Extremal properties of (epi)sturmian sequences and applications: a survey

J.-P. Allouche
CNRS, LRI
Bâtiment 490
F-91405 Orsay Cedex
France
allouche@lri.fr

1 Introduction

This extended abstract is the first announcement of a paper with A. Glen (Adelaide) in preparation.

A few months ago JPA came across a paper of Y. Bugeaud and A. Dubickas [5] where the authors were interested in describing all irrational numbers $\xi > 0$ such that the fractional parts $\{\xi b^n\}$, $n \geq 0$, all belong to an interval of length $1/b$, where $b \geq 2$ is a given integer. Furthermore they prove that $1/b$ is the minimal length having this property. An interesting and unexpected result in their paper is that, when the interval is closed and its length is exactly $1/b$, the irrational numbers are exactly the positive real numbers whose base b expansion is a characteristic Sturmian sequence on $\{k, k+1\}$, where $k \in \{0, 1, \ldots, b-2\}$.

2 More on Bugeaud-Dubickas' result

Looking at the proofs in [5] one sees that the core of the result is the following property:

Theorem 1 *A binary sequence $u := (u_n)_{n \geq 0}$ is a characteristic Sturmian sequence if and only if, for all $k \geq 0$,*

$$0u \leq T^k u \leq 1u$$

where T is the shift defined by $T((u_n)_{n\geq 0}) = (u_{n+1})_{n \geq 0}$ and the order is the lexicographical order.

Actually this theorem was known. It was indicated to JPA by G. Pirillo (who published it in [9]): JPA suggested that this could well be already in a paper by S. Gan [7] under a slightly disguised form (which is indeed the case). Also J.-P. Borel and F. Laubie proved one direction of the above theorem, namely that characteristic Sturmian sequences satisfy the inequalities $0u \leq T^k u \leq 1u$ [4].

3 Generalizations

Two directions for generalizations are possible. One is purely combinatorial and looks at generalizations of Sturmian sequences: in particular episturmian sequences have some aspects of Sturmian sequences and they have similar extremal properties [8, 10]. The other is number-theoretic and looks at distribution modulo 1 from a combinatorial point of view: several recent papers of Dubickas go in this direction, we cite one of them [6] since the Thue-Morse sequence appears in it.

4 The Thue-Morse sequence shows up

In the paper of Dubickas [6] the Thue-Morse sequence appears wheen studying the "small" and "large" limit points of $\|\xi(p/q)^n\|$ the distance to the nearest integer of the product of any nonzero real number ξ by the powers of a rational.

Interestingly enough this sequence appeared in 1983 in another question of distribution as a by-product of the combinatorial study of a set of sequences related to iterating continuous maps of the interval [1, 2, 3].

Theorem 2 *Define the set* Γ *by*

$$\Gamma := \{x \in [0,1],\ 1 - x \leq \{2^k x\} \leq x\}.$$

Then the smallest limit point of Γ *is the number* $\alpha := \sum a_n/2^n$*, where* $(a_n)_{n\geq 0}$ *is the Thue-Morse sequence. The set* Γ *contains only countably many elements less than* α *and they are all rational. Furthermore any segment on the right of* α *contains uncountably many elements of* Γ*. This structure around* α *repeats:* Γ *is a fractal set.*

The reader will have guessed that the above theorem is a by-product of the combinatorial study of the set

$$\Gamma := \{u \in \{0,1\}^{\mathbb{N}},\ \forall k \geq 0,\ \overline{u} \leq T^k u \leq u\}$$

where $\overline{u}$ is the sequence obtained by switching 0's and 1's in u.

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