{1}{N}\sum_{m=1}^{M}(\kappa_m\eta_m + C_{\eta_m}'(1))G_{j_m} + \mathcal{O}(N^{-\delta}).
\end{aligned}$$

Now note that

$$\sum_{m=1}^{M}(j_1 - j_m)\eta_m G_{j_m} \le a_1\sum_{j=0}^{j_1} jG_{j_1-j} = \mathcal{O}(\sum_{j=0}^{j_1} j\alpha^{j_1-j}) = \mathcal{O}(\alpha^{j_1}) = \mathcal{O}(N)$$

and that

$$\sum_{m=1}^{M}|\kappa_m|\eta_m G_{j_m} \le \sum_{m=1}^{M} Dma_1 G_{j_m} = \mathcal{O}(N).$$

Hence,

$$\mathbf{E}X_N = j_1\frac{\alpha'(1)}{\alpha(1)} + \mathcal{O}(1).$$

Finally, since $j_1 = \log N/\log\alpha + \mathcal{O}(1)$ the representation (5.14) for $\mathbf{E}X_N$ follows.

For the variance $\mathbf{V}X_N$ we have to be a little bit more careful. Let us start with the full expansion of

$$\begin{aligned}
N(c_N''(1) + c_N'(1)) - c_N'(1)^2 &= N\sum_{m=1}^{M}\left(\kappa_m^2\eta_m + 2\kappa_m C_{\eta_m}'(1) + C_{\eta_m}'(1) + C_{\eta_m}''(1)\right)G_{j_m} \\
&\quad + 2N\frac{\alpha'(1)}{\alpha(1)}\sum_{m=1}^{M}(\kappa_m\eta_m + C_{\eta_m}'(1))j_m G_{j_m} \\
&\quad - 2\frac{\alpha'(1)}{\alpha(1)}\sum_{m=1}^{M}(\kappa_m\eta_m + C_{\eta_m}'(1))G_{j_m}\sum_{k=1}^{M} j_k\eta_k G_{j_k}
\end{aligned}$$

$$
\begin{aligned}
&+N\sum_{m=1}^{M}\left(\frac{\alpha''(1)}{\alpha(1)}+\frac{\alpha'(1)}{\alpha(1)}-\frac{\alpha'(1)^2}{\alpha(1)^2}\right)j_m\eta_m G_{j_m}\\
&+N\frac{\alpha'(1)^2}{\alpha(1)^2}\sum_{m=1}^{M}j_m^2\eta_m G_{j_m}-\frac{\alpha'(1)^2}{\alpha(1)^2}\left(\sum_{m=1}^{M}j_m\eta_m G_{j_m}\right)^2\\
&-\left(\sum_{m=1}^{M}(\kappa_m\eta_m+C'_{\eta_m}(1))G_{j_m}\right)^2+\mathcal{O}(N^{2-\delta}).
\end{aligned}
$$

Now we apply the estimates

$$
\begin{aligned}
\sum_{m=1}^{M}m^2\eta_m G_{j_m} &= \mathcal{O}(N),\\
\sum_{m=1}^{M}|\kappa_m\eta_m+C'_{\eta_m}(1)|(j_1-j_m)G_{j_m} &= \mathcal{O}(N), \qquad \left(C'_{\eta_m}(1)\le\max_{0\le i\le a_1}C'_i(1)\right)\\
\sum_{m=1}^{M}(j_1-j_m)\eta_m G_{j_m} &= \mathcal{O}(N),
\end{aligned}
$$

and

$$
N\sum_{m=1}^{M}j_m^2\eta_m G_{j_m}-\left(\sum_{m=1}^{M}j_m\eta_m G_{j_m}\right)^2=\sum_{k,m=1}^{M}j_m(j_m-j_k)\eta_m\eta_k G_{j_m}G_{j_k}
$$

$$
=\sum_{k,m=1}^{M}(j_m-j_1)((j_m-j_1)-(j_k-j_1))\eta_m\eta_k G_{j_m}G_{j_k}=\mathcal{O}(N^2)
$$

and directly obtain

$$
\mathbf{V}X_N=j_1\left(\frac{\alpha''(1)}{\alpha(1)}+\frac{\alpha'(1)}{\alpha(1)}-\frac{\alpha'(1)^2}{\alpha(1)^2}\right)+\mathcal{O}(1).
$$

As above, $j_1=\log N/\log\alpha+\mathcal{O}(1)$ and so the theorem is proved. □

5.5 Global Limit Law

With help of Lemma 5.9 and Lemma 5.10 we can prove asymptotic normality of X_N. Observe that

$$
\frac{1}{N}c_N(e^{it})=\mathbf{E}e^{itX_N}
$$

is the characteristic function of X_N.

Proposition 5.4. *Suppose that $\sigma^2 \neq 0$ and set $\mu_N = \mathbf{E}X_N$ and $\sigma_N^2 = \mathbf{V}X_N$. Then for every $\varepsilon > 0$ we have uniformly for $|t| \le (\log N)^{1/2-\varepsilon}$*

$$e^{-it\mu_N/\sigma_N}\frac{1}{N}c_N(e^{it/\sigma_N}) = e^{-t^2/2} + \mathcal{O}((\log N)^{-1/2+\varepsilon}). \tag{5.15}$$

Proof. Set $f(z) = \log\alpha(e^z)$ in an open neighbourhood of $z = 0$. Then we have

$$\alpha(e^{it}) = \alpha e^{i\mu t - \sigma^2 t^2/2 + \mathcal{O}(|t|^3)},$$

with $\mu = f'(0) = \alpha'(1)/\alpha$ and $\sigma^2 = f''(0) = \alpha''(1)/\alpha + \mu - \mu^2$ (see Theorem 5.3). Hence, by using Lemma 5.9

$$c_{\eta G_j}(e^{it}) = \eta G_j e^{j(i\mu t - \sigma^2 t^2/2)} e^{\mathcal{O}(|t| + j|t|^3)} + \mathcal{O}\left(\alpha^{(1-\delta)j}\right)$$

in an open neighbourhood of $t = 0$ in $\mathbf{R}$.

Now suppose that $N = \sum_{m=1}^{M} \eta_m G_{j_m}$ with $j_1 > j_2 > \cdots > j_M$ and $\eta_m > 0$ is the G-ary expansion of N. Then by Lemma 5.11

$$\begin{aligned} c_N(e^{it}) &= \sum_{m=1}^{M} c_{\eta_m G_{j_m}}(e^{it}) e^{it\kappa_m} \\ &= \sum_{m=1}^{M} \eta_m G_{j_m} e^{ij_m\mu t - j_m\sigma^2 t^2/2} e^{\mathcal{O}(m|t| + j_m|t|^3)} + \mathcal{O}(N^{(1-\delta)}). \end{aligned}$$

Now observe that

$$\frac{it}{\sigma_N} = \frac{it}{\sigma\, j_1^{1/2}}\left(1 + \mathcal{O}(j_1^{-1})\right),$$

and that

$$e^{-it\mu_N/\sigma_N} = e^{-it(\mu/\sigma)j_1^{1/2}(1+\mathcal{O}(j_1^{-1}))}.$$

Hence

$$\begin{aligned} \mathbf{E}e^{it(X_N-\mu_N)/\sigma_N} &= e^{-it\mu_N/\sigma_N}\frac{1}{N}c_N(e^{it/\sigma_N}) \\ &= e^{-t^2/2}\frac{1}{N}\sum_{m=1}^{M}\eta_m G_{j_m} e^{it(j_m-j_1)/(\sigma j_1^{1/2}) - (j_m-j_1)t^2/(2j_1)} \\ &\qquad \times e^{\mathcal{O}\left(m|t|/j_1^{1/2} + t^2/j_1 + |t^3|/j_1^{1/2}\right)} + \mathcal{O}(N^{-\delta}). \end{aligned}$$

Let $\varepsilon > 0$ be a (small) real number and let τ be defined by $j_\tau > j_1 - j_1^\varepsilon \geq j_{\tau+1}$. Then $\tau/2 \leq j_1 - j_\tau < j_1^\varepsilon$ and consequently

$$\begin{aligned}
\mathbf{E} e^{it(X_N-\mu_N)/\sigma_N} &= e^{-t^2/2} \sum_{m=1}^{\tau} \frac{\eta_m G_{j_m}}{N} e^{\mathcal{O}\left(|t|\sigma j_1^{\varepsilon-1/2} + t^2 j_1^{\varepsilon-1} + |t|^3 j_1^{-1/2}\right)} \\
&\quad + \mathcal{O}\left(\sum_{m=\tau+1}^{M} \frac{\eta_m G_{j_m}}{N} \right) + \mathcal{O}(N^{-\delta}) \\
&= e^{-t^2/2} e^{\mathcal{O}\left(|t|\sigma j_1^{\varepsilon-1/2} + t^2 j_1^{\varepsilon-1} + |t|^3 j_1^{-1/2}\right)} + \mathcal{O}(\alpha^{-j_1^\varepsilon}).
\end{aligned}$$

Since $j_1 = (\log N)/(\log \alpha) + \mathcal{O}(1)$ this implies (5.15) directly for $|t| \leq (\log N)^{\varepsilon/3}$. Furthermore, since

$$e^{-t^2/2 + \mathcal{O}(|t^3| j_1^{-1/2})} \leq e^{-c j_1^{2\varepsilon/3}} = \mathcal{O}(j_1^{-1})$$

for $(\log N)^{\varepsilon/3} \leq |t| \leq (\log N)^{1/2-\varepsilon}$ and a sufficiently small $c > 0$ we finally obtain the full version of (5.15). □

We use Proposition 5.4 to prove the following theorem.

Theorem 5.4. *If $\sigma^2 \neq 0$, then for every $\varepsilon > 0$*

$$\frac{1}{N} |\{n < N : s_F(n) < \mathbf{E} X_N + x \mathbf{V} X_N\}| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt + \mathcal{O}((\log N)^{-1/2+\varepsilon}) \tag{5.16}$$

uniformly for all real x as $N \to \infty$.

Proof. Set

$$\Delta_N(t) = e^{-t^2/2} - \mathbf{E} e^{it(X_N-\mu_N)/\sigma_N}.$$

Then by Esseen's inequality [15, p. 32] we have

$$\frac{1}{N} |\{n < N : s_F(n) < \mathbf{E} X_N + x \mathbf{V} X_N\}| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt + \mathcal{O}\left(\frac{1}{T} + \int_{-T}^{T} \left| \frac{\Delta_N(t)}{t} \right| dt \right).$$

We choose $T = (\log N)^{1/2-\varepsilon}$ and use the estimate

$$e^{-it\mu_N/\sigma_N} \frac{1}{N} c_N(e^{it/\sigma_N}) = 1 + \mathcal{O}(t^2)$$

for $|t| \leq (\log N)^{-2}$. Combining this with Proposition 5.4 we directly get

$$\begin{aligned}
\int_{-T}^{T} \left| \frac{\Delta_N(t)}{t} \right| dt &= \int_{|t| \leq (\log N)^{-2}} \left| \frac{\Delta_N(t)}{t} \right| dt + \int_{(\log N)^{-2} < |t| \leq T} \left| \frac{\Delta_N(t)}{t} \right| dt \\
&= \mathcal{O}\left((\log N)^{-1/2+\varepsilon} (\log\log N) \right).
\end{aligned}$$

Hence, (5.16) follows. □

5.6 Local Limit Law

In order to prove a local limit law for X_N, we need more precise information on the behaviour of $c_{\eta G_j}(z)$.

Proposition 5.5. *Suppose that α is a simple α-number and that*

$$d = \gcd\{\kappa(\eta, B) : B \in \mathcal{B}_{L+1}\} = 1.$$

Then there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$c_{\eta G_j}(e^{it}) = C_\eta(e^{it})\alpha(e^{it})^j + \mathcal{O}\left(\alpha^{(1-\delta)j}\right) \tag{5.17}$$

uniformly for $|t| \le \varepsilon$, where $C_\eta(z)$ and $\alpha(z)$ are as in Lemma 5.9, and

$$c_{\eta G_j}(e^{it}) = \mathcal{O}\left(\alpha^{(1-\delta)j}\right) \tag{5.18}$$

uniformly for $\varepsilon \le |t| \le \pi$.

Proof. Obviously, (5.17) follows from Lemma 5.9 for some $\varepsilon > 0$.

For the proof of (5.18) we just have to observe that $d = 1$ implies that all solutions t of $G(t, z) = \det(t\mathbf{I} - \mathbf{A}_L(z)) = 0$ for $z = e^{i\varphi}$, $0 < \varphi < 2\pi$, are strictly bounded by α.

If $|z| = 1$ then all entries of $\mathbf{A}_L(z)$ are complex numbers whose abolute values are bounded by those of $\mathbf{A}_L(1)$. Hence by [33, theorem 2.1, p. 36] all eigenvalues β of $\mathbf{A}_L(z)$ are bounded by $|\beta| \le \alpha$. Furthermore, $|\beta| = \alpha$ if and only if there exists a complex number λ with $|\lambda| = 1$ and a diagonal matrix $D = \mathrm{diag}(\lambda_B)_{B \in \mathcal{B}_L}$ with complex numbers λ_B with $|\lambda_B| = 1$ such that

$$\mathbf{A}_L(z) = \lambda\, D\, \mathbf{A}_L(1)\, D^{-1}.$$

Without loss of generality we may assume that $\lambda_{00\cdots 0} = 1$.

We now show that in this case we have $\lambda = 1$ and $\lambda_B = 1$ for all $B \in \mathcal{B}_L$, i.e. $\mathbf{A}_L(z) = \mathbf{A}_L(1)$. First observe that $a_{00\cdots 0,00\cdots 0}(z) = 1$ for all z. Thus, $\lambda = 1$. More generally, if $a_{B,C}(z) = a_{B,C}(1) = 1$ then $\lambda_B = \lambda_C$. Obviously, we have $a_{B,C}(z) = a_{B,C}(1) = 1$ if $B' = {}'C$ and $\eta_B = 0$. Thus, if $B = (\eta_1, \dots, \eta_L)$ is any block in $\mathcal{B}_L$ then we can consider the sequence

$$B_0 = (0, 0, \dots, 0),\ B_1 = (0, \dots, 0, \eta_1),\ B_2 = (0, \dots, 0, \eta_1, \eta_2),\ \dots,\ B_{L-1} = B$$

and can conclude that

$$1 = \lambda_{B_0} = \lambda_{B_1} = \cdots = \lambda_B.$$

However, if $d = 1$ then for every φ, $0 < \varphi < 2\pi$, there exists $(\epsilon, B) \in \mathcal{B}_{L+1}$ with $e^{it\kappa(\epsilon,B)} \ne 1$. Consequently, if $d = 1$ then all eigenvalues β of $\mathbf{A}_L(e^{it})$ are strictly bounded by $|\beta| < \alpha$. $\square$

Remark 5.9. If $d > 1$ then we have $\mathbf{A}_L(z) = \mathbf{A}_L(1)$ if and only if z is a d-th root of unity, and therefore we get a periodic structure, i.e. a local limit law for every residue class of d.

With help of Proposition 5.5 it is possible to derive asymptotic expansions for the coefficients

$$c_{N,k} = |\{n < N : s_F(n) = k\}|$$

of

$$c_N(z) = \sum_{k\geq 0} c_{N,k} z^k$$

for $N = \eta G_j$, $1 \leq \eta \leq Q$, via saddle point approximations.

Proposition 5.6. *We have*

$$c_{\eta G_j,k} = \frac{\eta G_j}{\sqrt{2\pi j\sigma^2}}\left(\exp\left(-\frac{(k-j\mu)^2}{2j\sigma^2}\right) + \mathcal{O}(j^{-1/2})\right)$$

uniformly for all $j, k \geq 0$.

Proof. We again use Cauchy's formula

$$c_{\eta G_j,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} c_{\eta G_j}(e^{it})e^{-ikt}\,dt.$$

Since

$$\int_{\varepsilon\leq|t|\leq\pi} |c_{\eta G_j}(e^{it})|\,dt = \mathcal{O}\left(\alpha^{(1-\delta)j}\right) = \mathcal{O}(G_j/j)$$

we just have to evaluate

$$I = \frac{1}{2\pi}\int_{|t|\leq j^{-\nu}} c_{\eta G_j}(e^{it})e^{-ikt}\,dt + \frac{1}{2\pi}\int_{j^{-\nu}\leq|t|\leq\varepsilon} c_{\eta G_j}(e^{it})e^{-ikt}\,dt = I_1 + I_2,$$

where $0 < \nu < \frac{1}{6}$. From $\alpha(e^{it}) = \alpha e^{i\mu t - \sigma^2 t^2/2 + \mathcal{O}(|t|^3)}$ it follows that there exists a constant $c > 0$ such that $|\alpha(e^{it})| \leq e^{-ct^2}$ for $|t| \leq \varepsilon$. Hence,

$$I_2 \leq \frac{1}{\pi}\int_{j^{-\nu}}^{\infty} e^{-cjt^2}\,dt + \mathcal{O}\left(\alpha^{(1-\delta)j}\right) = \mathcal{O}\left(e^{-cj^{1-2\nu}}\right) + \mathcal{O}\left(\alpha^{(1-\delta)j}\right) = \mathcal{O}(G_j/j)$$

Finally,

$$I_1 = \frac{1}{2\pi}\int_{|t|\leq j^{-\nu}} C_\eta(1)\alpha^j e^{it(j\mu-k)-j\sigma^2t^2/2}\left(1+\mathcal{O}(j|t|^3+|t|)\right)dt + \mathcal{O}\left(\alpha^{(1-\delta)j}\right)$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} C_\eta(1)\alpha^j e^{it(j\mu-k)-j\sigma^2t^2/2}\,dt + \mathcal{O}\left(\int_{|t|>j^{-\nu}} C_\eta(1)\alpha^j e^{-j\sigma^2t^2/2}\,dt\right)\\
&\quad +\mathcal{O}\left(\int_{|t|\le j^{-\nu}} C_\eta(1)\alpha^j e^{-j\sigma^2t^2/2}(j|t|^3+|t|)\,dt\right) + \mathcal{O}\left(\alpha^{(1-\delta)j}\right)\\
&= \frac{C_\eta(1)\alpha^j}{\sqrt{2\pi j\sigma^2}}\exp\left(-\frac{(k-j\mu)^2}{2j\sigma^2}\right) + \mathcal{O}(\alpha^j/j)\\
&= \frac{\eta G_j}{\sqrt{2\pi j\sigma^2}}\exp\left(-\frac{(k-j\mu)^2}{2j\sigma^2}\right) + \mathcal{O}(G_j/j).
\end{aligned}
$$

□

Proposition 5.6 and Lemma 5.11 can be used to prove the following theorem.

Theorem 5.5. *If α is a simple α-number, $\sigma^2 \neq 0$, F just attains integer values and*

$$d = \gcd\{\kappa(\epsilon, B) : (\epsilon, B) \in \mathcal{B}_{L+1}\} = 1,$$

then for every $\varepsilon > 0$

$$|\{n < N : s_F(n) = k\}| = \frac{N}{\sqrt{2\pi \mathbf{V}X_N}}\left(\exp\left(-\frac{(k-\mathbf{E}X_N)^2}{2\mathbf{V}X_N}\right) + \mathcal{O}((\log N)^{-1/2+\varepsilon})\right)$$

uniformly for all non-negative integers k as $N \to \infty$.

Proof. As in the proof of Proposition 5.4 we suppose that $N = \sum_{m=1}^{M} \eta_m G_{j_m}$ (with $j_1 > j_2 > \cdots > j_M$ and $\eta_m > 0$) is the G-ary expansion of N. Furthermore, let $\varepsilon > 0$ be a (small) real number and let τ be defined by $j_\tau > j_1 - j_1^\varepsilon \geq j_{\tau+1}$. Then by (5.13)

$$
\begin{aligned}
c_{N,k} &= \sum_{m=1}^{M} c_{\eta_m G_{j_m}, k-\kappa_m}\\
&= \sum_{m=1}^{\tau} c_{\eta_m G_{j_m}, k-\kappa_m} + \mathcal{O}\left(\sum_{l=\tau+1}^{M} \frac{\eta_m G_{j_m}}{j_l^{1/2}}\right)\\
&= \sum_{m=1}^{\tau} \frac{\eta_m G_{j_m}}{\sqrt{2\pi j_m\sigma^2}}\exp\left(-\frac{(k-\kappa_m-j_m\mu)^2}{2j_m\sigma^2}\right) + \mathcal{O}(G_{j_1}/j_1).
\end{aligned}
$$

If $m < \tau$ and $|k-\mu_N| = \mathcal{O}(j_1^{1/2}\log j_1)$ then

$$
\begin{aligned}
\frac{(k-\mu_N)^2}{2\sigma_N^2} &- \frac{(k-\kappa_m-j_m\mu)^2}{2j_m\sigma^2} \\
&= \frac{(k-\mu_N)^2-(k-\kappa_m-j_m\mu)^2}{2\sigma_N^2} + (k-\kappa_m-j_m\mu)^2\left(\frac{1}{2\sigma_N^2}-\frac{1}{2j_m\sigma^2}\right) \\
&= \mathcal{O}\left(j_1^{\varepsilon-1/2}\log j_1\right) + \mathcal{O}\left(j_1^{-1}(\log j_1)^2\right),
\end{aligned}
$$

where we have used $\mu_N = j_1\mu + \mathcal{O}(1)$, $\sigma_N^2 = j_1\sigma^2 + \mathcal{O}(1)$, and $\kappa_m = \mathcal{O}(m)$. Hence, from

$$
\sqrt{\frac{j_1+\mathcal{O}(1)}{j_m}}\exp\left(\frac{(k-\mu_N)^2}{2\sigma_N^2} - \frac{(k-\kappa_m-j_m\mu)^2}{2j_m\sigma^2}\right) = 1 + \mathcal{O}\left(j_1^{\varepsilon-1/2}\log j_1\right)
$$

we obtain

$$
\begin{aligned}
c_{N,k} &= \frac{N}{\sqrt{2\pi\sigma_N^2}}\exp\left(-\frac{(k-\mu_N)^2}{2\sigma_N^2}\right)\sum_{m=0}^{\tau}\frac{\eta_m G_{j_m}}{N}\left(1+\mathcal{O}\left(j_1^{\varepsilon-1/2}\log j_1\right)\right) + \mathcal{O}(G_{j_1}/j_1) \\
&= \frac{N}{\sqrt{2\pi\sigma_N^2}}\left(\exp\left(-\frac{(k-\mu_N)^2}{2\sigma_N^2}\right) + \mathcal{O}\left(j_1^{\varepsilon-1/2}\log j_1\right)\right).
\end{aligned}
$$

If $|k-\mu_N| \ge j_1^{1/2}\log j_1$ then we have for $m < \kappa$

$$
\begin{aligned}
c_{\eta_m G_{j_m}, k-\kappa_m} &= \mathcal{O}\left(\alpha^{j_m} j_1^{-1/2}\exp\left(-\frac{(\log j_1)^2}{4\sigma^2}\right)\right) \\
&= \mathcal{O}\left(\alpha^{j_m} j_1^{-1}\right)
\end{aligned}
$$

which finally gives

$$
\begin{aligned}
c_{N,k} &= \mathcal{O}(\left(\alpha^{j_1} j_1^{-1}\right) + \mathcal{O}\left(\sum_{m=\tau+1}^{M}\frac{G_{j_m}}{j_m^{1/2}}\right) \\
&= \mathcal{O}(G_{j_1}/j_1).
\end{aligned}
$$

□

Remark 5.10. The case $d > 1$ can be treated in a similar way. (See Remark 5.9.) However, for the sake of shortness and simplicity we just formulated Theorem 5.5 for $d = 1$.

Remark 5.11. It is an open problem if corresponding theorems to Theorem 5.4 and Theorem 5.5 hold for real numbers α which are not α-numbers and simple α-numbers respectively. The case $L = 0$ is discussed in [11].

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