VPSPACE and a transfer theorem over the complex numbers

The question "$\mathrm{P} = \mathrm{PSPACE}$?" in algebraic complexity

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Introduction

- Decision problems
  Languages (over $\mathbb{C}$), Blum-Shub-Smale model
  Example: decide whether a system of multivariate polynomials has a solution ($\mathrm{NP}_{\mathbb{C}}$-complete)

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- Evaluation problems
  Families of polynomials, Valiant's model
  Example: compute the permanent of a matrix (VNP-complete)

Outline

1. P and PSPACE (boolean case)

2. P and PSPACE in BSS model

3. P and PSPACE in Valiant's model

4. Sign condition

if $\mathrm{VP} = \mathrm{VPSPACE}$ then $\mathrm{P}_{\mathbb{C}} = \mathrm{PAR}_{\mathbb{C}}$

P and PSPACE (boolean case)

- P: languages over $\{0,1\}$ recognized in polynomial time by a Turing machine.
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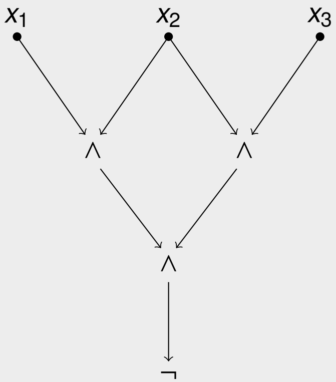
Turing machines

$\longleftrightarrow$

boolean circuits

(gates $\wedge$, $\vee$, $\neg$).



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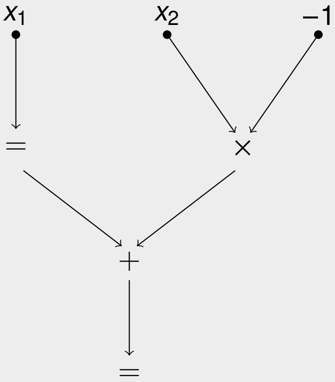
- P: languages recognized by boolean circuits of polynomial size (+ uniformity).
- PSPACE: languages recognized by boolean circuits of polynomial *depth* (of possibly exponential size) (+ uniformity).

P and PSPACE in BSS model

Algebraic circuits: gates $+$, $-$, $\times$ and $=$.



P and PSPACE in BSS model

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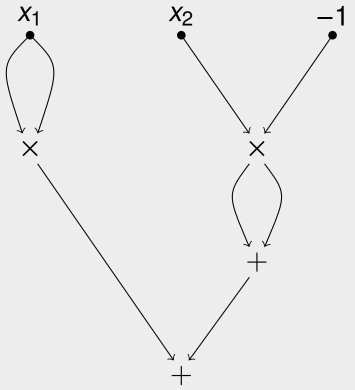
P and PSPACE in Valiant's model

Arithmetic circuits: gates $+$, – and $\times$, inputs $x_1, \ldots, x_n$ and constant 1 $\longrightarrow$ multivariate polynomial with integer coefficients.



P and PSPACE in Valiant's model

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- VPSPACE: families of polynomials computed by arithmetic circuits of polynomial *depth* (+ uniformity).

Recapitulation

- Decision problems over $\{0, 1\}$: boolean circuits (gates $\wedge$, $\vee$ et $\neg$).
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- P: circuits of polynomial size.
- PSPACE: circuits of polynomial depth.

Other characterizations of $\mathrm{VPSPACE}$

- Original definition: coefficient function in PSPACE.

$$f_n(\bar{x}) = \sum_{\alpha} a(\alpha)\bar{x}^{\alpha}$$

Function $a : \{0,1\}^* \rightarrow \mathbb{Z}$ computable bit by bit in polynomial space.

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- Proposition: $\mathrm{VPSPACE} = \mathrm{VP} \Longrightarrow \mathrm{PSPACE} = \mathrm{P}$.

Transfer theorem

If $\mathrm{VPSPACE} = \mathrm{VP}$ then $\mathrm{PAR}_{\mathbb{C}} = \mathrm{P}_{\mathbb{C}}$.

Outline of the proof:

- Goal: for $A \in \mathrm{PAR}_{\mathbb{C}}$, decide in polynomial time (with $\mathrm{VPSPACE}$ tests) whether $\bar{x} \in A$.
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- Simulate the circuit on this sign condition.

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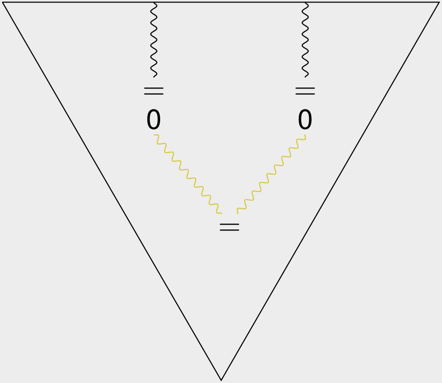
- Find the sign condition of $\bar{x}$
  - enumeration of the satisfiable sign conditions (Fichtas, Galligo, Morgenstern);
  - binary search.
- Simulate the circuit on this sign condition.

Polynomials tested by a circuit

Test gate: $f(\bar{x}) = 0$?

If the results of the preceding tests are fixed, f is a polynomial.

→ enumeration of all possible polynomials (polynomial space): family $f_1, \ldots, f_s$.



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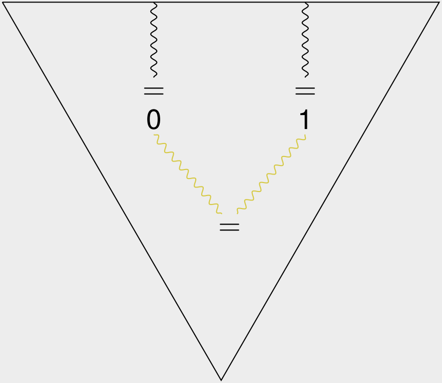

Sign conditions

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- If $\bar{x}$ and $\bar{y}$ have the same sign condition then every test gives the same result $\longrightarrow$ $\bar{x}$ and $\bar{y}$ are simultaneously in the language or outside of the language.
- It is enough to study the sign condition (boolean object).

Satisfiable sign conditions

- Sign condition $S \in \{0, 1\}^s$: sign of the polynomials $f_1, \ldots, f_s$.
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- Example: x and $x + 1$ cannot be both zero, hence $(0, 0)$ is not satisfiable.

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Theorem (Fichtas, Galligo, Morgenstern 1990)

- *There are $N = (sd)^{O(n)}$ satisfiable sign conditions (s: number of polynomials, n: number of variables, d: max degree).*
- *Satisfiable sign conditions can be enumerated in* PSPACE.

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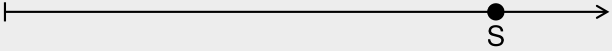

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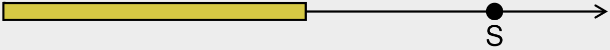

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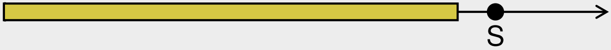

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- Over $\mathbb{R}$, easy thanks to $\mathrm{VPSPACE}$ tests

$$\prod_{j \leq i} \left(\sum_{S_k^{(j)}=0} f_k(\bar{x})^2 \right) = 0 \ \ \text{(true iff } S \leq i\text{)}$$

Membership to a variety

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Lemma

Let $V \in \mathbb{C}^n$ be a variety defined by s polynomials $f_1, \ldots, f_s$. Then V is defined by $n+1$ generic linear combinations $g_1, \ldots g_{n+1}$ of the f_i.

"generic": $g_i = \sum_{j=1}^{s} \alpha_{i,j} f_j$ where the $\alpha_{i,j}$ are algebraically independent.

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- Problem: we can only use integers.

Constructive tests

Lemma (Nonconstructive, reminder)

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Replace transcendant numbers by integers growing sufficiently fast.

Lemma

Let $\phi(x_1, \ldots, x_n)$ be a first order formula which is true on any algebraically independent coefficients $\alpha_1, \ldots, \alpha_n$. Then $\phi(\beta_1, \ldots, \beta_n)$ is true for any integers β_i growing sufficiently fast.

Proof idea: lack of “big” roots of multivariate polynomials.

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- By the first lemma, $\phi(\bar{\alpha})$ is true for all algebraically independent coefficients $\bar{\alpha}$.
- By the second lemma, $\phi(\bar{\beta})$ is true for integers $\bar{\beta}$ growing sufficiently fast: V is then defined by the $n + 1$ linear combinations of the f_i with coefficients $\bar{\beta}$.
- Hence $n + 1$ polynomials to test to zero.

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- $\rightarrow$ Divide and conquer.
- We can perform the binary search for the sign condition in polynomial time (with $\mathrm{VPSPACE}$ tests).

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In order to show that $\mathrm{VPSPACE} = \mathrm{VP} \Rightarrow \mathrm{PAR}_{\mathbb{C}} = \mathrm{P}_{\mathbb{C}}$:

- For $A \in \mathrm{PAR}_{\mathbb{C}}$ we want to decide in polynomial time (with $\mathrm{VPSPACE}$ tests) whether $\bar{x} \in A$.
- We enumerate all the polynomials possibly tested in the cricuit (polynomial space).
- Thanks to $\mathrm{VPSPACE}$ tests, a binary search gives the sign condition of $\bar{x}$.
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Main ideas:

1. sign conditions;
2. binary search thanks to tests of membership to varieties;
3. integers instead of transcendant numbers.

Conclusion

- Study of the question $\mathrm{P} = \mathrm{PSPACE}$ in different contexts (boolean, BSS, Valiant).
- Similar results over $\mathbb{R}$ but different techniques: we have to take into account the sign ($\rightarrow$ a vector orthogonal to roughly half a collection of vectors).
- Converse? Nullstellensatz $\Rightarrow$ work only up to a multiple.

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