Pushdown compression

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Motivations

- New compression algorithms for structured documents (XML): behaviour depending on the current tag
  $\rightarrow$ use of a stack to push and pop tags.

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- Very simple algorithms $\rightarrow$ pushdown automata.
- Easy to compress and decompress.
  Practical tests: better performances than zip (Lempel-Ziv).

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- New compression algorithms for structured documents (XML): behaviour depending on the current tag
  $\rightarrow$ use of a stack to push and pop tags.
- Very simple algorithms $\rightarrow$ pushdown automata.
- Easy to compress and decompress.
  Practical tests: better performances than zip (Lempel-Ziv).
- Need for a theoretical study.

Outline

1. Introduction (LZ, FS)
2. Pushdown compression
3. Pushdown beats LZ
4. LZ beats pushdown
5. Conclusion

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- Compression ratio on a finite word x:

$$\rho_f(x) = \frac{|f(x)|}{|x|}.$$

Compression ratio on an infinite sequence S:

$$\rho_f(S) = \limsup_{n\to\infty} \rho_f(S[1..n]).$$

Lempel-Ziv

Text to be compressed:

0 1 0 0 0 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:

Lempel-Ziv

Text to be compressed:

0

ϵ/0 1 0 0 0 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:

ϵ;

Lempel-Ziv

Text to be compressed:

0 1

ϵ/0/1 0 0 0 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:

$\epsilon;(0,0);$

Lempel-Ziv

Text to be compressed:

0 1 2

ϵ/0/1/0 0 0 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:

ϵ;$(0, 0)$;$(0, 1)$;

Lempel-Ziv

Text to be compressed:

0 1 2 3

ϵ/0/1/0 0/0 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:

ϵ;$(0, 0)$;$(0, 1)$;$(1, 0)$;

Lempel-Ziv

Text to be compressed:

0 1 2 3   4

ϵ/0/1/0 0/0 1/0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:

$\epsilon;(0,0);(0,1);(1,0);(1,1);$

Lempel-Ziv

Text to be compressed:

0 1 2 3   4   5

ϵ/0/1/0 0/0 1/0 1 1/1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:

$\epsilon;(0,0);(0,1);(1,0);(1,1);(4,1);$

Lempel-Ziv

Text to be compressed:

0 1 2 3 4 5 6 7 8 9 10

ϵ/0/1/0 0/0 1/0 1 1/1 0/1 0 0/1 0 0 0/0 0 1/1 1

Compression result:

ϵ;$(0,0)$;$(0,1)$;$(1,0)$;$(1,1)$;$(4,1)$;$(2,0)$;$(6,0)$;$(7,0)$;$(3,1)$;$(2,1)$

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Text to be compressed:

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ϵ/0/1/0 0/0 1/0 1 1/1 0/1 0 0/1 0 0 0/0 0 1/1 1

Compression result:

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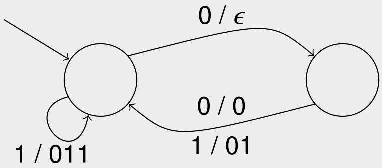
Lemma

- *If p is the number of phrases, then $|LZ(x)| = p \log p$.*
- *For all x, the compression ratio $\rho_{LZ}(x)$ satisfies*

$$\frac{\log |x|}{\sqrt{|x|}} \leq \rho_{LZ}(x) \leq 1 + o(1).$$

Finite-state compression (1)

Finite-state transducer: finite-state automaton that outputs letters at each transition
$\to$ function $f : \{0,1\}^* \to \{0,1\}^*$.



$$00 \mapsto 0, \quad 01 \mapsto 01, \quad 1 \mapsto 011.$$

Example: 00 00 01 1 1 00 $\mapsto$ 0 0 01 011 011 0.

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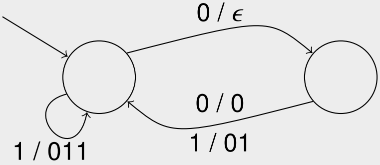

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Example: 00 00 01 1 1 00 $\mapsto$ 0 0 01 011 011 0.

Finite-state compressor: injective finite-state transducer (taking into account the final state).

Finite-state compression (2)

For a finite-state compressor C: compression ratio of an infinite sequence S

$$\rho_C(S) = \limsup_{n \to \infty} \frac{|C(S[1..n])|}{n}.$$

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Finite-state compression ratio:

$$\rho_{FS}(S) = \inf_{C \in FS} \rho_C(S).$$

LZ better than FS

Theorem (Lempel, Ziv, 1979)

On every infinite sequence $S \in \{0,1\}^{\mathbb{N}}$, Lempel-Ziv is better than any finite-state compressor, that is,

$$\rho_{LZ}(S) \leq \rho_{FS}(S).$$

Outline

Pushdown transducers

Pushdown compressor = finite-state transducer with a stack.

The transition is done according both to the symbol read and to the topmost symbol of the stack.

Each transition either pushes or pops symbols from the stack.

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PD compression ratio: $\rho_{PD}(S) = \inf_{C \in PD} \rho_C(S)$.

Two variants: with or without endmarkers
$\rightarrow C(x)$ or $C(x\#)$ (enables to empty the stack).

Example

Proposition

Let $S = 0^\infty$.

- *The compression ratio on* S *of a finite-state compressor with* k *states is* $\geq 1/k$.
- *There exists a pushdown compressor with* k *states whose compression ratio on* S *is* $\leq 1/k^2$ *(with endmarkers).*

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Proof.

1. Let C be a FS compressor with k states.

Then C must output at least one symbol every k letters.

Otherwise there would exist u such that for all $i_0 \leq i \leq i_0 + k$, all the $u[1..i]$ have the same image.

Since there are only k states, this contradicts injectivity.

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Proof.

2. Let C be the following pushdown compressor on input 0^n:
   - it pushes $0^{n/k}$ on the stack (by counting modulo k);
   - at the end it pops the stack and outputs one symbol every k (by counting modulo k).

□

Remarks

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- LZ on $S = 0^\infty$ has compression ratio $0\ldots$
- but FS also!

$$\rho_{FS}(S) = \inf_{C \in FS} \rho_C(S) \leq 1/k \text{ for all } k.$$

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Pushdown beats LZ

Theorem

There exists a sequence S such that

$$\rho_{PD}(S) = 1/2 \text{ (without endmarkers)}$$

and

$$\rho_{LZ}(S) = 1.$$

The idea

Pushdown compresses palindromes with ratio $\simeq 1/2$...

but LZ not always.

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but LZ not always.

$\rightarrow$ build a sequence of the form

$$S = u_1 \bar{u}_1 u_2 \bar{u}_2 \ldots$$

with well-chosen words u_i (here $\bar{u}$ stands for the mirror of u).

Proof

Let $E_n \subset \{0,1\}^n$ be the set of words of size n that are not palindromes. Let $u_1, \ldots, u_{|E_n|/2}$ be $|E_n|/2$ words of E_n such that $\forall i, j, u_i \neq \bar{u}_j$. Then

$$u_1 \ldots u_{|E_n|/2} \ \bar{u}_{|E_n|/2} \ldots \bar{u}_1$$

is LZ-incompressible but 1/2-PD-compressible.
$\rightarrow$ repeat this for all sizes n to obtain the infinite sequence S. $\square$

Outline

LZ beats pushdown

Theorem

There exists a sequence S such that

$$\rho_{LZ}(S) = 0$$

and

$$\rho_{PD}(S) = 1 \text{ (with endmarkers).}$$

The idea

LZ compresses repetitions very well (ratio tends to 0). . .

but pushdown not always.

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LZ compresses repetitions very well (ratio tends to 0)...

but pushdown not always.

- Show that some repetitions are not compressed by pushdown (→ pumping lemma);
- build a sequence of the form

$$S = u_1^{n_1} u_2^{n_2} \ldots$$

  for well chosen u_i and n_i.

LZ and repetitions

Lemma

Let u be a word.
The compression ratio of LZ on u^n is $O(\frac{\log n}{\sqrt{n}})$ (and thus tends to 0 when $n \to \infty$).

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The compression ratio of LZ on u^n is $O(\frac{\log n}{\sqrt{n}})$ (and thus tends to 0 when $n \to \infty$).

Proof.
For all k, there are at most $|u|$ different words of size k in u^n.
Call p the number of phrases in the parsing of u^n by LZ algorithm.
Let t_k be the number of phrases of size k. We have:

$$|u^n| = \sum_{k \geq 1} t_k \geq \sum_{k=1}^{p/|u|} k|u| \geq \frac{p^2}{2|u|}.$$

Thus $p = O(\sqrt{n})$ and $|LZ(x)| = p \log p$. □

PD and repetitions

Let C be a pushdown compressor.

Suppose there is a pumping lemma: on input uv^nw, C has each time the same behaviour on v.

If v is not compressible, then $C(uv^nw) \geq n|v|$,
thus $\rho_C(uv^nw) \to 1$.

Pumping lemma

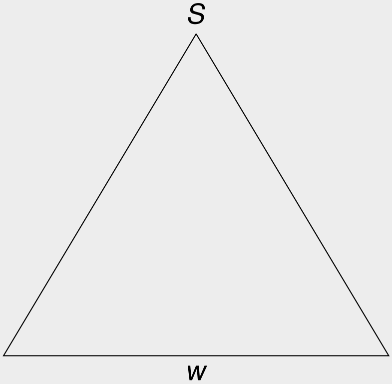
Theorem

Let A be a pushdown transducer (working without endmarkers). There exist two constants $\alpha, \beta > 0$ such that all word w can be cut in three pieces $w = tuv$ satisfying:

- $|u| \geq \lfloor \alpha |w|^{\beta} \rfloor$;
- *if* $C(tuv) = xyz$ *then* $C(tu^n) = xy^n$.

Acceptors: reminder

Equivalence pushdown automata / context-free grammars.



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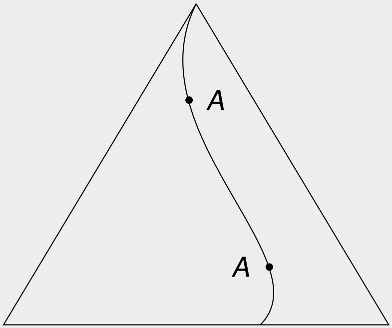

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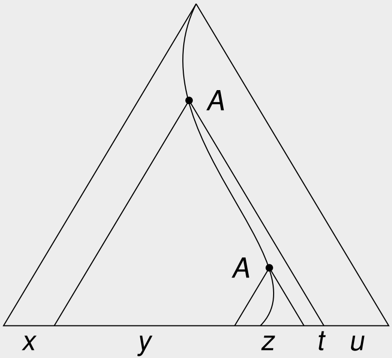

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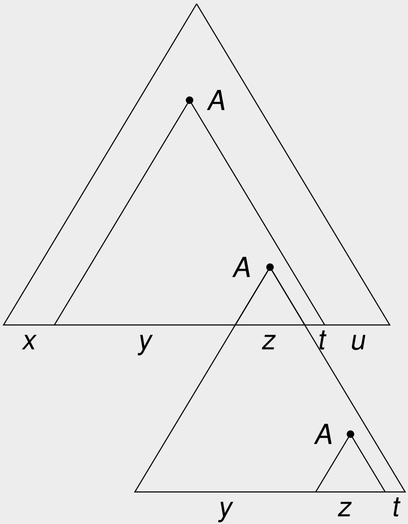

Transducers: proof (1)

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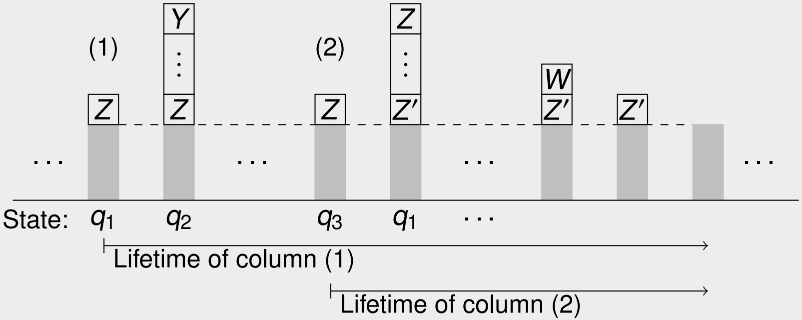
- Lifetime of a column: never go below its top symbol.
- Equivalent columns: c' in the lifetime of c and same state/top symbol.

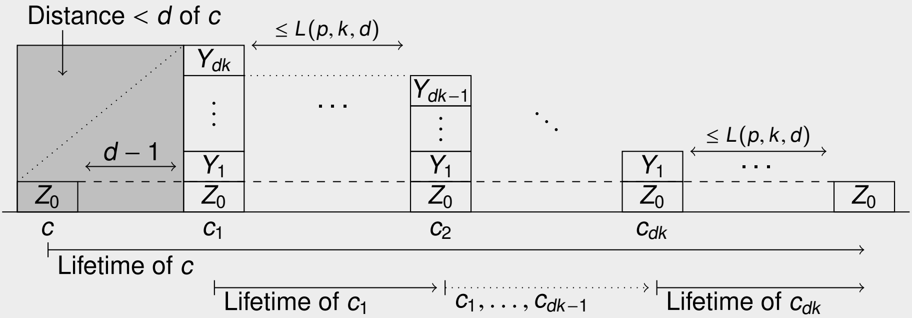


Transducers: proof (2)

- p: number of pairs state/top symbol;
- k: max number of symbols pushed by one rule;
- $L(p, k, d)$: maximum lifetime of a column during which no pair of equivalent columns are at distance $\geq d$.

Transducers: proof (2)

- p: number of pairs state/top symbol;
- k: max number of symbols pushed by one rule;
- $L(p,k,d)$: maximum lifetime of a column during which no pair of equivalent columns are at distance $\geq d$.
- $L(p+1,k,d) = d + kdL(p,k,d)$.



The endmarker

Theorem

Let A be a pushdown transducer (working with endmarkers). There exist two constants $\alpha, \beta > 0$ such that all word w can be cut in three pieces $w = tuv$ satisfying:

- $|u| \geq \lfloor \alpha |w|^{\beta} \rfloor$;
- *there are five words x, x', y, y', z such that* $C(tu^n v\#) = xy^n zy'^n x'$.

Remark.

The same is true with an initially nonempty stack.

LZ beats PD

Theorem

There exists a sequence S such that

$$\rho_{LZ}(S) = 0 \text{ and } \rho_{PD}(S) = 1 \text{ (with endmarkers).}$$

LZ beats PD

Theorem

There exists a sequence S such that

$$\rho_{LZ}(S) = 0 \text{ and } \rho_{PD}(S) = 1 \text{ (with endmarkers).}$$

Proof.
Let w_i be a sufficiently big Kolmogorov-random word
$\rightarrow$ cut it in three pieces $w_i = t_i u_i v_i$, with u_i big enough (thus incompressible), according to the i-th pushdown transducers. Then

$$S = t_1 u_1^{n_1} t_2 u_2^{n_2} \dots$$

(for sufficiently large integers n_i) is LZ-compressible but not PD-compressible. □

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- Strictly better than finite-state compression.

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- Introduction of pushdown compression.
- Strictly better than finite-state compression.
- Better than Lempel-Ziv on some sequences (palindromes: compression ratio 1/2 instead of 1).
- Worse than Lempel-Ziv on some sequences (repetitions: compression ratio 1 instead of 0).

Future work

- Lower bound on the compression ratio of a PD compressor with k states with endmarkers?
- Better separation for "PD beats LZ"?

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