Exponential time vs probabilistic polynomial time

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Introduction

Probabilistic algorithms:

- can toss a coin
- polynomial time (worst case)
- probability of error $< \epsilon$

$\rightarrow$ class BPP.

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Common belief:
they can be *derandomized*

(true if some
circuit lower bounds hold)

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Even if it is believed that

$$\mathsf{BPP} = \mathsf{P}$$

it is open whether

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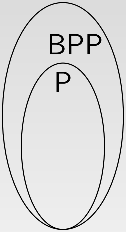

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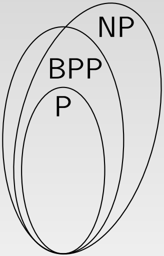

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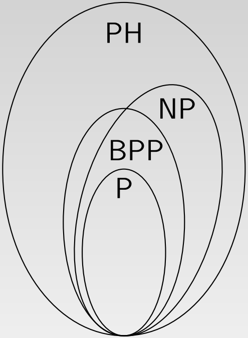

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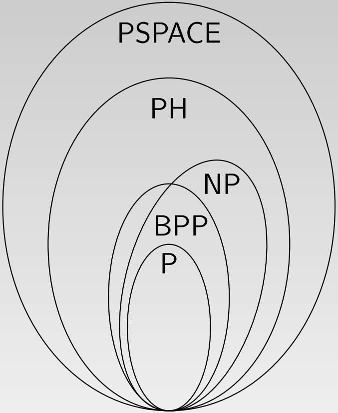

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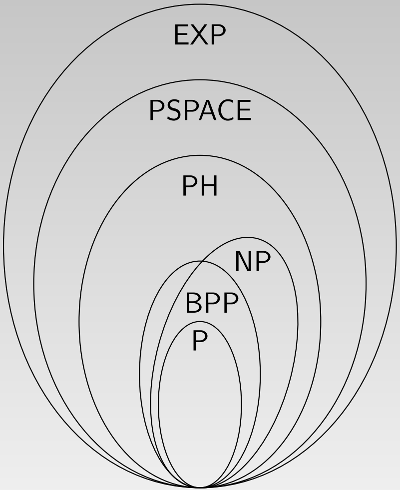

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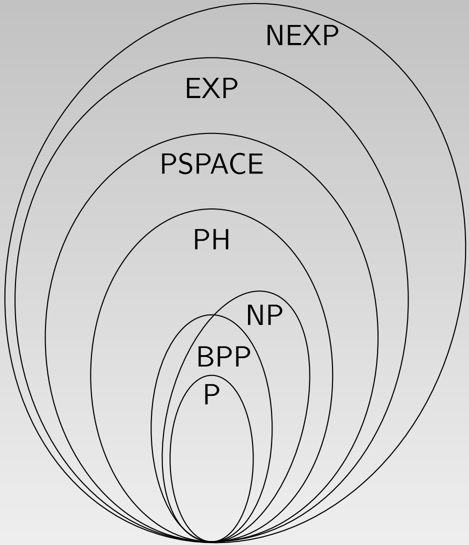

Outline

1. Problem
2. Resource-bounded Kolmogorov complexity
3. Interactive protocols

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BPP: class of languages A such that
there is a polynomial-time Turing machine M satisfying

- if $x \in A$ then $\Pr_{r \in \{0,1\}^{p(n)}}(M(x, r) = 1) \geq 2/3$
- if $x \notin A$ then $\Pr_{r \in \{0,1\}^{p(n)}}(M(x, r) = 0) \geq 2/3$

$\to M$ gives the correct answer with probability $\geq 2/3$.

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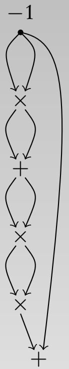
- polynomial time whatever the random bits
- the probability of error $1/3$ can be reduced
  by repeting the algorithm
- BPP has circuits of polynomial size (Adleman 1978)

Example

Integer testing:

Input: an arithmetic circuit C computing an integer N

Question: $N = 0$?



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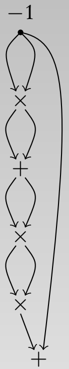

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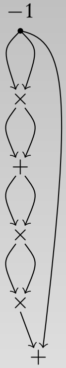

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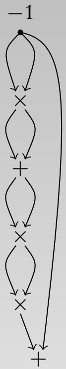

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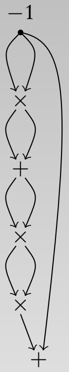

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$$C() = 0 \iff \forall i, C() \equiv 0 \mod m_i, \qquad (|C| = n)$$

then $\prod_i m_i$ doesn't have circuits of size n: lower bound!

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Open:

- $\mathsf{NEXP} \not\subset \mathsf{P/poly}$ (polynomial size circuits)?
- $\boxed{\mathsf{NEXP} \neq \mathsf{BPP}?}$

Why doesn't simple diagonalization work?

In order to simulate deterministically
a probabilistic TM with $|r|$ random bits:

- run $M(x, r)$ for all r
- take the majority answer

$\rightarrow$ time $\geq 2^{|r|}$

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One strategy for $\mathsf{NEXP} \neq \mathsf{BPP}$:
diagonalize over probabilistic machines working in time $n^{\log n}$
$\rightarrow$ time $2^{n^{\log n}}$, outside NEXP.

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Kolmogorov complexity:

$C(x) =$ minimal size of a program printing x

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- Other variants studied:
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In this talk: more basic stuffs!

Example of a link with our problem

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Theorem (Lee and Romashchenko 2005, P. 2007)

If (SI) holds with polynomial-time bounds, then $\mathsf{EXP} \neq \mathsf{BPP}$ and even $\mathsf{EXP} \not\subset \mathsf{P/poly}$.

A classical observation

Let M be a BPP machine for a language A
working in time t with error $2^{-\epsilon t}$.

Suppose $x \in A$.

- Then there are $2^{(1-\epsilon)t}$ words r such that $M(x, r) = 0$.
- Hence if $M(x, r) = 0$ then $C^{t2^t}(r|x) \leq (1 - \epsilon)t$.

In other words,
if $C^{t2^t}(r|x) > (1 - \epsilon)t$ then $M(x, r)$ gives the correct result.

The converse?

[Cheating…]

If $M(x,r) = 0$ then $C^{t2^t}(r|x) \leq (1-\epsilon)t$.

Since there are:

- $2^{(1-\epsilon)t}$ words r such that $M(x,r) = 0$
- $\leq 2^{(1-\epsilon)t}$ words r such that $C^{t2^t}(r|x) \leq (1-\epsilon)t$

we have

$$C^{t2^t}(r|x) \leq (1-\epsilon)t \iff M(x,r) = 0 \quad (!)$$

Going outside BPP

Then for an enumeration (M_n) of probabilistic machines on input x, in NEXP:

- guess r of size $n^{\log n}$,
- check it has high complexity,
- simulate $M_n(x, r)$ for $n^{\log n}$ steps and take the opposite result.

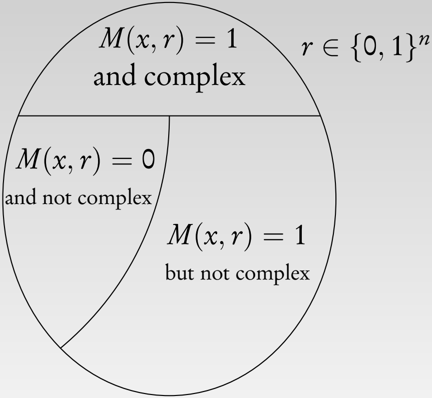
Hence $\mathsf{NEXP} \neq \mathsf{BPP}$!

The converse?

If $M(x,r) = 0$ then $C^{t2^t}(r|x) \leq (1-\epsilon)t + \alpha$.

Hence:

- if $C^{t2^t}(r|x) \geq (1-\epsilon)t + \alpha$ then $M(x,r) = 1$;
- for a fraction $2^{-\alpha}$ of the remaining words r, $M(x,r) = 0$.



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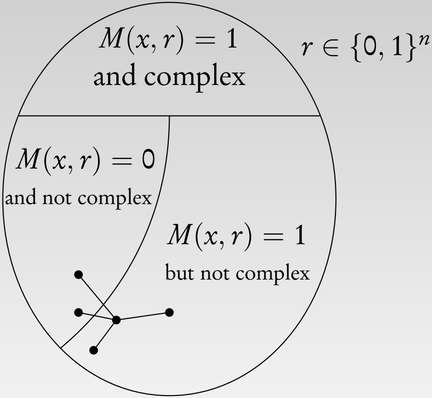

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Quantifier alternation

THEOREM (Kannan 1982)

$\mathsf{NEXP}^{\mathsf{NP}}$ does not have circuits of polynomial size

Idea (in fact more subtle): in $\mathsf{NEXP}^{\mathsf{NP}}$ express

“there is a circuit A_0 of size $n^{2\log n}$ such that:
for all circuit A of size $n^{\log n}$, $A(x) \neq A_0(x)$ for some x”

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(even better: $\mathsf{MA}_{\mathsf{EXP}} \not\subset \mathsf{P}/\mathsf{poly}$
Buhrman, Fortnow, Thierauf 1998)

Interactive protocol for NEXP

$\mathsf{NEXP} = \mathsf{MIP}$ (Babai, Fortnow, Lund 1991) or
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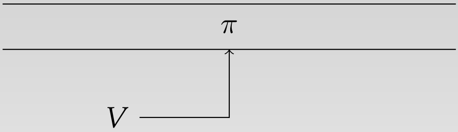
Exponentially long proof π, BPP-style verifier V that can read a polynomial number of bits from the proof.



- if $x \in A$ then $\exists \pi$ st $V^{\pi}(x)$ accepts with proba 1;
- if $x \in A$ then $\forall \pi$, $V^{\pi}(x)$ rejects with proba $\geq 2/3$.

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Question: what can be done in $\mathsf{PCP}(\mathsf{poly}, \mathsf{poly})^{\mathsf{NEXP}}$?

Conclusion

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Last question:

$$\text{if } M \text{ is a BPP-machine and } C^{2^n}(r) = |r|,$$
$$\text{does } M(x, r) \text{ give the correct answer?}$$

(time bound 2^n instead of $t2^t$)