Separating multilinear branching programs and formulas

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Introduction

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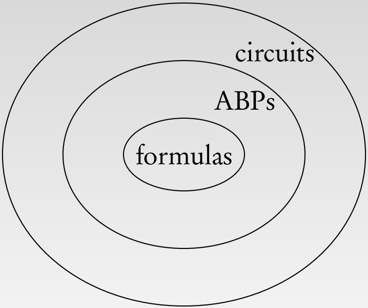
- No strong lower bounds for general circuits.

  (Best nontrivial lower bound for "explicit" polynomials: $\Omega(n \log n)$, Baur-Strassen)

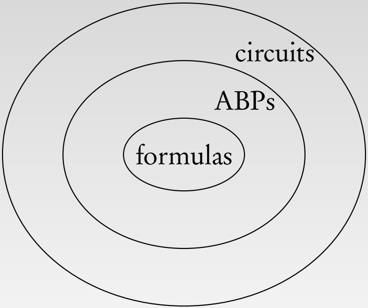
Introduction

- Arithmetic circuits: model for computing polynomials.
- Algebraic variants of P vs NP.
- No strong lower bounds for general circuits.
- Other weaker models:
  formulas and algebraic branching programs (ABPs).



Introduction

- Important restriction = multilinear computation.
- Raz 2004: separation of multilinear circuits and formulas.



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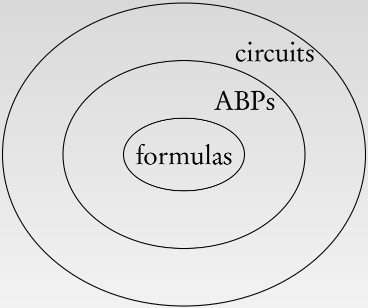
Here: $\boxed{\text{separation of multilinear ABPs and formulas.}}$



Outline

1. Formulas, ABPs and multilinearity

2. The rank technique

3. Our separation

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Multilinearity

- A polynomial is multilinear if
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- Example: $x_1x_2 + x_1x_3 + x_2 + 1.$
- Counter-example: $x^2y + xyz.$

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  the degree of each variable is at most 1.
- Example: $x_1x_2 + x_1x_3 + x_2 + 1.$
- Counter-example: $x^2y + xyz.$
- Important multilinear polynomials:
  determinant, permanent…

$$\det(x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{n,n}) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} x_{i,\sigma(i)}.$$

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Arithmetic formula

- Weakest model:
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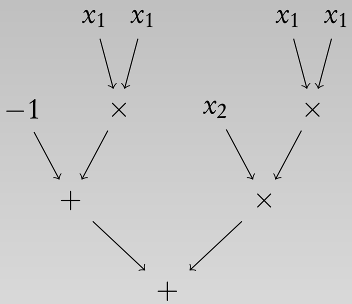
- Underlying graph = tree.



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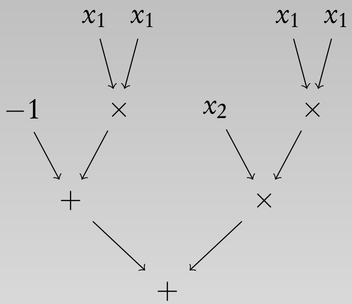

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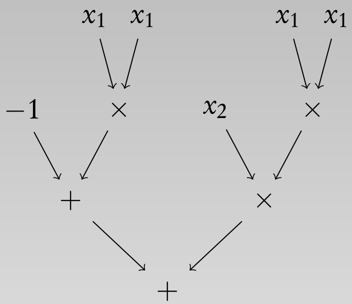

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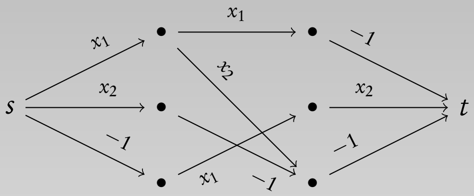
- Underlying graph = tree.
- Formulas can be parallelized (logarithmic depth) =
  efficient parallel algorithm.
- Multilinear =
  each gate computes a multilinear polynomial.

Algebraic Branching Program (ABP)

- DAG from a source s to a sink t with arcs labelled by constants or variables.



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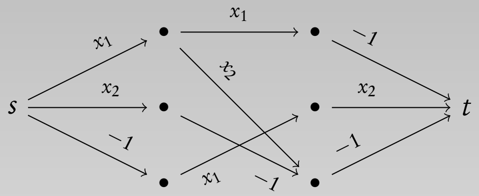

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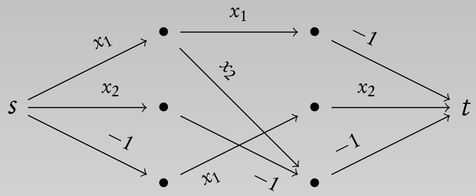

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- However no multilinear polynomial-size ABP known for the determinant.

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The coefficient matrix

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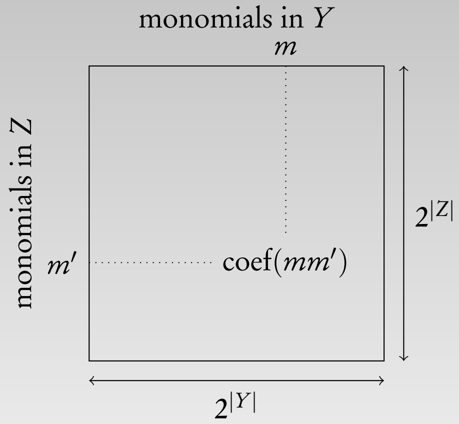
- $M(f_\Pi)$: coefficient matrix of f according to Π.



Example

- $X = \{x_1, x_2, x_3, x_4\}$
- $f(x_1, x_2, x_3, x_4) = 2x_2x_3x_4 + x_1x_2 + 5x_1x_3 - 2x_4 - 3$

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- $Y = \{y_1, y_2\}, \quad Z = \{z_1, z_2\}$
- $\Pi : x_1 \mapsto y_1, \quad x_2 \mapsto y_2,$
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$$M(f_\Pi) = \begin{array}{c|cccc} & 1 & y_1 & y_2 & y_1y_2 \\ \hline 1 & -3 & 0 & 0 & 1 \\ z_1 & 0 & 5 & 0 & 0 \\ z_2 & -2 & 0 & 0 & 0 \\ z_1z_2 & 0 & 0 & 2 & 0 \end{array}$$

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- if $Y(f)$ and $Z(f)$ are the numbers of Y and Z variables appearing in f_Π, then

$$\mathrm{rank}(M(f_\Pi)) \leq 2^{\min(Y(f),Z(f))}.$$

The rank technique

Separation of multilinear circuits and formulas (Raz 2004):

- build a polynomial f such that:
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- Probabilistic method: g_Π is not full rank if Π is chosen at random.

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The result

Theorem

There exists a polynomial-size *multilinear* ABP computing a polynomial P that has no *multilinear* formula of size $n^{o(\log n)}$.

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Lower bound:

- it suffices that polynomials computed by formulas
  are not full-rank for a single partition;
- however probabilistic argument:
  not full-rank for most partitions.

Pairings (1)

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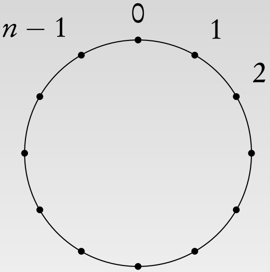
- Set of variables $X \equiv \{0, 1, \ldots, n-1\}$
  seen as the n-cycle C_n.



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Random pairing: iterative process

- First pair: $\{\mathtt{0}, 1\}$.
- At any given step,
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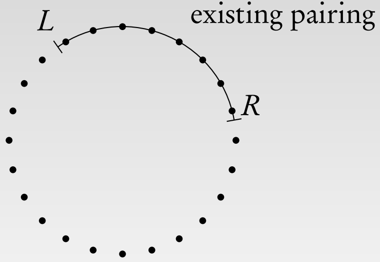

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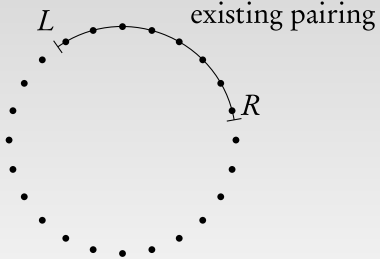

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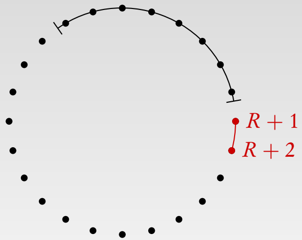

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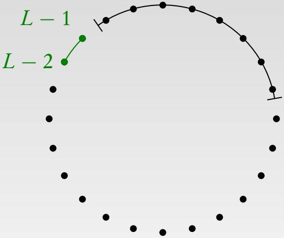

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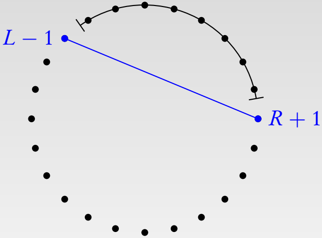

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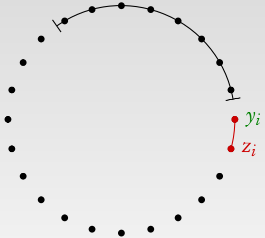

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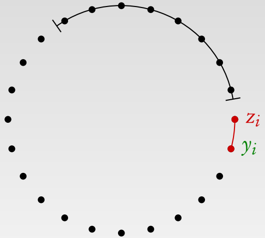

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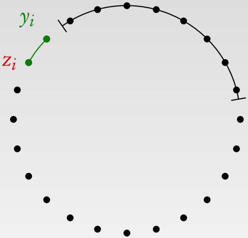

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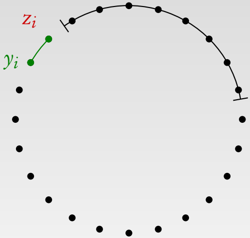

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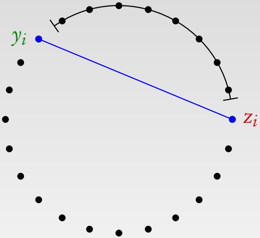

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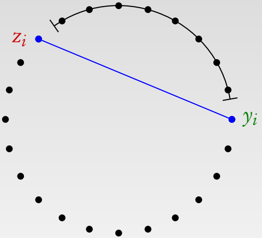

The branching program

The ABP is built according to the iterative process of pairing:

- vertices = arcs $[L, R]$ of the pairing
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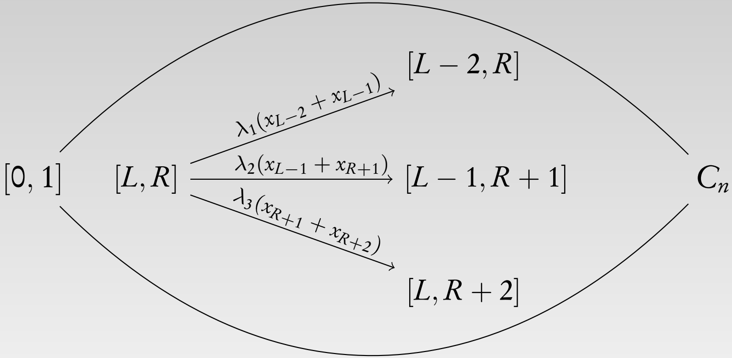

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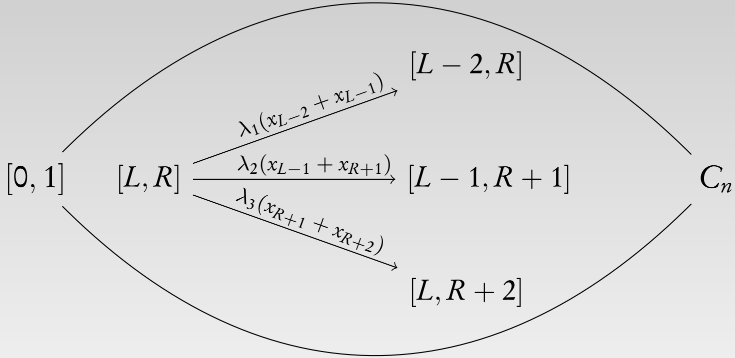

Independent paths, two choices per edge $\rightarrow$ full rank.

(K, T)-products

Definition

A polynomial $g(x_1, \ldots, x_n)$ is a (K, T)-product if $g = g_1 g_2 \cdots g_K$ where:

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Example: $(x_1 x_2 - 3x_1)(x_3 + 1)(5x_5 x_6 - x_6)$

is a $(3, 2)$-product.

Restricting to (K, T)-products

Lemma (Shpilka&Yehudayoff)

If $f(x_1, \ldots, x_n)$ is computed by a formula of size s, then $f = f_1 + \cdots + f_{s+1}$ where f_i is a $(\frac{\log n}{100}, n^{7/8})$-product.

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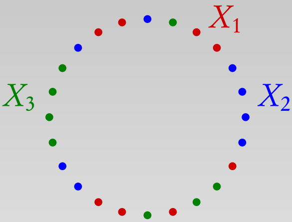
Since $\mathrm{rank}(M((f_i + f_j)_\Pi)) \leq \mathrm{rank}(M((f_i)_\Pi)) + \mathrm{rank}(M((f_j)_\Pi))$,

we restrict the study to one (K, T)-product $g = g_1 g_2 \cdots g_K$

$\rightarrow$ we must argue that g is low rank
(instead of only "not full rank")

Combinatorics

- K disjoint subsets of the variables = K colors.



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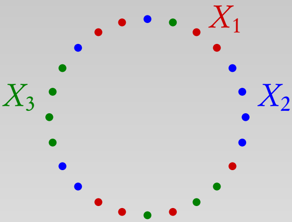

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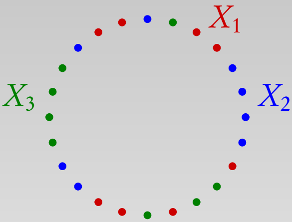

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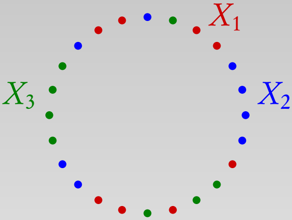

From now on the argument is only combinatorial.

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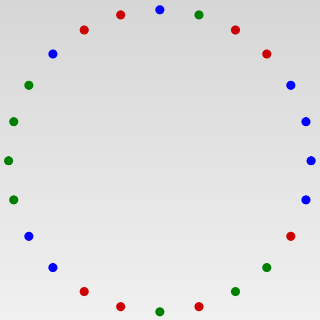
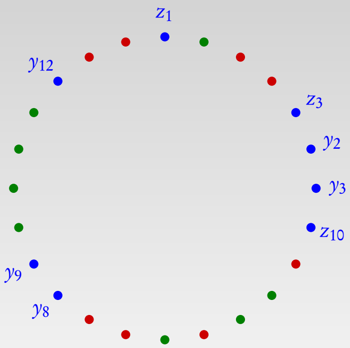


Unbalanced colors

Definition

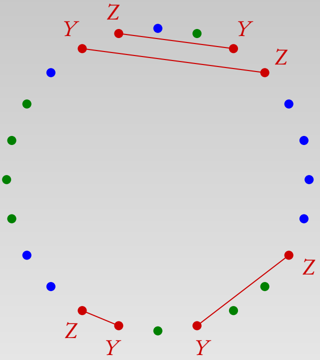
For a given partition Π, a color is balanced if it has roughly the same number of Y and Z variables.

$\rightarrow$ We want to show that some colors are unbalanced.



A sufficient condition of balance

If all pairs containing a red vertex has its other vertex red, then color red is balanced.



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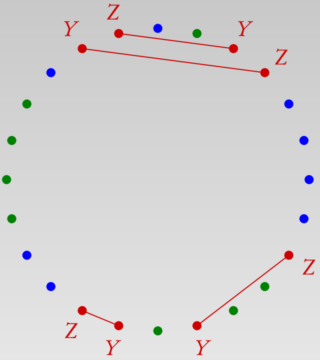
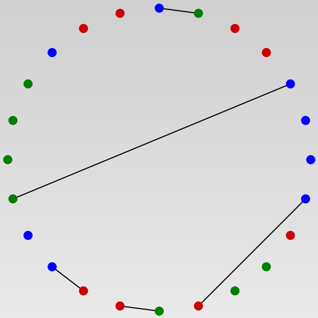


$\rightarrow$ look for pairs whose vertices have different colors

= "violations"

Violations

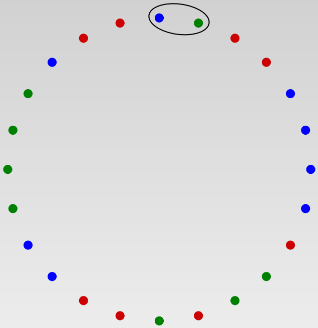
Examples of violations in a pairing:



Jumps

Violation for consecutive vertices = "jump"

(=change of color)



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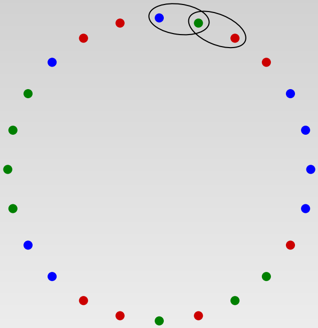

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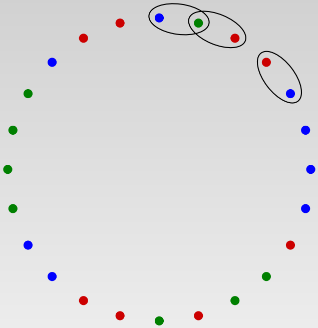

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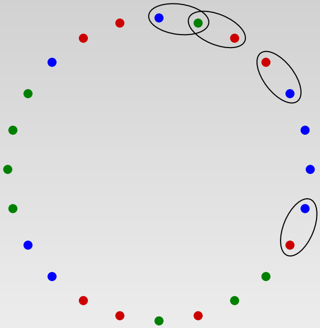

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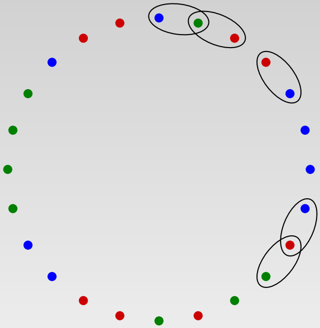

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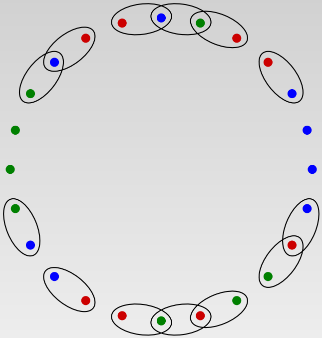
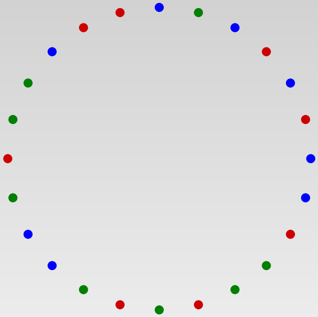


First case: Many colors with many jumps

- If $[R, R+1]$ is a jump,
  it is chosen in the pairing with probability 1/3.
- Many jumps
  $\rightarrow$ one third of them yield violations.



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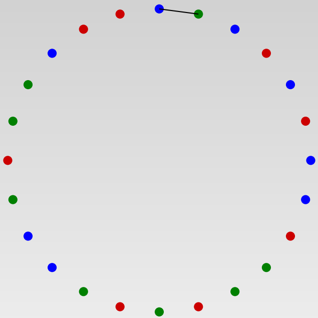

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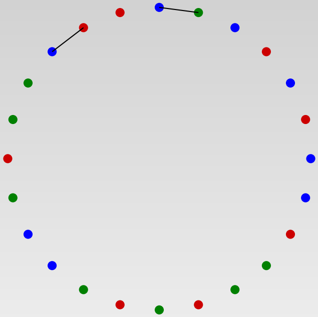

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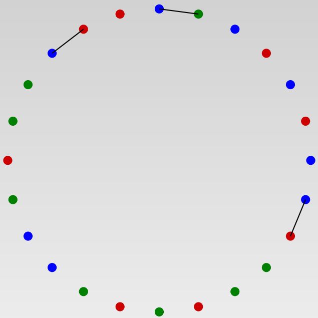

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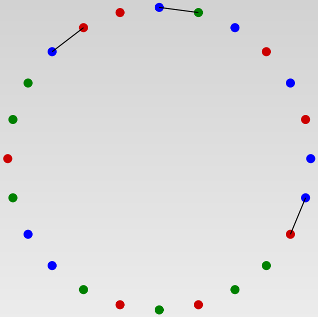
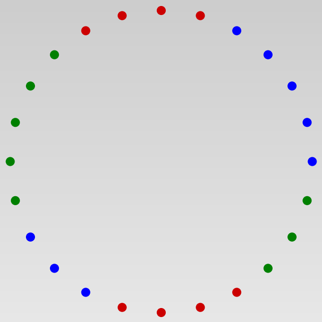


$\rightarrow$ the color is unbalanced with sufficiently high probability.

Second case: Many colors with few jumps

- Large monochromatic arcs.
- $\rightarrow$ A large number of cords give violations since each violating cord is chosen with probability 1/3.



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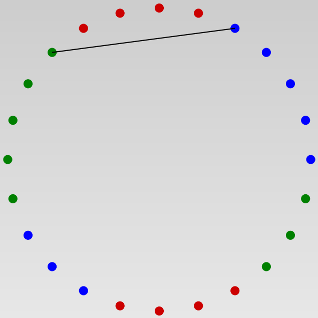

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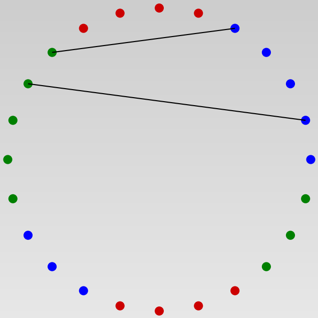

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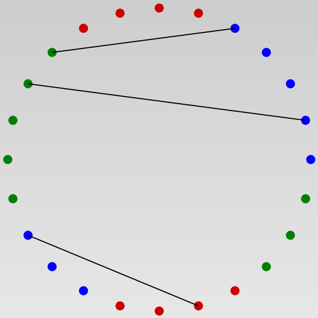

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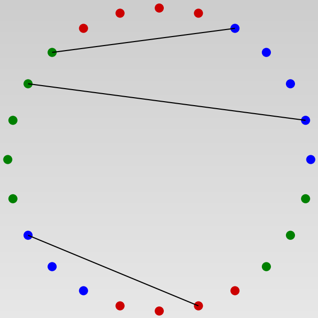

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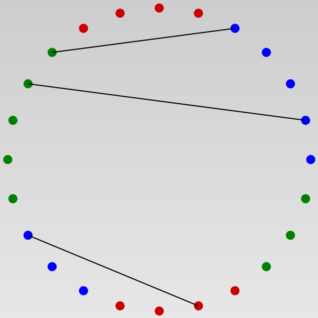

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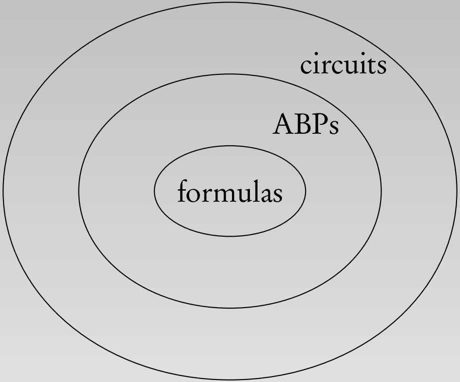
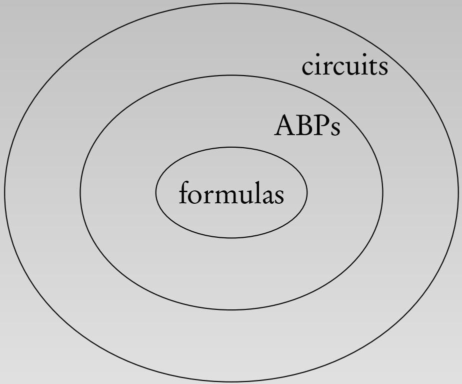
(Formal analysis = 2D random walk on a chessboard.)

Future directions



- Separate multilinear circuits and ABPs?

Future directions



- Separate multilinear circuits and ABPs?
- Are there polynomial-size multilinear ABPs for the determinant?