# Symmetry of information and nonuniform lower bounds

Sylvain Perifel

LIP, ENS Lyon

Ekaterinburg, September 4, 2007

# Outline

1. Complexity classes
2. Advices of size $n^c$
3. Symmetry of information
4. Polynomial-size advices

# Two complexity classes

- $\mathrm{EXP}$: set of languages recognized in exponential time by a deterministic Turing machine

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- Open question: $\mathrm{EXP} \subset \mathrm{P/poly}$?

- Main result: polynomial-time symmetry of information implies $\mathrm{EXP} \not\subset \mathrm{P/poly}$.

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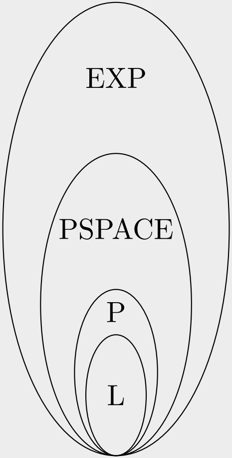
- Space complexity version:

$$\mathrm{PSPACE} \subset \mathrm{NC/poly}?$$

- Even the question "$\mathrm{EXP} \subset \mathrm{L/poly}$?" is open.

# Some classes



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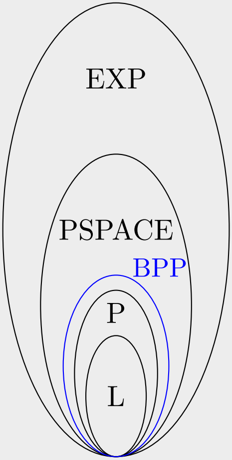

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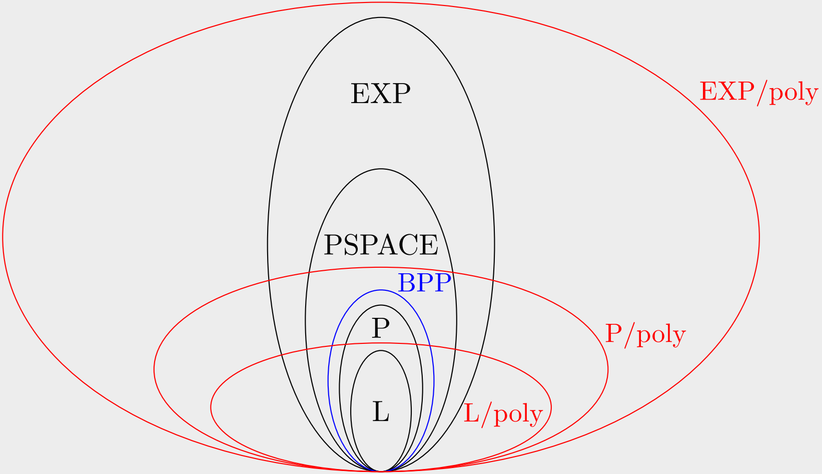

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- If $\mathcal{C}$ is a complexity class and $a : \mathbb{N} \to \mathbb{N}$ a function, then $\mathcal{C}/a(n)$ is the set of languages $A$ such that there exists $B \in \mathcal{C}$ and a function $c : \mathbb{N} \to \{0,1\}^*$ satisfying:
  - $\forall n, |c(n)| \leq a(n)$;
  - $\forall x \in \{0,1\}^*, x \in A \iff (x, c(|x|)) \in B$.
- "The class $\mathcal{C}$ is helped by the advice $c(|x|)$" (the same for all words of each length).

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  $\mathrm{P}/\mathrm{poly}$: conversion advice $\longleftrightarrow$ boolean circuit.
- $\mathrm{EXP} \subset \mathrm{P}/\mathrm{poly} \iff \mathrm{EXP}/\mathrm{poly} = \mathrm{P}/\mathrm{poly}$.

# Links with derandomization

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  - Babai, Fortnow, Nisan & Wigderson 1993: if $\mathrm{EXP} \not\subset \mathrm{P}/\mathrm{poly}$ then $\mathrm{BPP}$ has subexponential-time deterministic algorithms.

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  - Babai, Fortnow, Nisan & Wigderson 1993: if $\mathrm{EXP} \not\subset \mathrm{P/poly}$ then $\mathrm{BPP}$ has subexponential-time deterministic algorithms.
- For the other direction, Kabanets & Impagliazzo 2002: if $\mathrm{P} = \mathrm{BPP}$ then $\mathrm{NEXP}$ does not have polynomial-size circuits.

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- Homer & Mocas 1995: $\forall c > 0, \mathrm{EXP} \not\subset \mathrm{P}/n^c$.
- Here: symmetry of information $(\mathrm{SI_p}) \Rightarrow \mathrm{EXP} \not\subset \mathrm{P/poly}$;
- Lee & Romashchenko 2004: $(\mathrm{SI_p}) \Rightarrow \mathrm{EXP} \neq \mathrm{BPP}$
  (remark: $\mathrm{BPP} \subset \mathrm{P/poly}$, Adleman 1978).

## Advices of size $n^c$

- Words of $\{0,1\}^n$ are ordered lexicographically $x_1 < x_2 < \cdots < x_{2^n}$.
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**Lemma**

*If $A \in \mathrm{P}/n^c$ then there exists a constant $k$ and a family $(p_n)$ of programs of size $k + n^c$ such that*

- $\mathcal{U}(p_n, x) = 1$ *iff* $x \in A$;
- $\mathcal{U}(p_n, x)$ *works in polynomial time.*

# Advices of size $n^c$ (continued)

## Proposition (warm-up)

*For all constants* $c_1, c_2 \geq 1$*, there exists a sparse language* $A$ *in* $\mathrm{DTIME}(2^{n^{1+c_1 c_2}})$ *but not in* $\mathrm{DTIME}(2^{n^{c_1}})/n^{c_2}$*.*

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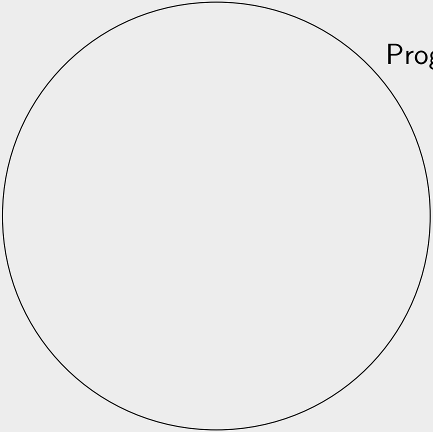

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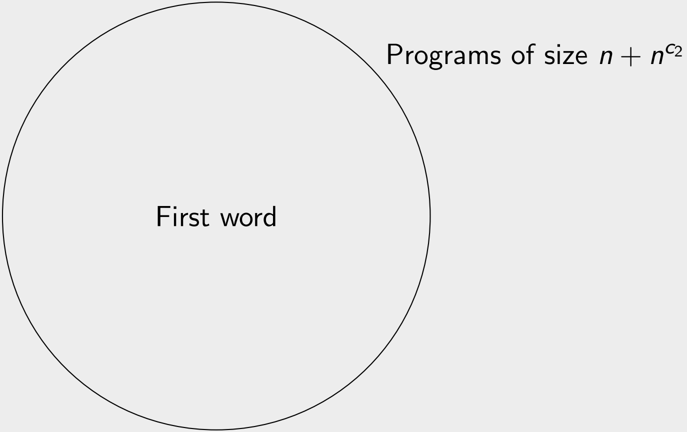

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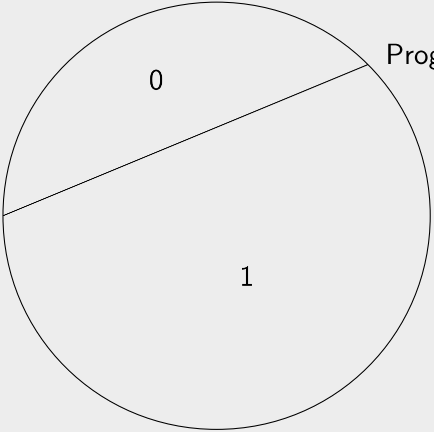


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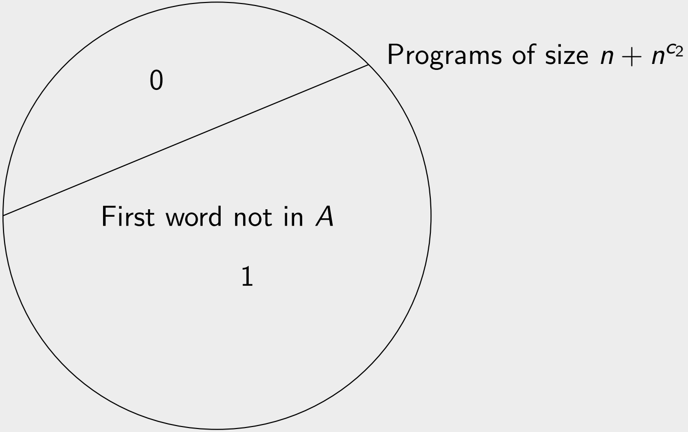

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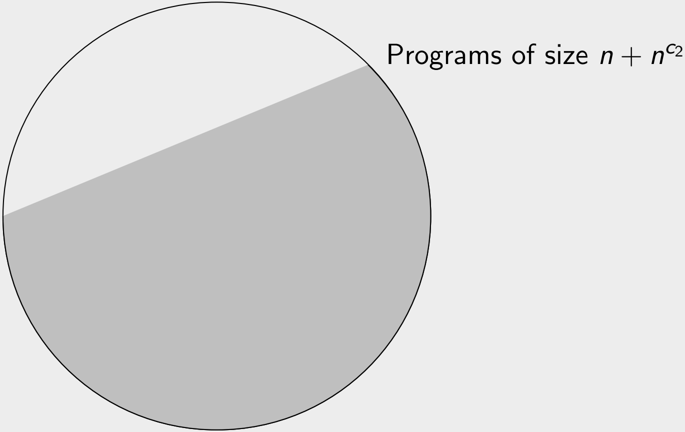

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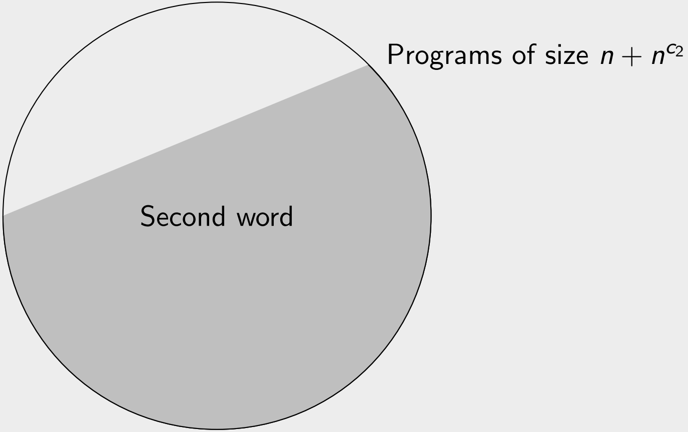

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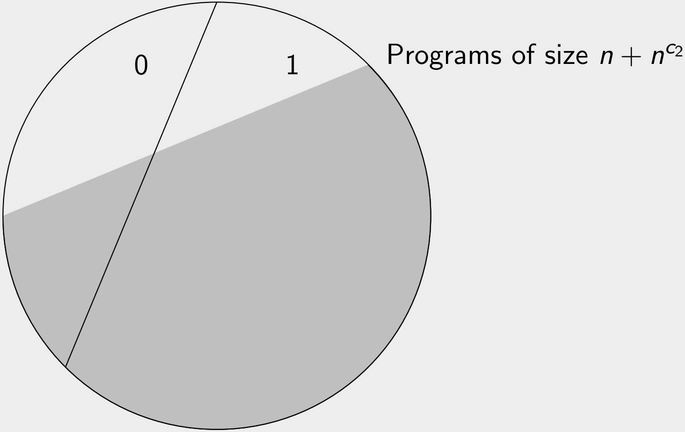

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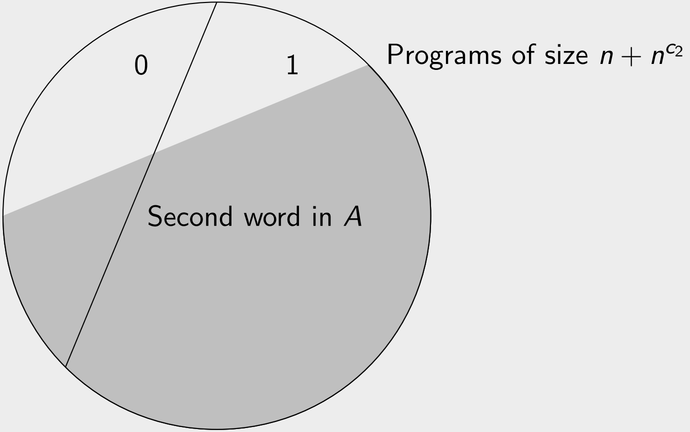

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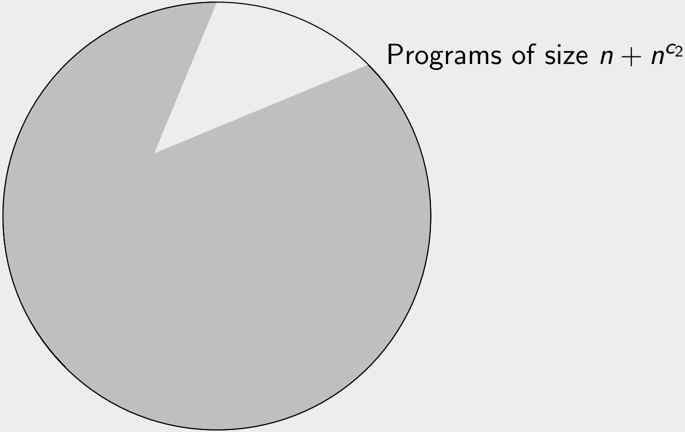

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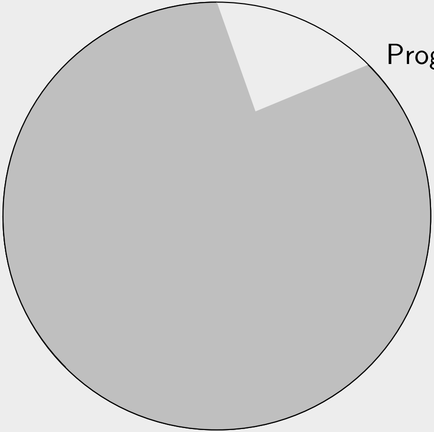

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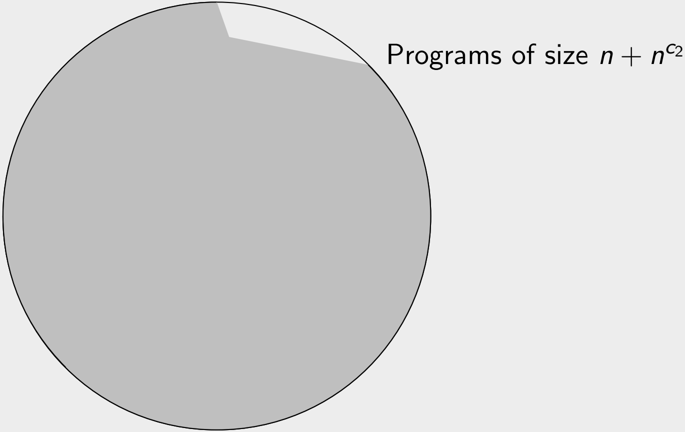

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## Corollary

*For all constant* $c > 0$, $\mathrm{EXP} \not\subset \mathrm{P}/n^c$ *and* $\mathrm{PSPACE} \not\subset (\cup_k \mathrm{DSPACE}(\log^k n)/n^c)$.

# Kolmogorov complexity

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- Typical time bound: polynomial or exponential. There could also be a space bound.

# Links Kolmogorov/nonuniform complexity

Characteristic string $\chi^n \in \{0,1\}^{2^n}$ of $A^{=n}$:

$$\chi_i^n = 1 \iff x_i \in A^{=n}.$$

**Lemma**

*Suppose that for all $n$ and some $1 \leq i \leq 2^n$ we have*

$$C^{ir(n)}(\chi^n[1..i]) > n + a(n).$$

*Then $A \notin \mathrm{DTIME}(r(n))/a(n)$.*

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## Proof

If $A \in \mathrm{DTIME}(r(n))/a(n)$ then $\chi^n[1..i]$ is computed in time $ir(n)$ with a program of size $a(n) + O(1)$. $\square$

# Symmetry of information

**Theorem (symmetry of information, Levin & Kolmogorov)**

*Given $x$ and $y$, $x$ contains as much information on $y$ as $y$ on $x$*

$$C(y) - C(y|x) \simeq C(x) - C(x|y).$$

- The (equivalent) version we will use:

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- Exponential time bounds $\rightarrow$ still true.
- Polynomial-time symmetry of information: easy direction still holds; hard direction is open!
  (true if $\mathrm{P} = \mathrm{NP}$, Longpré & Watanabe 1995).

# Symmetry of information

Hypothesis $(\mathrm{SI_p})$

There exist a polynomial $q$ and a constant $\alpha > 1/2$ such that for all $t$ and all words $x, y$ of size $n$:

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# Iterations of $(\mathrm{SI_p})$

**Lemma**

*Suppose* $(\mathrm{SI_p})$ *holds.*
*Let* $u_1, \ldots, u_n$ *be words of size* $s$. *Let* $m = ns$. *Suppose there exists* $k$ *such that for all* $j \leq n$,

$$C^{tq(m)^{\log n}}(u_j | u_1, \ldots, u_{j-1}) \geq k.$$

*Then* $C^t(u_1, \ldots, u_n) \geq n^{\log(2\alpha)} k$.

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### Proof sketch

$$C^t(u_1, \ldots, u_n) \geq \alpha(C^{tq(m)}(u_1, \ldots, u_{n/2}) +$$
$$C^{tq(m)}(u_{n/2+1}, \ldots, u_n | u_1, \ldots, u_{n/2})). \quad \square$$

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- Other point of view: thanks to $(\mathrm{SI_p})$, build a characteristic string of high Kolmogorov complexity.

# Main result

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Outline of the proof: feedback with previously defined segments.

**Proof**

We build $A$ by input sizes and word by word. Let $t(n) = n^{\log^3 n}$.
Let us fix $n$ and define $A^{=n}$:

$$x_1 \in A \iff \begin{array}{l} \text{for at least half of the programs } p \text{ of size } \leq n, \\ \mathcal{U}^{t(n)}(p, x_1) = 0. \end{array}$$

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Let $V_1$ be the set of programs giving the right answer for $x_1$.

# Proof (continued)

We go on like this, discarding half of the remaining programs at each step, until $x_n$:

$$x_n \in A \iff \begin{array}{l}\text{for at least half of the programs } p \in V_{n-1},\\ \mathcal{U}^{t(n)}(p, x_n) = 0.\end{array}$$

We call $u^{(1)}$ the $n$ first bits of the characteristic string of $A^{=n}$ just defined.

# Proof (continued)

Then:

$$x_{n+1} \in A \iff \begin{array}{l}\text{for at least half of the programs } p \text{ of size } \leq n, \\ \mathcal{U}^{t(n)}(p, u^{(1)}, x_{n+1}) = 0.\end{array}$$

(at least half of the programs are wrong on $x_{n+1}$, even with the advice $u^{(1)}$).

# Proof (continued)

Then:

$$x_{n+1} \in A \iff \begin{array}{l} \text{for at least half of the programs } p \text{ of size } \leq n, \\ \mathcal{U}^{t(n)}(p, u^{(1)}, x_{n+1}) = 0. \end{array}$$

(at least half of the programs are wrong on $x_{n+1}$, even with the advice $u^{(1)}$).

Keep going: call $V_1$ the set of programs that where right at the preceding step.

$$x_{n+2} \in A \iff \begin{array}{l} \text{for at least half of the programs } p \in V_1, \\ \mathcal{U}^{t(n)}(p, u^{(1)}, x_{n+2}) = 0. \end{array}$$

# Proof continued

And so on, until the next segment $u^{(2)}$ of size $n$ is defined. Then:

$$x_{2n+1} \in A \Longleftrightarrow \begin{array}{l} \text{for at least half of the programs } p \text{ of size } \leq n, \\ \mathcal{U}^{t(n)}(p, u^{(1)}, u^{(2)}, x_{2n+1}) = 0. \end{array}$$

(at least half of the programs give the wrong answer for $x_{2n+1}$, even with the advice $u^{(1)}, u^{(2)}$).

We define $n^{\log n}$ segments of size $n$ and decide that $x_j \notin A^{=n}$ for $j > n \times n^{\log n}$.

# Proof continued

- $A \notin \mathrm{P/poly}$ because for all $j$, $C^{t(n)}(u^{(j)} | u^{(1)}, \ldots, u^{(j-1)}) \geq n - 1$. Thus by iteratively applying $(\mathrm{SI_p})$, $C^t(\chi^n[1..n^{1+\log n}]) \geq n^{\Omega(\log n)}$.
- $A \in \mathrm{EXP}$. $\square$

# Proof continued

- $A \notin \mathrm{P/poly}$ because for all $j$, $C^{t(n)}(u^{(j)} | u^{(1)}, \ldots, u^{(j-1)}) \geq n - 1$. Thus by iteratively applying $(\mathrm{SI_p})$, $C^t(\chi^n[1..n^{1+\log n}]) \geq n^{\Omega(\log n)}$.
- $A \in \mathrm{EXP}$. $\square$

## Corollary

*If* $(\mathrm{SI_p})$ *holds, then there exists a constant* $c > 0$ *such that*

$$\mathrm{BPP} \subseteq \mathrm{DTIME}(2^{\log^c n}).$$

# Conclusion

- $(\mathrm{SI_p})$ is a central (and hard) question: if true, then $\mathrm{EXP} \not\subset \mathrm{P/poly}$; if false, then $\mathrm{P} \neq \mathrm{NP}\dots$
- What about the usual version of $(\mathrm{SI_p})$ (with time bound $q(t)$ instead of $tq(n)$)?
- Can we obtain unconditionnal results by using variants of Kolmogorov complexity ? (for instance CAMD, a version based on the class AM).

# Outline

1. Complexity classes
2. Advices of size $n^c$
3. Symmetry of information
4. Polynomial-size advices