# On digit patterns in expansions of rational numbers with prime denominator

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August 24, 2012

**Abstract**

We show that, for any fixed $\varepsilon > 0$ and almost all primes $p$, the $g$-ary expansion of any fraction $m/p$ with $\gcd(m,p) = 1$ contains almost all $g$-ary strings of length $k < (17/72 - \varepsilon)\log_g p$. This complements a result of J. Bourgain, S. V. Konyagin, and I. E. Shparlinski that asserts that, for almost all primes, all $g$-ary strings of length $k < (41/504 - \varepsilon)\log_g p$ occur in the $g$-ary expansion of $m/p$.

## 1 Introduction

Let us fix some integer $g \geq 2$. It is well-known that if $\gcd(n, gm) = 1$ then the $g$-ary expansion of the rational fractions $m/n$ is purely periodic with period $t_n$, which is independent of $m$ and equals the multiplicative order of $g$ modulo $n$, see [9]. In the series of works [3, 8, 9], the distribution of digit patterns in such expansions has been studied. In particular, for positive integers $k$ and $m < n$ with $\gcd(n, gm) = 1$, we denote by $T_{m,n}(k)$ the number of distinct $g$-ary strings $(d_1, \ldots, d_k) \in \{0, 1, \ldots, g-1\}^k$ that occur among the first $t_n$ strings $(\delta_r, \ldots, \delta_{r+k-1})$, $r = 1, \ldots, t_n$, from the $g$-ary expansion

$$\frac{m}{n} = \sum_{r=1}^{\infty} \delta_r g^{-r}, \quad \delta_r \in \{0, 1, \ldots, g-1\}. \tag{1}$$

Motivated by applications to pseudorandom number generators, see [1], we are interested in describing the conditions under which $T_{m,n}(k)$ is close to its trivial upper bound

$$T_{m,n}(k) \le \min\{t_n, g^k\}.$$

Since $t_n \le n$, it is clear that only values $k \le \lceil \log_g n \rceil$ are of interest. It has been shown in [8, Theorem 11.1] that, for any fixed $\varepsilon > 0$ and for almost all primes $p$ (that is, for all but $o(x/\log x)$ primes $p \le x$), we have $T_{m,p}(k) = g^k$, provided that $k \le (3/37 - \varepsilon) \log_g p$. The coefficient 3/37 has been increased up to 41/504 in [3, Corollary 8]. Here we show that, for almost all primes $p$, we have $T_{m,p}(k) = (1 + o(1))g^k$ for much larger string lengths $k$.

**Theorem 1.** *For any fixed $\varepsilon > 0$, for almost all primes $p$, we have*

$$T_{m,p}(k) = (1 + o(1))g^k$$

*as $p \to \infty$, provided that $k \le (17/72 - \varepsilon) \log_g p$.*

Our arguments depend on the reduction of the problem to the study of intersections of intervals and multiplicative groups modulo $p$ generated by $g$, that has been established in [8]. In turn, the question about the intersections of intervals and subgroups in residue rings has been studied in a number of works [3, 4, 8]. In particular, the results of [3, Corollary 8] and [8, Theorem 11.1] are based on estimates of the length of the longest interval that is not hit by a subgroup of the multplicative group $\mathbb{F}_p^*$ of the field $\mathbb{F}_p$ of $p$ elements. To prove Theorem 1, we use the results and ideas of [3] to estimate the total number of intervals of a given length that do not intersect a given subgroup of $\mathbb{F}_p^*$.

Throughout the paper, the implied constants in the symbols '$O$', '$\ll$' and '$\gg$' may occasionally, where obvious, depend on the small real parameter $\varepsilon > 0$. We recall that the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are all equivalent to the assertion that the inequality $|U| \le c|V|$ holds for some constant $c > 0$.

## 2 Multiplicative orders

We recall the following well-known implication of the classical result of [5].

**Lemma 2.** *For almost all primes $p$, the multiplicative order $t$ of $g$ modulo $p$ satisfies $t > p^{1/2}$.*

## 3 Bounds for some exponential sums

Let $p$ be prime and let $\mathcal{G} \subseteq \mathbb{F}_p^*$ be a subgroup of order $t$, where $\mathbb{F}_p$ is a finite field of $p$ elements.

We denote

$$\mathbf{e}_p(z) = \exp(2\pi i z/p)$$

and define exponential sums

$$S_\lambda(p; \mathcal{G}) = \sum_{v \in \mathcal{G}} \mathbf{e}_p(\lambda v).$$

Using [6, Lemma 3] (see also [8, Lemma 3.3]) if $t < p^{2/3}$, and the well known bounds

$$|S_\lambda(p; \mathcal{G})| \le p^{1/2} \quad \text{and} \quad \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^2 \le pt$$

(see [8, Equations (3.4) and (3.15)]) if $t \ge p^{2/3}$, we derive:

**Lemma 3.** *For any prime $p$ and a subgroup $\mathcal{G} \subseteq \mathbb{F}_p^*$ of order $t$, we have*

$$\sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^4 \ll pt^{5/2}.$$

Finally, for small values of $t$ we use the following bound of Shkredov [11, Theorem 34].

**Lemma 4.** *For any prime $p$ and a subgroup $\mathcal{G} \subseteq \mathbb{F}_p^*$ of order $t \ll p^{6/11}$, we have*

$$\sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^4 \ll pt^{22/9} (\log t)^{2/3}.$$

## 4 Intervals avoiding subgroups

As before, let $p$ be prime and let $\mathcal{G} \subseteq \mathbb{F}_p^*$ be a subgroup of order $t$.

Let $\mathcal{U}(p;\mathcal{G},H)$ be the set of $u \in \mathbb{F}_p$ such the congruence

$$v \equiv u + x \pmod p, \qquad v \in \mathcal{G},\ 0 \le x < H,$$

has no solution.

**Lemma 5.** *Assume that $\mathcal{G}$ is of order $t > p^{1/2}$. Then, for any fixed integer $\nu \ge 1$, we have*

$$\#\mathcal{U}(p;\mathcal{G},H) \le p^{2-1/4(\nu+1)+o(1)} H^{-1/2} t^{-5/4+(2\nu+1)/4\nu(\nu+1)} + p^{5/2-1/2\nu+o(1)} H^{-1} t^{-5/4+1/2\nu}.$$

*Proof.* Let us fix some $\varepsilon > 0$. We put

$$s = \left\lceil \frac{3}{2}(1+\varepsilon^{-1}) \right\rceil, \qquad h = \left\lceil p^{1+\varepsilon}/H \right\rceil, \qquad Z = \lceil H/s \rceil .$$

We can assume that $h < p/2$, as otherwise the bound is trivial (for example, it follows immediately from the bound of Heath-Brown and Konyagin [6, Theorem 1]). Obviously

$$\mathcal{U}(p;\mathcal{G},H) \subseteq \mathcal{W}_s(p;\mathcal{G},Z), \qquad (2)$$

where $\mathcal{W}_s(p;\mathcal{G},Z)$ is the set of $u \in \mathbb{F}_p$ such the congruence

$$v \equiv u + x_1 + \ldots + x_s \pmod p, \quad v \in \mathcal{G}, \quad 0 \le x_1,\ldots,x_s < Z, \qquad (3)$$

has no solution.

For the number $Q_s(p;\mathcal{G},Z,u)$ of solutions to the congruence (3), exactly as in the proof of [8, Lemma 7.1], we obtain

$$Q_s(p;\mathcal{G},Z,u) = \frac{1}{p} \sum_{|a|<p/2} \mathbf{e}_p(-au) \left( \sum_{0\le x<Z} \mathbf{e}_p(ax) \right)^s S_a(p;\mathcal{G}),$$

where the sums $S_a(p;\mathcal{G})$ are defined in Section 3.

Separating the term $tZ^s p^{-1}$ corresponding to $a = 0$ and summing over all $u \in \mathcal{W}_s(p; \mathcal{G}, Z)$ yields

$$0 = \sum_{u \in \mathcal{W}_s(p;\mathcal{G},Z)} Q_s(p; \mathcal{G}, Z, u) \geq \frac{tWZ^s}{p} - \frac{\sigma}{p},$$

where

$$W = \#\mathcal{W}_s(p; \mathcal{G}, Z)$$

and

$$\sigma = \sum_{1 \leq |a| < p/2} \left| \sum_{u \in \mathcal{W}_s(p;\mathcal{G},Z)} \mathbf{e}_p(au) \right| \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^s |S_a(p; \mathcal{G})| .$$

Using the Cauchy inequality, and then the orthogonality relation for exponential functions, we obtain

$$\begin{aligned} \sigma^2 &\leq \sum_{1 \leq |a| < p/2} \left| \sum_{u \in \mathcal{W}_s(p;\mathcal{G},Z)} \mathbf{e}_p(au) \right|^2 \sum_{1 \leq |a| < p/2} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2 \\ &\leq pW \sum_{1 \leq |a| < p/2} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2 . \end{aligned}$$

Hence

$$W \leq \frac{p}{t^2 Z^{2s}} \Sigma, \tag{4}$$

where

$$\Sigma = \sum_{1 \leq |a| < p/2} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2 .$$

Following the idea of the proof of [8, Lemma 7.1], we write

$$\Sigma = \Sigma_1 + \Sigma_2, \tag{5}$$

where

$$\Sigma_1 = \sum_{1 \leq |a| \leq h} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2 ,$$

$$\Sigma_2 = \sum_{h < |a| < p/2} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2 .$$

For $1 \le |a| \le h$, we use the trivial estimate

$$\left| \sum_{0 \le x < Z} \mathbf{e}_p(ax) \right| \le Z$$

and derive

$$\begin{aligned} \Sigma_1 &\le Z^{2s} \sum_{1 \le |a| \le h} |S_a(p; \mathcal{G})|^2 = \frac{Z^{2s}}{t} \sum_{1 \le |a| \le h} \sum_{w \in \mathcal{G}} |S_{aw}(p; \mathcal{G})|^2 \\ &= \frac{Z^{2s}}{t} \sum_{\lambda \in \mathbb{F}_p^*} M_\lambda(p; \mathcal{G}, h) \ |S_\lambda(p; \mathcal{G})|^2 , \end{aligned}$$

where $M_\lambda(p; \mathcal{G}, h)$ denotes the number of solutions to the congruence

$$\lambda \equiv aw \pmod p, \quad 1 \le |a| \le h, \quad w \in \mathcal{G}.$$

Hence, by the Cauchy inequality

$$\Sigma_1 \le \frac{Z^{2s}}{t} \left( \sum_{\lambda \in \mathbb{F}_p^*} M_\lambda(p; \mathcal{G}, h)^2 \right)^{1/2} \left( \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^4 \right)^{1/2} .$$

As in [3, Section 3.3], we have

$$\sum_{\lambda \in \mathbb{F}_p^*} M_\lambda(p; \mathcal{G}, h)^2 \le t N(p; \mathcal{G}, h),$$

where $N(p; \mathcal{G}, h)$ is the number of solutions of the congruence

$$ux \equiv y \pmod p, \quad 0 < |x|, |y| \le h, \quad u \in \mathcal{G}.$$

Therefore,

$$\Sigma_1 \le \frac{Z^{2s}}{t^{1/2}} N(p; \mathcal{G}, h)^{1/2} \left( \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^4 \right)^{1/2} . \qquad (6)$$

It is shown in [3, Theorem 1] that if $t \ge p^{1/2}$ then for any fixed integer $\nu$ and any positive number $h$, we have

$$N(p; \mathcal{G}, h) \le h t^{(2\nu+1)/2\nu(\nu+1)} p^{-1/2(\nu+1)+o(1)} + h^2 t^{1/\nu} p^{-1/\nu+o(1)}. \qquad (7)$$

Therefore, using Lemma 3 and the bound (7) we derive from (6) that

$$\Sigma_1 \le p^{1/2} t^{3/4} Z^{2s} \left( h^{1/2} t^{(2\nu+1)/4\nu(\nu+1)} p^{-1/4(\nu+1)+o(1)} + h t^{1/2\nu} p^{-1/2\nu+o(1)} \right). \quad (8)$$

If $h < |a| < p/2$, then we use the bound

$$\sum_{0 \le x < Z} \mathbf{e}_p(ax) \ll \frac{p}{|a|},$$

see [7, Bound (8.6)]. From the trivial bound

$$|S_a(p; \mathcal{G})| \le t,$$

recalling the choice of $h$, we obtain

$$\Sigma_2 \ll \sum_{h<|a|<p/2} \left( \frac{p}{|a|} \right)^{2s} t^2 \ll t^2 \frac{p^{2s}}{h^{2s-1}} \ll t^2 \frac{Z^{2s} h}{p^{2s\varepsilon}} \le \frac{Z^{2s} p^3}{p^{2s\varepsilon}} \ll Z^{2s},$$

as $2s\varepsilon > 3$ for our choice of $s$. Thus the bound on $\Sigma_2$ is dominated by the bound (8) on $\Sigma_1$. Using (4) and (5), we obtain

$$W \le p^{3/2} t^{-5/4} \left( h^{1/2} t^{(2\nu+1)/4\nu(\nu+1)} p^{-1/4(\nu+1)+o(1)} + h t^{1/2\nu} p^{-1/2\nu+o(1)} \right).$$

Recalling (2), the choice of $h$ and that $\varepsilon$ is arbitrary, after simple calculations, we obtain the result. □

Similarly, for small values of $t$ we can use Lemma 4 instead of Lemma 3 and derive

**Lemma 6.** *Assume that $\mathcal{G}$ is of order $p^{6/11} \gg t > p^{1/2}$. Then, for any fixed integer $\nu \ge 1$, we have*

$$\#\mathcal{U}(p; \mathcal{G}, H) \le p^{2-1/4(\nu+1)+o(1)} H^{-1/2} t^{-23/18+(2\nu+1)/4\nu(\nu+1)} + p^{5/2-1/2\nu+o(1)} H^{-1} t^{-23/18+1/2\nu}.$$

We now derive from Lemmas 5 and 6:

**Corollary 7.** *Assume that $\mathcal{G}$ is of order $t > p^{1/2}$. Then for any $\varepsilon > 0$ and*

$$H \ge p^{55/72+\varepsilon}$$

*we have*

$$\#\mathcal{U}(p; \mathcal{G}, H) = o(p).$$

*Proof.* For $t > p^{6/11}$, by Lemma 5, we have, for any fixed integer $\nu \geq 1$,

$$\#\mathcal{U}(p; \mathcal{G}, H) \leq p^{29/22+(\nu+6)/44\nu(\nu+1)+o(1)} H^{-1/2} + p^{20/11-5/22\nu+o(1)} H^{-1}.$$

Taking $\nu = 2$, we obtain

$$\#\mathcal{U}(p; \mathcal{G}, H) \leq p^{89/66+o(1)} H^{-1/2} + p^{75/44+o(1)} H^{-1} = o\big(p^{1531/1584}\big)$$

in this case.

For $p^{6/11} \gg t > p^{1/2}$, by Lemma 6, we have, for any fixed integer $\nu \geq 1$,

$$\#\mathcal{U}(p; \mathcal{G}, H) \leq p^{49/36+1/8\nu(\nu+1)+o(1)} H^{-1/2} + p^{67/36-1/4\nu+o(1)} H^{-1}.$$

Taking again $\nu = 2$ gives $\#\mathcal{U}(p; \mathcal{G}, H) = o(p)$, which concludes the proof. $\square$

# 5 Proof of Theorem 1

By Lemma 2 it is enough to consider prime $p$ for which the multiplicative order $t$ of $g$ modulo $p$ satisfies $t > p^{1/2}$.

We now take a positive integer $k \leq (17/72 - \varepsilon) \log_g p$ and consider the intervals $\left[\frac{D}{g^k}, \frac{D+1}{g^k}\right)$. As in the proof of [8, Theorem 11.1], we observe that, for any integer $\ell \geq 0$ and any $g$-ary string $(d_1, \ldots, d_k)$, we have $\delta_{\ell+i} = d_i$, $i = 1, \ldots, k$, if and only if

$$\frac{mg^\ell}{p} - \left\lfloor \frac{mg^\ell}{p} \right\rfloor \in \left[\frac{D}{g^k}, \frac{D+1}{g^k}\right),$$

where $D = d_1 g^{k-1} + d_2 g^{k-2} + \cdots + d_k$ and the $\delta_r$, $r = 1, 2, \ldots$, are defined by (1) with $n = p$. Thus, if a string $(d_1, \ldots, d_k)$ is not present in the $g$-ary expansion of $m/p$, then each interval $[u, u + H)$ with

$$u = \left\lceil \frac{D}{g^k} p \right\rceil, \ldots, \left\lfloor \frac{D + 1/2}{g^k} p \right\rfloor \quad \text{and} \quad H = \left\lfloor \frac{1}{2g^k} p \right\rfloor$$

contains no element of the conjugacy class $m\mathcal{G}_p$ of the group $\mathcal{G}_p$ generated by $g$ modulo $p$. Clearly, different strings $(d_1, \ldots, d_k)$ correspond to different intervals of the values of $u$, and each of them contains

$$\left\lceil \frac{D + 1/2}{g^k} p \right\rceil - \left\lceil \frac{D}{g^k} p \right\rceil \gg \frac{p}{g^k}$$

values of $u$. Therefore, the number of missing strings $(d_1, \ldots, d_k)$ satisfies

$$g^k - T_{m,p}(k) \ll \frac{g^k}{p} \# \mathcal{U}(p; \mathcal{G}_p, H).$$

Since $g^k \le p^{17/72-\varepsilon}$, we infer from Corollary 7 that $\#\mathcal{U}(p; \mathcal{G}_p, H) = o(p)$, which proves Theorem 1.

# 6 Comments

We note that the constant 41/504 of [3, Corollary 8] is based only on Lemma 3. Certainly using the recent bound of Lemma 4, one can improve this value.

It is quite likely that one can also study $T_{m,n}(k)$ for almost all composite $n$ by supplementing the ideas of this work with those of [2] (to get an analogue of Lemma 3) and also using the result of [10] that gives an analogue of Lemma 2.

# Acknowledgements

The authors are very grateful to Ilya Shkredov for very useful comments and especially for the suggestion to use [11] to improve our original results.

The second author wishes to express his heartfelt thanks to the members of the Department of Computing of the Macquarie University for their hospitality during his stay as a visiting academic.

During the preparation of this work the first author was supported in part by the Australian Research Council Grant DP1092835.

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