THE CENTER OF SMALL GAUSSIAN GROUPS

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Abstract. Small Gaussian groups are a natural generalization of spherical Artin groups, namely groups of fractions of monoids in which the existence of least common multiples is kept as an hypothesis, but the relations between the generators are not supposed to necessarily be of Coxeter type. Here we completely describe the center of small Gaussian groups by constructing a minimal generating set for the quasi-center. We deduce that every small Gaussian group is an iterated crossed product of small Gaussian groups with a cyclic center.

Key words: center; quasi-center; crossed product; decomposition; Artin groups.

MSC 2000: 20F05, 20F36, 20F12.

Introduction

Define a small Gaussian monoid to be a cancellative monoid where 1 is the only invertible element, in which least common multiples exist, and which admits a finite generating set closed under $\backslash$, where $\backslash$ is the operation defined such that $a(a\backslash b)$ is the right lcm of a and b. A small Gaussian group is defined to be the group of fractions of a small Gaussian monoid. Small Gaussian groups have been introduced in [11] and [12] as a natural generalization for spherical Artin groups, *i.e.*, Artin groups associated with finite Coxeter groups.

In this paper, we construct a minimal generating set of the quasi-center of every small Gaussian monoid. Moreover, we define a notion of Δ-purity and a crossed product for small Gaussian monoids, and we prove

Proposition A. *The center of every Δ-pure small Gaussian group is an infinite cyclic subgroup.*

Proposition B. *Every small Gaussian monoid is an iterated crossed product of some Δ-pure small Gaussian monoids.*

These results extend similar statements established by Brieskorn, Saito [7] and Deligne [13] in the special case of spherical Artin groups.

This paper is organized as follows. In Section 1, we gather earlier results of [11] and [12] about small Gaussian groups. In Section 2, we introduce what we call local Delta's, and compute a minimal generating set of the quasi-center of every small Gaussian monoid. A convenient notion of crossed product for small Gaussian monoids is studied in Section 3. Finally, in Section 4, we define Δ-purity, and prove Propositions A and B.

1. Preliminaries

In this section, we list some basic properties of small Gaussian monoids and small Gaussian groups.

Assume that M is a monoid. We say that M is *conical* if 1 is the only invertible element in M. For a, b in M, we say that b is a left divisor of a—or that a is a right multiple of b—if $a = bd$ holds for some d in M. An element c is a right lower common multiple—or a right lcm—of a and b if it is a right multiple of both a and b, and every common right multiple of a and b is a right multiple of c. Right divisor, left multiple, and left lcm are defined symmetrically. For a, b in M, we say that b *divides* a—or that b is a divisor of a—if $a = cbd$ holds for some c, d in M.

If c, c' are two right lcm's of a and b, necessarily c is a left divisor of c', and c' is a left divisor of c. If we assume M to be conical and cancellative, we have $c = c'$. In this case, the unique right lcm of a and b is denoted by $a \vee b$. If $a \vee b$ exists, and M is left cancellative, there exists a unique element c satisfying $a \vee b = ac$. This element is denoted by $a\backslash b$. We define the *left lcm* $\widetilde{\vee}$ and the left operation $/$ symmetrically. In particular, we have

$$a \vee b = a(a\backslash b) = b(b\backslash a), \quad \text{and} \quad a \mathbin{\widetilde{\vee}} b = (b/a)a = (a/b)b.$$

Let us mention that cancellativity plus conicity simply means that left and right divisibility are order relations.

Definition. [11] A monoid M is said to be *Gaussian* if it is conical, cancellative, and every pair of elements in M admits a left lcm and a right lcm. A Gaussian monoid M is said to be *small* if there exists a finite subset that generates M and is closed under $\backslash$.

Example 1.1. The monoid M_0 with presentation $\langle\, x, y : xyyxyxyyx = yxyyxy \,\rangle$ is a small Gaussian monoid.

If M is a (small) Gaussian monoid, then M satisfies Ore's conditions [8], and it embeds in a group of right fractions, and, symmetrically, in a group of left fractions. In this case, by construction, every right fraction ab^{-1} with a, b in M can be expressed as a left fraction $c^{-1}d$, and conversely. Therefore, the two groups coincide, and there is no ambiguity in speaking of *the* group of fractions of a small Gaussian monoid.

Definition. A group G is a *small Gaussian group* if there exists a small Gaussian monoid of which G is the group of fractions.

By [7], all spherical Artin monoids are small Gaussian monoids. The braid monoids of the complex reflection groups $G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}$ and G_{22} given in [6], some monoids for torus knot or link groups [20][19], the Birman-Ko-Lee monoids of spherical Artin groups [4][2][18][1] are also small Gaussian monoids.

Lemma 1.2. [11] *Assume that M is a Gaussian monoid. Then the following identities holds in M:*

$$(ab) \vee (ac) = a(b \vee c), \tag{1.1}$$

$$c\backslash(ab) = (c\backslash a)((a\backslash c)\backslash b), \qquad (ab)\backslash c = b\backslash(a\backslash c), \tag{1.2}$$

$$(a \vee b)\backslash c = (a\backslash b)\backslash(a\backslash c) = (b\backslash a)\backslash(b\backslash c), \qquad c\backslash(a \vee b) = (c\backslash a) \vee (c\backslash b). \tag{1.3}$$

Lemma 1.3. [12] *Assume that M is a small Gaussian monoid. Then the following equivalent assertions hold:*

(i) There exists a mapping μ from M into the integers satisfying $\mu(a) > 0$ for every $a \neq 1$ in M, and satisfying $\mu(ab) \geq \mu(a) + \mu(b)$ for every a, b in M;

(ii) For every set X that generates M and for every a in M, the lengths of the decompositions of a as products of elements in X have a finite upper bound.

Definition. [12] A monoid is said to be *atomic* if it satisfies the equivalent conditions of Lemma 1.3. The *norm* function $\|.\|$ of an atomic monoid M is defined such that, for every a in M, $\|a\|$ is the upper bound of the lengths of the decompositions of a as products of atoms.

By the previous lemma, every element in a small Gaussian monoid has only finitely many left divisors, then, for every pair of elements (a, b), the common left divisors of a and b admit a right lcm, which is therefore the *left gcd* of a and b. This left gcd will be denoted by $a \wedge b$. We define the *right gcd* $\tilde{\wedge}$ symmetrically.

The following property essentially expresses the connection between the operations $\vee, \wedge, \tilde{\vee}$ and $\tilde{\wedge}$.

Lemma 1.4. *Assume that M is a small Gaussian monoid. Then, for a, b, c, d in M satisfying $ab = cd$, we have $ab = (a \vee c)(b \,\tilde{\wedge}\, d) = (a \wedge c)(b \,\tilde{\vee}\, d) = cd$.*

Proof. There exists g in M satisfying $ab = (a \vee c)g = cd$. We deduce $b = (a\backslash c)g$ and $(c\backslash a)g = d$. In particular, there exists h in M satisfying $b \,\tilde{\wedge}\, d = hg$. Therefore, h is a right divisor of both $a\backslash c$ and $c\backslash a$. By definition of the operation $\backslash$, we find $h = 1$, hence $ab = (a \vee c)(b \,\tilde{\wedge}\, d) = cd$. The equality $ab = (a \wedge c)(b \,\tilde{\vee}\, d) = cd$ is obtained symmetrically. □

Lemma 1.5. [11] *Assume that M is a small Gaussian monoid. Then it admits a finite generating subset that is closed under $\backslash, /, \vee, \wedge, \widetilde{\vee}$ and $\widetilde{\wedge}$.*

An *atom* is defined to be a non trivial element a such that $a = bc$ implies $b = 1$ or $c = 1$. Every small Gaussian monoid admits a finite set of atoms, and this set is the minimal generating set [12]. The hypothesis that there exists a finite generating subset that is closed under $\backslash$ implies that the closure of the atoms under $\backslash$ is finite—its elements are called *right primitive elements*. In particular, the closure of the atoms under $\backslash$ and $\vee$ is finite—its elements are called *simple elements*, and their right lcm is denoted by Δ. It turns out that the set of the simple elements is also the closure of atoms under $/$ and $\widetilde{\vee}$. So, the element Δ is both the right and the left lcm of the simple elements, and it is called the *Garside element* of the monoid. If M is a small Gaussian monoid and S is the set of simple elements in M, then $(S, \wedge, \vee, 1, \Delta)$ is a finite lattice.

Proposition 1.6. [11] *Assume that M is a small Gaussian monoid, S is the set of its simple elements, and Δ is its Garside element.*

(i) Let k be a nonnegative integer. Then, S^k is both the set of all left divisors of Δ^k and the set of all right divisors of Δ^k.

(ii) The functions $a \mapsto (a\backslash\Delta)\backslash\Delta$ and $a \mapsto \Delta/(\Delta/a)$ from S into itself extend into automorphisms ϕ and $\widetilde{\phi}$ of M that map S^k into itself for every k, and the equalities

$$a\Delta = \Delta\phi(a), \quad \text{and} \quad \Delta a = \widetilde{\phi}(a)\Delta$$

hold for every a in M.

Definition. Assume that M is a small Gaussian monoid. The order of the automorphisms ϕ and $\widetilde{\phi}$ of M is called the *exponent* of M.

Our main subject here will be the study of the center. Let us first recall some basic notions.

Definition. Assume that M is a small Gaussian monoid, A is its set of atoms, and G is its group of fractions. Then the *quasi-center* of M (*resp.* the *quasi-centralizer of A in G*) is the submonoid $\{b \in M \; ; \; Ab = bA\}$ of M (*resp.* the subgroup $\{b \in G \; ; \; Ab = bA\}$ of G).

Lemma 1.7. *Assume that M is a small Gaussian monoid. Then, for every element a and every quasi-central element b in M, the following are equivalent:*

(i) a divides b;

(ii) a is a left divisor of b;

(iii) a is a right divisor of b.

The study of the center of small Gaussian groups reduces to the study of the center and quasi-center of small Gaussian monoids:

Lemma 1.8. *Assume that M is a small Gaussian monoid, A is the set of its atoms, and G is its group of fractions. Then*

(i) the quasi-centralizer of A in G is the group of fractions of the quasi-center of M;

(ii) the center of G is the group of fractions of the center of M.

Proof. Let c be an element in G. There exist an integer p and an element c' in M satisfying $c = \Delta^p c'$, see [11][17].
(i) Assume c in the quasi-centralizer of A in G. Then, the element $\Delta^{|p|}$ of M being quasi-central by Proposition 1.6, c' is quasi-central. Every element in the quasi-centralizer of A in G is so the quotient of two quasi-central elements of M.
(ii) There exist integers q, r satisfying $p = qe + r$ and $r \geq 0$, where e denotes the exponent of M. Assume c central. The element $\Delta^{|q|e}$ of M being central by definition, the element $\Delta^r c'$ belongs to the center of M. Every central element in G is thus the quotient of two central elements of M. $\square$

2. A local Delta for each element

Assume that M is a small Gaussian monoid. Here we associate with each element a in M a distinguished quasi-central element Δ_a which behaves like a sort of local Garside element. The main result is that the family of all Δ_x's for x an atom generates the quasi-center of M.

Notation. Assume that M is a Gaussian monoid. For $X, Y \subseteq M$, we denote by $Y \backslash X$ the set of the elements $b \backslash a$ for a in X, b in Y. We write $Y \backslash a$ for $Y \backslash \{a\}$ and $b \backslash X$ for $\{b\} \backslash X$.

Lemma 2.1. *Assume that M is a small Gaussian monoid, and S is its set of simples. Then, for every a in M, we have $M \backslash a = S^q \backslash a$ for some q (depending on a).*

Proof. Let $a \in M$. As S generates M, a belongs to S^p for some p. Now, a direct computation gives $M \backslash S^p = S^p$. In particular, we have $M \backslash a \subseteq S^p$, hence

$$S \backslash a \subseteq S^2 \backslash a \subseteq S^3 \backslash a \subseteq \ldots \subseteq S^p.$$

As S is finite, there exists $q \leq \mathrm{card}(S^p)$ satisfying $S^q \backslash a = S^{q+1} \backslash a$. We show using induction on j that, for every $j \leq 1$, we have $S^q \backslash a = S^{q+j} \backslash a$. The result is vacuously true for $j = 1$. Assume $j > 1$. Let $b \in S^{q+j-1}$ and $c \in S$. By induction hypothesis, there exists d in S^q satisfying $d\backslash a = b\backslash a$. By using Identity (1.2) of Lemma 1.2, we find $(bc)\backslash a = c\backslash(b\backslash a) = c\backslash(d\backslash a) = (dc)\backslash a$, so $(bc)\backslash a$ belongs to $S^{q+1}\backslash a$, *i.e.*, to $S^q\backslash a$, which completes the induction. Finally, we obtain $S^q\backslash a = M\backslash a$. □

Definition. Assume that M is a small Gaussian monoid. For every a in M, we define

$$\Delta_a = \bigvee\{b\backslash a \ ; \ b \in M\}.$$

By Lemma 2.1, the element Δ_a is well defined and effectively computable for every a in M. Symmetrically, we define $\widetilde{\Delta}_a = \widetilde{\bigvee}\{a/b \ ; \ b \in M\}$. Let us remark that, for every a in M, the equality $1\backslash a = a$ (*resp.* $a/1 = a$) implies a to be a left divisor of Δ_a (*resp.* a right divisor of $\widetilde{\Delta}_a$), and that, having $b\backslash 1 = 1 = 1/b$ for every b in M, we obtain $\Delta_1 = 1 = \widetilde{\Delta}_1$.

For instance, in the small Gaussian monoid M_0 of Example 1.1, we compute $S_0\backslash x \subsetneq S_0^2\backslash x = M_0\backslash x$ and $S_0\backslash y \subsetneq S_0^2\backslash y = M_0\backslash y$, where S_0 denotes the set of simple elements in M_0. The considered sets are displayed in Figure 1. We find $\Delta_x = \Delta_y = \Delta$. The current example shows that the sets $M\backslash x$ with x an atom need not be the whole set of primitive elements in M.

We are going to prove:

Proposition 2.2. *Assume that M is a small Gaussian monoid. Then, for every a in M, the element Δ_a is quasi-central. More precisely, the application $a \mapsto \Delta_a$ is a surjection from M onto the quasi-center of M.*

The proof of this result relies on several preliminary statements.

Lemma 2.3. *Assume that M is a small Gaussian monoid. Then every quasi-central element a in M satisfies $\Delta_a = a = \widetilde{\Delta}_a$.*

Proof. Let $b \in M$. As a is quasi-central, we have $ba = ab'$ for some b' in M. Therefore, ba is a right multiple of $b \vee a$, which is $b(b\backslash a)$, and, by left cancellation, a is a right multiple of $b\backslash a$. So, a is a right multiple of Δ_a—which is the right lcm of all $b\backslash a$'s. Now, a being a left divisor of Δ_a, cancellativity and conicity imply $\Delta_a = a$. The equality $\widetilde{\Delta}_a = a$ is obtained symmetrically. □

Lemma 2.4. *Assume that M is a small Gaussian monoid. Then, for every a in M, the following are equivalent:*

(i) $\Delta_a = a$ holds;

(ii) for every b in M, a is a left divisor of ba.

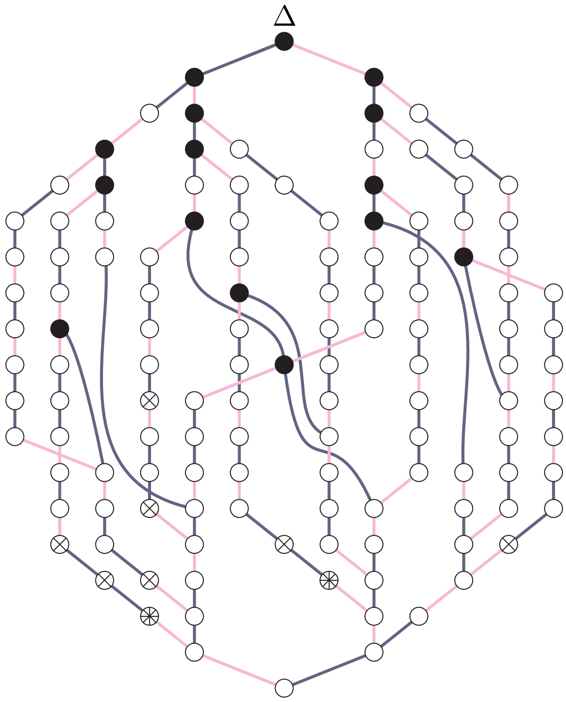

Figure 1. The lattice of simple elements in $M_0 = \langle\ x,y\ :\ xyyxyxyyx = yxyyxy\ \rangle$. The light edges represent x, while the dark ones represent y. The white points represent the primitive elements in M_0, while the black points represent the non-primitive simple elements in M_0. The elements of $M_0\backslash x$ (*resp.* of $M_0\backslash y$) are those represented by all white points except those marked '×' (*resp.* '+').

Proof. Assume (i). Let $b \in M$. From $\bigvee(M\backslash a) = a$, we deduce that $b\backslash a$ is a left divisor of a. Therefore, $b(b\backslash a)$ is a left divisor of ba. Now, by definition, $b(b\backslash a)$ is $a(a\backslash b)$, which implies (ii). Conversely, assume (ii). Then, for every b in M, $a \vee b$—which is $b(b\backslash a)$ by definition—is a left divisor of ba, and so, by left cancellation, $b\backslash a$ is a left divisor of a. This implies that $\bigvee(M\backslash a)$ is a left divisor of a, and, a being a left divisor of Δ_a, cancellativity and conicity yield (i) . □

Lemma 2.5. *Assume that M is a small Gaussian monoid. Then, for every a in M, $\Delta_a = a$ is equivalent to $\widetilde{\Delta}_a = a$.*

Proof. Let G be the group of fractions of M. We consider the injective endomorphism $h_a : b \mapsto a^{-1}ba$ of G. Assume $\Delta_a = a$. Then, by Lemma 2.4, for every b in M, a is a left divisor of ba : we deduce $h_a(M) \subseteq M$. Let S be the set of simples in M and e be the exponent of M. According to Proposition 1.6, for every c in S^e, there exists an element d in S^e satisfying $\Delta^e = cd$. We obtain $h_a(\Delta^e) = h_a(c)h_a(d)$, and, Δ^e being central, $\Delta^e = h_a(c)h_a(d)$, which implies $h_a(c) \in S^e$ (and $h_a(d) \in S^e$). As, by hypothesis, the set S is finite, the injective endomorphism h_a restricted to S^e is an automorphism. In particular, $h_a(M)$ includes the atoms of M, and we deduce $h_a(M) = M$. The endomorphism h_a is then an automorphism of M. Therefore, for every b in M, a is a right divisor of ab, and, by the left counterpart of Lemma 2.4, we deduce $\widetilde{\Delta}_a = a$. The converse implication is obtained symmetrically. $\square$

Lemma 2.6. *Assume that M is a small Gaussian monoid. Then every element a in M satisfying $\Delta_a = a$ is quasi-central.*

Proof. Let x be an atom of M. By Lemma 2.4, the hypothesis $\Delta_a = a$ implies that there exists d in M satisfying $xa = ad$. By right cancellativity, we have $d \neq 1$, and there exist a positive integer n and atoms $z_1, \ldots, z_n$ satisfying $d = z_1 \cdots z_n$. By Lemma 2.5, $\widetilde{\Delta}_a = a$ holds, and, by the left counterpart of Lemma 2.4, for every atom z_i with $1 \leq i \leq n$, there exists an element c_i in M satisfying $az_i = c_i a$. By left cancellativity, we have $c_i \neq 1$ for $1 \leq i \leq n$. We obtain

$$xa = ad = az_1 \cdots z_n = c_1 \cdots c_n a,$$

hence, by right cancellation, $x = c_1 \cdots c_n$. As x is an atom, we must have $n = 1$, *i.e.*, d is an atom. So, there exists a mapping f_a from the atoms of M into themselves such that $xa = af_a(x)$ holds for every atom x. By cancellativity, f_a is injective, hence surjective : a is quasi-central by definition. $\square$

Proof of Proposition 2.2. Let us show that $a \mapsto \Delta_a$ is idempotent. Let $a \in M$. By Lemma 2.1, there exists an integer n satisfying $M\backslash a = S^n\backslash a$ and $M\backslash\Delta_a = S^n\backslash\Delta_a$. Let $S^n = \{q_1, \ldots, q_r\}$. By using Lemma 1.2, we find

$$\begin{aligned}\Delta_{\Delta_a} &= (q_1\backslash(q_1\backslash a \vee \ldots \vee q_r\backslash a)) \vee \ldots \vee (q_r\backslash(q_1\backslash a \vee \ldots \vee q_r\backslash a))\\ &= ((q_1q_1\backslash a) \vee \ldots \vee (q_rq_1\backslash a)) \vee \ldots \vee ((q_1q_r\backslash a) \vee \ldots \vee (q_rq_r\backslash a)).\end{aligned}$$

Now, one of the q_i's is 1, and, therefore, we obtain $\Delta_{\Delta_a} = \Delta_a \vee \bigvee(S'\backslash a)$, where S' is some subset of S^{2n}. We deduce that $\Delta_{\Delta_a} = \Delta_a$ holds for every a in M. Therefore, by Lemma 2.6, Δ_a is quasi-central for every a in M. $\square$

Lemma 2.7. *Assume that M is a small Gaussian monoid. Then, for every element a and every quasi-central element b in M, a dividing b implies Δ_a and $\widetilde{\Delta}_a$ dividing b.*

Proof. By hypothesis, there exists an element d in M satisfying $b = ad$. As b is quasi-central, for every c in M, there exists an element c' in M satisfying $cb = adc'$. In particular, for every c in M, $c\vee a$—which is $c(c\backslash a)$—is a left divisor of cb, and, by left cancellation, $c\backslash a$ is a left divisor of b. Therefore, by definition, Δ_a divides b. $\square$

Proposition 2.8. *Assume that M is a small Gaussian monoid. Let A be its set of atoms. Then the quasi-center of M is generated by the set $\{\Delta_x \ ; \ x \in A\}$.*

Proof. Let b be a quasi-central element in M. We show using induction on $\|b\|$ that there exist an integer n and atoms $x_1, \dots, x_n$ satisfying $b = \Delta_{x_1} \cdots \Delta_{x_n}$. For $\|b\| = 0$, n is 0. Assume now $\|b\| > 0$. Then there exist an atom x and an element b' in M satisfying $b = xb'$. By Lemma 2.7, we have $b = \Delta_x b''$ for some b'' in M with $\|b''\| < \|b\|$. By Proposition 2.2, the element Δ_x is quasi-central, hence so is b''. By induction hypothesis, there exist an integer m and atoms $y_1, \dots, y_m$ satisfying $b'' = \Delta_{y_1} \cdots \Delta_{y_m}$. We obtain $b = \Delta_x \Delta_{y_1} \cdots \Delta_{y_m}$. □

For instance, in the case of the small Gaussian monoid M_0 of Example 1.1, Proposition 2.8 implies that its quasi-center is generated by Δ. As its exponent is 1, the center of M_0 coincides with the quasi-center.

We now prove that the generating set $\{\Delta_x \ ; \ x \in A\}$ is minimal.

Lemma 2.9. *Assume that M is a small Gaussian monoid. Then, for all atoms x, y in M, we have either $\Delta_x = \Delta_y$ or $\Delta_x \wedge \Delta_y = 1$.*

Proof. We first prove that, for all atoms x, y and every b in M, $\Delta_x = \Delta_y b$ implies $b = 1$. As $1 \backslash x = x$ holds, we have $\Delta_x = xd$ for some d in M. By using Lemma 1.4, we obtain

$$\Delta_x = xd = \Delta_y \ b = (x \wedge \Delta_y)(d \,\widetilde{\vee}\, b).$$

Assume $x \wedge \Delta_y = 1$. Then we find $\Delta_x = xd = \Delta_y \ b = d \,\widetilde{\vee}\, b = (b/d)d$, hence, by right cancellation, $x = b/d$. Therefore, x divides $\widetilde{\bigvee}(b/M)$, which, by definition, is $\widetilde{\Delta}_b$. Now, by hypothesis, b is quasi-central, and Lemma 2.3 implies $\widetilde{\Delta}_b = b$. By Lemma 2.7, Δ_x divides b, which, by cancellativity and conicity, implies $\Delta_y = 1$, a contradiction. Assume $x \wedge \Delta_y \neq 1$. Then, by atomicity, x divides Δ_y, and, by Lemma 2.7, Δ_x divides Δ_y, which, by cancellativity and conicity, implies $b = 1$.

Now, let x, y be atoms in M. Assume $\Delta_x \wedge \Delta_y \neq 1$. Then there exists an atom z in M dividing both Δ_x and Δ_y. By Lemma 2.7, Δ_z divides both Δ_x and Δ_y, which, by the result above, implies $\Delta_x = \Delta_z = \Delta_y$. □

Proposition 2.10. *Assume that M is a small Gaussian monoid. Let A be its set of atoms. Then $\{\Delta_x \ ; \ x \in A\}$ is a minimal generating set of the quasi-center of M.*

Proof. By Proposition 2.8, the set $\{\Delta_x \ ; \ x \in A\}$ generates the quasi-center of M. Let x be an atom, and a, b be quasi-central elements in M. We have to show that $\Delta_x = ab$ implies either $a = 1$ or $b = 1$. Assume $a \neq 1$. Then we have $a = ya'$

for some atom y and some a' in M. As a is quasi-central, by Lemma 2.7, Δ_y is a left divisor of a, and, therefore, Δ_y is a left divisor of Δ_x. We have $\Delta_y \neq 1$, hence, by Lemma 2.9, $\Delta_y = \Delta_x$. Cancellativity and conicity imply then $b = 1$. $\square$

We give now a new characterization of the function $a \mapsto \Delta_a$. We have seen that every element Δ_a is quasi-central, and, by construction, Δ_a is a right multiple of a. We prove that Δ_a is minimal with these properties. This new point of view will allow us to show that Δ_a and $\widetilde{\Delta}_a$ always coincide.

Lemma 2.11. *Assume that M is a small Gaussian monoid. Then, for all quasi-central elements a, b in M, the elements $a \wedge b$ and $a \mathbin{\widetilde{\wedge}} b$ are quasi-central.*

Proof. By Lemma 2.7, the element $\Delta_{a\wedge b}$ divides both a and b. Therefore, $\Delta_{a\wedge b}$ is a left divisor of $a \wedge b$. Now, $a \wedge b$ being a left divisor of $\Delta_{a\wedge b}$, we deduce $\Delta_{a\wedge b} = a \wedge b$ by using cancellativity and conicity. By Lemma 2.6, $a \wedge b$ is therefore quasi-central. Symmetrically, $a \mathbin{\widetilde{\wedge}} b$ is quasi-central. $\square$

Proposition 2.12. *Assume that M is a small Gaussian monoid, and QZ is its quasi-center. Then, for every a in M, we have*

$$\Delta_a = \bigwedge(QZ \cap aM) \qquad \textit{and} \qquad \widetilde{\Delta}_a = \widetilde{\bigwedge}(QZ \cap Ma).$$

Proof. Let $a \in M$. By definition, Δ_a is a right multiple of a, and, by Proposition 2.2, Δ_a is quasi-central. Therefore, Δ_a belongs to $QZ \cap aM$, and $\bigwedge(QZ \cap aM)$ divides Δ_a. Now, as $QZ \cap aM$ is nonempty, $\bigwedge(QZ \cap aM)$ is a right multiple of a. Moreover, by Lemma 2.11, $\bigwedge(QZ \cap aM)$ is quasi-central, and so, by Lemma 2.7, Δ_a divides $\bigwedge(QZ \cap aM)$. Cancellativity and conicity allow to conclude. The equality $\widetilde{\Delta}_a = \widetilde{\bigwedge}(QZ \cap Ma)$ is obtained symmetrically. $\square$

Corollary 2.13. *Assume that M is a small Gaussian monoid. Then, for every a in M, we have $\Delta_a = \widetilde{\Delta}_a$.*

Proof. Let $a \in M$. By Lemma 1.7, we have

$$QZ \cap aM = QZ \cap MaM = QZ \cap Ma.$$

By using Proposition 2.12, we deduce $\Delta_a = \bigwedge(QZ \cap MaM)$ and $\widetilde{\Delta}_a = \widetilde{\bigwedge}(QZ \cap MaM)$. Now, Δ_a belongs to $QZ \cap MaM$, and, therefore, $\widetilde{\Delta}_a$ is a right divisor of Δ_a. Symmetrically, Δ_a is a left divisor of $\widetilde{\Delta}_a$. Cancellativity and conicity allow to conclude. $\square$

We conclude the current section with the observation that the quasi-center of every small Gaussian monoid is a free abelian submonoid.

Lemma 2.14. *Assume that M is a small Gaussian monoid. Then, for all elements a, b in M, we have $\Delta_a \vee \Delta_b = \Delta_{a \vee b}$.*

Proof. First, let us show that, for all quasi-central elements a, b in M, the element $a \vee b$ is quasi-central. Let S be the set of simples in M. As S generates M, there exists a positive integer n such that a, b belong to S^n, and, by Proposition 1.6, there exist elements a', b' in S^n satisfying $\Delta^n = aa' = bb'$. As, by definition, Δ^n is quasi-central, both a', b' are quasi-central. Now, Lemma 1.4 gives $\Delta^n = aa' = bb' = (a \vee b)(a' \mathbin{\tilde{\wedge}} b')$. As, by Lemma 2.11, $a' \mathbin{\tilde{\wedge}} b'$ is quasi-central, we deduce that $a \vee b$ is quasi-central.

As $a \vee b$ divides $\Delta_{a \vee b}$, a divides $\Delta_{a \vee b}$. By Proposition 2.2 and Lemma 2.7, Δ_a divides $\Delta_{a \vee b}$, and, symmetrically, Δ_b divides $\Delta_{a \vee b}$. So, $\Delta_a \vee \Delta_b$ divides $\Delta_{a \vee b}$, and the equality follows from the result above and Proposition 2.12. □

Proposition 2.15. *Assume that M is a small Gaussian monoid. Let QZ be its quasi-center. Then QZ is a free abelian submonoid of M, and the function $a \mapsto \Delta_a$ is a surjective semilattice homomorphism from $(M, \vee)$ onto $(QZ, \vee)$.*

Proof. Let A be the set of atoms in M. By Proposition 2.10, QZ is the submonoid generated by $\{\Delta_x \,;\, x \in A\}$. So, in order to prove that QZ is free abelian, it suffices to show that $\Delta_x \backslash \Delta_y = \Delta_y$ holds for all x, y in A with $\Delta_x \neq \Delta_y$. Assume $\Delta_x \neq \Delta_y$. Then Lemma 2.9 implies $\Delta_x \backslash \Delta_y \neq 1$. As Δ_{Δ_y} is Δ_y (see the proof of Proposition 2.2), $\Delta_x \backslash \Delta_y$ divides Δ_y. Now, by Lemma 2.14, the element $\Delta_x \backslash \Delta_y$ is quasi-central, and Proposition 2.10 implies $\Delta_x \backslash \Delta_y = \Delta_y$. The second part of the assertion follows then from Lemma 2.14. □

Remark. Assume that M is a small Gaussian monoid. Let QZ be its quasi-center. The function $a \mapsto \Delta_a$ need not be a semilattice homomorphism from $(M, \wedge)$ onto $(QZ, \wedge)$. Indeed, for a, b in M, $\Delta_{a \wedge b}$ divides $\Delta_a \wedge \Delta_b$ (as $a \wedge b$ divides both Δ_a and Δ_b, $a \wedge b$ divides $\Delta_a \wedge \Delta_b$, which is quasi-central by Lemma 2.11, and so, by Lemma 2.7, $\Delta_{a \wedge b}$ divides $\Delta_a \wedge \Delta_b$), but there is no equality in general. We shall see in Section 4 a necessary and sufficient condition for this.

3. Crossed products

In this section, we define the notion of a crossed product for small Gaussian groups. As the latter are groups of fractions, we first define the notion for small Gaussian monoids.

Definition. Assume that $M_1, \ldots, M_n$ are small Gaussian monoids—or, more generally, cancellative conical monoids with finitely many atoms. Let A_i be the set of atoms in M_i for $1 \leq i \leq n$. Assume that $\vec{\Theta} = (\Theta_{ij})_{1 \leq i \neq j \leq n}$ is a family of functions $\Theta_{ij} : M_i \times M_j \to M_j$. We say that $\vec{\Theta}$ satisfies Condition (#) if, for every a in M_i, the restriction $\Theta_{ij}(a, .)$ of Θ_{ij} to $\{a\} \times M_j$ is a bijection of M_j, and, in addition, we have

$$\Theta_{ij}(ab, c) = \Theta_{ij}(b, \Theta_{ij}(a, c)), \tag{\#1}$$

$$\Theta_{ij}(a, cd) = \Theta_{ij}(a, c)\, \Theta_{ij}(\Theta_{ji}(c, a), d), \tag{\#2}$$

$$\Theta_{jk}(\Theta_{ij}(a, c), \Theta_{ik}(a, e)) = \Theta_{ik}(\Theta_{ji}(c, a), \Theta_{jk}(c, e)), \tag{\#3}$$

for a, b in M_i, c, d in M_j, e in M_k with $1 \leq i \neq j \neq k \neq i \leq n$. The *crossed product* $\bowtie_i^{\vec{\Theta}} M_i$ is then defined to be the quotient of the free product of the M_i's by the congruence generated by all pairs $(x\Theta_{ij}(x, y)\ ,\ y\Theta_{ji}(y, x))$ with $x \in A_i$, $y \in A_j$ and $1 \leq i < j \leq n$. For $n = 2$, we write $M_1 \bowtie_{\vec{\Theta}} M_2$.

The current notion of crossed product is reminiscent of the crossed product of groups as defined in [15] and [21] of which it is a monoidal version.

Example 3.1. Let us say that a family $\vec{\Theta}$ is *trivial* if, for $1 \leq i \neq j \leq n$, $\Theta_{ij}(a, .)$ is the identity for every a in M_i : $\vec{\Theta}$ is then a family satisfying Condition (#), and the crossed product $\bowtie_i^{\vec{\Theta}} M_i$ is the direct product $M_1 \times \ldots \times M_n$.

Lemma 3.2. *Assume that $M_1, \ldots, M_n$ are small Gaussian monoids—or, more generally, cancellative conical monoids with finitely many atoms. Let A_i be the set of atoms in M_i for $1 \leq i \leq n$. Then, for every family $\vec{\Theta}$ of functions satisfying Condition (#), for $1 \leq i \neq j \leq n$, for every a in M_i, the restriction $\Theta_{ij}(a, .)$ of Θ_{ij} to $\{a\} \times A_j$ is a permutation of A_j.*

Proof. First, by taking $b = 1$ in (#1) and using surjectivity, we find

$$\Theta_{ij}(1, d) = d, \tag{3.1}$$

for every d in M_j. Next, by taking $c = d = 1$ in (#2) and using both (3.1) and cancellativity, we obtain

$$\Theta_{ij}(a, 1) = 1, \tag{3.2}$$

for every a in M_i. Now, the restriction of Θ_{ij} to $\{a\} \times M_j$ is a surjection onto M_j : in particular, for every atom z of A_j, there exists c in M_j satisfying $\Theta_{ij}(a,c) = z$. We claim $c \in A_j$. Indeed, (3.2) implies $c \neq 1$, so we have $c = yc'$ for some $y \in A_j$ and $c' \in M_j$. By applying (#2), we find $\Theta_{ij}(a,y)\Theta_{ij}(\Theta_{ji}(y,a),c') = z$. Both injectivity of $\Theta_{ij}(a,.)$ and (3.2) imply $\Theta_{ij}(a,y) \neq 1$. As z is an atom, we obtain $\Theta_{ij}(a,y) = z$, and, by using injectivity of $\Theta_{ij}(a,.)$, we find $c = y \in A_j$. $\square$

Assume that $M_1, \ldots, M_n$ are small Gaussian monoids. Let A_i be the set of atoms in M_i for $1 \leq i \leq n$. Then every family $\vec{\Theta}$ satisfying Condition (#) for $M_1, \ldots, M_n$ is completely determined by the induced permutations $\Theta(x,.)$ of A_j for x in A_i and $1 \leq i \neq j \leq n$ (see Lemma 3.2). Now, conversely, not every such family of atom permutations extends into a family satisfying Condition (#) for $M_1, \ldots, M_n$. For instance, let us consider the small Gaussian monoids $\langle\ x, y : xyx = y^2\ \rangle$ and $\langle\ z :\quad \rangle$. The family of the atoms permutations $\Theta(x,.) = \Theta(y,.) = \left(\begin{smallmatrix} x & y & z \\ x & y & z \end{smallmatrix}\right)$ and $\Theta(z,.) = \left(\begin{smallmatrix} x & y & z \\ y & x & z \end{smallmatrix}\right)$ does not extend into a family satisfying Condition (#). Indeed, by using (#1) for instance, we would find $\Theta(z,y^2) = x^2$ and $\Theta(z,xyx) = yxy$, but $x^2 \neq yxy$ holds. See also Examples 3.3, 3.7, 3.9 and 3.12.

Definition. Assume that $M_1, \ldots, M_n$ are small Gaussian monoids—or, more generally, cancellative conical monoids with finitely many atoms. Let A_i be the set of atoms in M_i for $1 \leq i \leq n$, and A be the disjoint union $A_1 \sqcup \ldots \sqcup A_n$. Assume that $\vec{\theta} = (\theta_x)_{x \in A}$ is a family of permutations of A. We say that $\vec{\theta}$ satisfies Condition (#) if, for every x, θ_x is a permutation of A which globally preserves every A_j for $1 \leq j \leq n$, and, in addition, the θ_x's can be extended into a (necessary unique) family of functions satisfying Condition (#). The corresponding crossed product is then denoted by $\bowtie_i^{\vec{\theta}} M_i$. The latter does not depend on the value of $\theta_x(y)$ for x, y in A_j and $1 \leq j \leq n$, and we can assume that θ_x is the identity on A_j for every x in A_j and $1 \leq j \leq n$.

Example 3.3. Let us consider the small Gaussian monoids $M_1 = \langle\ x_1, x_2, x_3 : x_1x_2 = x_2x_3 = x_3x_1\ \rangle$ and $M_2 = \langle\ y, z : y^3 = z^3\ \rangle$. Let $\vec{\theta}$ be defined by $\theta_{x_1} = \theta_{x_2} = \theta_{x_3} = \left(\begin{smallmatrix} x_1 & x_2 & x_3 & y & z \\ x_1 & x_2 & x_3 & z & y \end{smallmatrix}\right)$, and $\theta_y = \theta_z = \left(\begin{smallmatrix} x_1 & x_2 & x_3 & y & z \\ x_3 & x_1 & x_2 & y & z \end{smallmatrix}\right)$. Then $\vec{\theta}$ is a family satisfying Condition (#) for M_1, M_2, and the monoid $M_1 \bowtie_{\vec{\theta}} M_2$ admits the presentation $\langle\ x_1, x_2, x_3, y, z : x_1x_2 = x_2x_3 = x_3x_1\ ,\ y^3 = z^3\ ,\ x_1z = yx_3\ ,\ x_1y = zx_3\ ,\ x_2z = yx_1\ ,\ x_2y = zx_1\ ,\ x_3y = zx_2\ ,\ x_3z = yx_2\ \rangle$.

Lemma 3.4. *Assume that $M_1, \ldots, M_n$ are small Gaussian monoids—or, more generally, cancellative atomic monoids with finitely many atoms. Then, for every family $\vec{\Theta}$ of functions satisfying Condition (#), for $1 \leq i \neq j \leq n$, for every a in M_i and every b in M_j, we have $\|\Theta_{ij}(a,b)\| = \|b\|$.*

Proof. We show by induction on $\|b\|$ that, for $1 \leq i \neq j \leq n$, for every a in M_i and every b in M_j, we have $\|\Theta_{ij}(a,b)\| \geq \|b\|$. For $\|b\| = 0$, the result follows from (3.2).

Assume $\|b\|>0$. Then there exist an atom x and an element d in M_j satisfying both $b = xd$ and $\|b\|= 1+\|d\|$. Lemma 3.2 implies $\|\Theta_{ij}(a, x)\|= 1$. By applying the induction hypothesis, we obtain

$$\begin{aligned}\|\Theta_{ij}(a,b)\|= \|\Theta_{ij}(a,xd)\| &\overset{(3.2)}{=} \|\Theta_{ij}(a,x)\Theta_{ij}(\Theta_{ji}(x,a),d)\|\\ &\geq \|\Theta_{ij}(a,x)\| + \|\Theta_{ij}(\Theta_{ji}(x,a),d)\|\\ &= 1+ \|\Theta_{ij}(\Theta_{ji}(x,a),d)\|\\ &\overset{(\mathrm{IH})}{\geq} 1+\|d\|=\|b\|,\end{aligned}$$

which completes the induction.

Now, for every a in M_i with $1 \leq i \leq n$, we denote by $\widetilde{\Theta}_{ij}(a, .)$ the inverse of the bijection $\Theta_{ij}(a, .)$. By definition, we have

$$\Theta_{ij}(a, \widetilde{\Theta}_{ij}(a,b)) = b = \widetilde{\Theta}_{ij}(a, \Theta_{ij}(a,b)), \tag{3.3}$$

for a in M_i, b in M_j and $1 \leq i \neq j \leq n$. From (3.3), (#1) and (#2), we deduce the following identities

$$\widetilde{\Theta}_{ij}(ab, c) = \widetilde{\Theta}_{ij}(a, \widetilde{\Theta}_{ij}(b,c)), \tag{3.4}$$
$$\widetilde{\Theta}_{ij}(a, cd) = \widetilde{\Theta}_{ij}(a,c)\widetilde{\Theta}_{ij}(\Theta_{ji}(\widetilde{\Theta}_{ij}(a,c),a),d), \tag{3.5}$$

for a, b in M_i, c, d in M_j and $1 \leq i \neq j \leq n$. An induction similar to the previous one gives $\|\widetilde{\Theta}_{ij}(a,c)\|\geq\|c\|$ for every a in M_i, every c in M_j with $1 \leq i \neq j \leq n$. We obtain

$$\|\Theta_{ij}(a,b)\|\geq\|b\|=\|\widetilde{\Theta}_{ij}(a,\Theta_{ij}(a,b))\|\geq\|\Theta_{ij}(a,b)\|,$$

which implies $\|\Theta_{ij}(a,b)\|=\|b\|$. □

Lemma 3.5. *Assume that $M_1, \ldots, M_n$ are small Gaussian monoids—or, more generally, cancellative atomic monoids with finitely many atoms. Then, for every family $\vec{\Theta}$ of functions satisfying Condition (#), for $1 \leq i \neq j \leq n$, for every a in M_i and every b in M_j,*

$$a\Theta_{ij}(a,b) = b\Theta_{ji}(b,a) \tag{3.6}$$

holds in $\bowtie_i^{\vec{\Theta}} M_i$.

Proof. We use an induction on $\|a\|\|b\|$. For $\|a\|\|b\|= 0$, the result follows from (3.1) and (3.2). Assume $\|a\|\|b\|> 0$. We have $a = xc$ and $b = yd$ for some atom x

and some element c in M_i, some atom y and some element d in M_j. By using Lemma 3.4, we obtain

$$\begin{aligned}
a\Theta_{ij}(a,b) &= xc\Theta_{ij}(xc,yd)\\
&\overset{(3.1)}{=} xc\Theta_{ij}(c,\Theta_{ij}(x,yd))\\
&\overset{(3.2)}{=} xc\Theta_{ij}(c,\Theta_{ij}(x,y)\Theta_{ij}(\Theta_{ji}(y,x),d))\\
&\overset{(3.2)}{=} xc\Theta_{ij}(c,\Theta_{ij}(x,y))\Theta_{ij}(\Theta_{ji}(\Theta_{ij}(x,y),c),\Theta_{ij}(\Theta_{ji}(y,x),d))\\
&\overset{(\text{IH})}{=} x\Theta_{ij}(x,y)\Theta_{ji}(\Theta_{ij}(x,y),c)\Theta_{ij}(\Theta_{ji}(\Theta_{ij}(x,y),c),\Theta_{ij}(\Theta_{ji}(y,x),d))\\
&\overset{(\text{IH})}{=} x\Theta_{ij}(x,y)\Theta_{ij}(\Theta_{ji}(y,x),d)\Theta_{ji}(\Theta_{ij}(\Theta_{ji}(y,x),d),\Theta_{ji}(\Theta_{ij}(x,y),c))\\
&\overset{\text{def}}{=} y\Theta_{ji}(y,x)\Theta_{ij}(\Theta_{ji}(y,x),d)\Theta_{ji}(\Theta_{ij}(\Theta_{ji}(y,x),d),\Theta_{ji}(\Theta_{ij}(x,y),c))\\
&\overset{(\text{IH})}{=} yd\Theta_{ji}(d,\Theta_{ji}(y,x))\Theta_{ji}(\Theta_{ij}(\Theta_{ji}(y,x),d),\Theta_{ji}(\Theta_{ij}(x,y),c))\\
&\overset{(3.2)}{=} yd\Theta_{ji}(d,\Theta_{ji}(y,x)\Theta_{ji}(\Theta_{ij}(x,y),c))\\
&\overset{(3.2)}{=} yd\Theta_{ji}(d,\Theta_{ji}(y,xc))\\
&\overset{(3.1)}{=} yd\Theta_{ji}(yd,xc) = b\Theta_{ji}(b,a),
\end{aligned}$$

which completes the induction. □

Proposition 3.6. *Assume that $M_1,\dots,M_n$ are small Gaussian monoids—or, more generally, cancellative atomic monoids with finitely many atoms. Then, for every family $\vec{\Theta}$ of functions satisfying Condition (#), $\bowtie_i^{\vec{\Theta}} M_i$ is set-theoretically equal to $M_1\times\ldots\times M_n$.*

Proof. Let $M = \bowtie_i^{\vec{\Theta}} M_i$. By definition, every element in M admits a decomposition as a product of elements in $M_1,\dots,M_n$. We have to show that such a decomposition is unique. Here, a *decomposition* of a non-trivial element a in M is a finite sequence $(b_1,\dots,b_m)$ satisfying $a = b_1\cdots b_m$ with $b_i \in M_{\mu_i}\setminus\{1\}$ for some sequence $(\mu_1,\dots,\mu_m)$ with values in $\{1,\dots,n\}$. The associated finite sequence $(\mu_1,\dots,\mu_m)$ is called the *support* of the decomposition. We order supports using the *ShortLex ordering* on sequences of integers : $(\mu_1,\dots,\mu_m) <^{\mathit{ShortLex}} (\nu_1,\dots,\nu_r)$ holds if $m < r$ does, or we have $m = r$ and $(\mu_1,\dots,\mu_m)$ precedes $(\nu_1,\dots,\nu_m)$ in the lexicographical extension of the standard order of the integers.
For $1 \le i \ne j \le n$, for every a in M_i, we denote by $\widetilde{\Theta}_{ij}(a,.)$ the inverse bijection of $\Theta_{ij}(a,.)$. Formula (3.6) is then equivalent to

$$ab = \widetilde{\Theta}_{ij}(a,b)\Theta_{ji}(\widetilde{\Theta}_{ij}(a,b),a), \tag{3.7}$$

for every $a \in M_i$, $b \in M_j$ and $1 \leq i \neq j \leq n$. Applying (3.7) inside some decomposition yields another decomposition (of the same element) : for $\mu_i \neq \mu_{i+1}$, we define

$$T_i(b_1, \ldots, b_i, b_{i+1}, \ldots, b_m) \\ = (b_1, \ldots, b_{i-1}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1}), \Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1}), b_i), b_{i+2}, \ldots, b_m).$$

Now, any two decompositions of an element a in M can be connected one to the other by a finite sequence of *elementary transformations* T, C and C^-, where C is defined by

$$C_i(b_1, \ldots, b_i, b_{i+1}, \ldots, b_m) = (b_1, \ldots, b_{i-1}, b_i b_{i+1}, b_{i+2}, \ldots, b_m),$$

for $\mu_i = \mu_{i+1}$, and C^- is the inverse (non-functionnal) transformation of C. The problem is that, starting with any decomposition of a, several transformations may be applied. We shall prove that, no matter the transformations are chosen, they lead to a unique final decomposition with $<^{\mathit{ShortLex}}$-minimal support. Let us say that an elementary transformation is *decreasing* if the support of the transformed decomposition is $<^{\mathit{ShortLex}}$-smaller than the initial support. So, applied to some transformation with support $(\mu_1, \ldots, \mu_m)$, T_i (*resp.* C_i) is decreasing whenever $\mu_i > \mu_{i+1}$ (*resp.* $\mu_i = \mu_{i+1}$) holds, while C_i^- is never decreasing. Now, $<^{\mathit{ShortLex}}$ is a wellordering on the supports of a given element, hence there exist no infinite sequence of decreasing transformations from a given decomposition. So, in order to prove that any sequence of decreasing transformations leads to a unique final decomposition with $<^{\mathit{ShortLex}}$-minimal support, it suffices to prove that, for every pair (D_1, D_2) of decreasing transformations applied to $\vec{b}$, there exist finite sequences D_1' and D_2' of decreasing transformations satisfying $D_1' \circ D_1(\vec{b}) = D_2' \circ D_2(\vec{b})$ ("confluence property", see [16]).

Claim 1. Assume the confluence property proved. Then any two decompositions with $<^{\mathit{ShortLex}}$-minimal supports of a given element are equal.

Proof. For a given decomposition $\vec{d}$, let $N(\vec{d})$ denote the unique decomposition obtained from $\vec{d}$ as above, *i.e.*, with $<^{\mathit{ShortLex}}$-minimal support. Let $\vec{a} = (a_1, \ldots, a_n)$ be a decomposition with a $<^{\mathit{ShortLex}}$-minimal support. We show that every decomposition $\vec{b}$ of $a_1 \cdots a_n$ satisfies $N(\vec{b}) = \vec{a}$ by using an induction on the number of T, C, C^- needed to transform $\vec{a}$ into $\vec{b}$. Thus, the point is to show that, if $\vec{d}$ is obtained in one step from $\vec{b}$, then we have $N(\vec{d}) = N(\vec{b})$. If the support of $\vec{d}$ is $<^{\mathit{ShortLex}}$-smaller than that of $\vec{b}$, $N(\vec{d}) = N(\vec{b})$ follows from confluence directly. Assume that the support of $\vec{d}$ is $<^{\mathit{ShortLex}}$-greater than that of $\vec{b}$, then $\vec{d}$ is obtained from $\vec{b}$ using either T or C^-; now, on the one hand, T_i is an involution by definition, and, on the other hand, in whatever way C_i^- is applied to $\vec{b}$, we have $C_i(\vec{d}) = \vec{b}$, so we find again $N(\vec{d}) = N(\vec{b})$. Thus, if $\vec{b}$ is a decomposition with a $<^{\mathit{ShortLex}}$-minimal support obtained from $\vec{a}$, we have $N(\vec{b}) = \vec{a}$. Now, by construction, we have $N(\vec{b}) = \vec{b}$, so $\vec{a}$ is the unique decomposition of $a_1 \cdots a_n$ with a $<^{\mathit{ShortLex}}$-minimal support. $\square$

It remains to prove the confluence property of the decreasing transformations. Three types of pairs (D_1, D_2) are to be considered. Let us fix an element a in M and a decomposition $(b_1, \ldots, b_m)$ of a with support $(\mu_1, \ldots, \mu_m)$.

Claim 2. Confluence holds for a pair of type (C_i, C_j).

Proof. Assume that both C_i and C_j are decreasing and applied to $(b_1, \ldots, b_m)$, *i.e.*, assume $\mu_i = \mu_{i+1}$ and $\mu_j = \mu_{j+1}$. Then we have $C_j(C_i(b_1, \ldots, b_m)) = C_i(C_j(b_1, \ldots, b_m))$, and confluence is verified. □

Claim 3. Confluence holds for a pair of type (T_i, C_j).

Proof. Assume that both T_i and C_j are decreasing and applied to $(b_1, \ldots, b_m)$, *i.e.*, assume $\mu_i > \mu_{i+1}$ and $\mu_j = \mu_{j+1}$. Then have $i \neq j$. For $|i-j| > 1$, we have $C_j(T_i(b_1, \ldots, b_m)) = T_i(C_j(b_1, \ldots, b_m))$, and confluence is verified. Assume $i - j = 1$ (the case $j - i = 1$ is similar). We show

$$C_i(T_{i-1}(T_i(b_1, \ldots, b_m))) = T_{i-1}(C_{i-1}(b_1, \ldots, b_m)).$$

By hypothesis, we have $\mu_{i-1} = \mu_i > \mu_{i+1}$, and we find

$$\begin{aligned}
&C_i(T_{i-1}(T_i(b_1, \ldots, b_m))) \\
&\quad = C_i(T_{i-1}(b_1, .., b_{i-1}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1}), \Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1}), b_i), b_{i+2}, .., b_m)) \\
&\quad = C_i(b_1, \ldots, b_{i-2}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1})), \\
&\qquad\qquad \Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1})), b_{i-1}), \\
&\qquad\qquad \Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1}), b_i), b_{i+2}, \ldots, b_m) \\
&\quad = (b_1, \ldots, b_{i-2}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1})), \\
&\qquad \Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1})), b_{i-1})\Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1}), b_i), \\
&\qquad\qquad b_{i+2}, \ldots, b_m),
\end{aligned}$$

and

$$\begin{aligned}
&T_{i-1}(C_{i-1}(b_1, \ldots, b_m)) \\
&\qquad = T_{i-1}(b_1, \ldots, b_{i-2}, b_{i-1}b_i, b_{i+1}, \ldots, b_m) \\
&\qquad = (b_1, \ldots, b_{i-2}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}b_i, b_{i+1}), \\
&\qquad\qquad \Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}b_i, b_{i+1}), b_{i-1}b_i), b_{i+2}, \ldots, b_m).
\end{aligned}$$

Now, the equality

$$\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1})) = \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}b_i, b_{i+1}) \tag{3.8}$$

follows from (#1), while the equality

$$\begin{aligned}
\Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}, \widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1})), b_{i-1})\Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_i, b_{i+1}), b_i)& \\
= \Theta_{\mu_{i+1}\mu_i}(\widetilde{\Theta}_{\mu_i\mu_{i+1}}(b_{i-1}b_i, b_{i+1}), b_{i-1}b_i)&
\end{aligned}$$

follows from (3.8) and (#2). □

Claim 4. Confluence holds for a pair of type (T_i, T_j).

Proof. Assume that both T_i and T_j are decreasing and applied to $(b_1, \dots, b_m)$, *i.e.*, assume $\mu_i > \mu_{i+1}$ and $\mu_j > \mu_{j+1}$. The case $i = j$ is trivial. For $|i - j| > 1$, we have $T_j(T_i(b_1, \dots, b_m)) = T_i(T_j(b_1, \dots, b_m))$, and confluence is verified. Assume $i - j = 1$ (the case $j - i = 1$ is symmetric). We show

$$T_{i-1}(T_i(T_{i-1}(b_1, \dots, b_m))) = T_i(T_{i-1}(T_i(b_1, \dots, b_m))).$$

Let $p = \mu_{i-1}$, $q = \mu_i$, $r = \mu_{i+1}$ and $a_p = b_{i-1}$, $a_q = b_i$, $a_r = b_{i+1}$. By hypothesis, we have $p > q > r$. We obtain

$$\begin{aligned}
&T_{i-1}(T_i(T_{i-1}(b_1, \dots, a_p, a_q, a_r, \dots, b_m)))\\
&\quad = T_{i-1}(T_i(b_1, \dots, \widetilde{\Theta}_{pq}(a_p, a_q), \Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p), a_r, \dots, b_m))\\
&\quad = T_{i-1}(b_1, \dots, \widetilde{\Theta}_{pq}(a_p, a_q), \widetilde{\Theta}_{pr}(\Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p), a_r),\\
&\qquad\quad \Theta_{rp}(\widetilde{\Theta}_{pr}(\Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p), a_r), \Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p)), \dots, b_m)\\
&\quad = (b_1, \dots, \widetilde{\Theta}_{qr}(\widetilde{\Theta}_{pq}(a_p, a_q), \widetilde{\Theta}_{pr}(\Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p), a_r)),\\
&\qquad\quad \Theta_{rq}(\widetilde{\Theta}_{qr}(\widetilde{\Theta}_{pq}(a_p, a_q), \widetilde{\Theta}_{pr}(\Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p), a_r)), \widetilde{\Theta}_{pq}(a_p, a_q)),\\
&\qquad\quad \Theta_{rp}(\widetilde{\Theta}_{pr}(\Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p), a_r), \Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p)), \dots, b_m)
\end{aligned}$$

and

$$\begin{aligned}
&T_i(T_{i-1}(T_i(b_1, \dots, a_p, a_q, a_r, \dots, b_m)))\\
&\quad = T_i(T_{i-1}(b_1, \dots, a_p, \widetilde{\Theta}_{qr}(a_q, a_r), \Theta_{rq}(\widetilde{\Theta}_{qr}(a_q, a_r), a_q), \dots, b_m))\\
&\quad = T_i(b_1, \dots, \widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)), \Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)), a_p),\\
&\qquad\qquad\qquad \Theta_{rq}(\widetilde{\Theta}_{qr}(a_q, a_r), a_q), \dots, b_m)\\
&\quad = (b_1, \dots, \widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)),\\
&\qquad\quad \widetilde{\Theta}_{pq}(\Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)), a_p), \Theta_{rq}(\widetilde{\Theta}_{qr}(a_q, a_r), a_q)),\\
&\qquad\quad \Theta_{qp}(\widetilde{\Theta}_{pq}(\Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)), a_p), \Theta_{rq}(\widetilde{\Theta}_{qr}(a_q, a_r), a_q)),\\
&\qquad\qquad \Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)), a_p)), \dots, b_m).
\end{aligned}$$

We are left with the task of proving the following equalities, which will prove confluence in this case:

$$\widetilde{\Theta}_{qr}(\widetilde{\Theta}_{pq}(a_p, a_q), \widetilde{\Theta}_{pr}(\Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p), a_r)) = \widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)), \tag{E_r}$$

$$\begin{aligned}
&\Theta_{rq}(\widetilde{\Theta}_{qr}(\widetilde{\Theta}_{pq}(a_p, a_q), \widetilde{\Theta}_{pr}(\Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p), a_r)), \widetilde{\Theta}_{pq}(a_p, a_q))\\
&\quad = \widetilde{\Theta}_{pq}(\Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)), a_p), \Theta_{rq}(\widetilde{\Theta}_{qr}(a_q, a_r), a_q)),
\end{aligned} \tag{E_q}$$

$$\begin{aligned}
&\Theta_{rp}(\widetilde{\Theta}_{pr}(\Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p), a_r), \Theta_{qp}(\widetilde{\Theta}_{pq}(a_p, a_q), a_p))\\
&\quad = \Theta_{qp}(\widetilde{\Theta}_{pq}(\Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)), a_p), \Theta_{rq}(\widetilde{\Theta}_{qr}(a_q, a_r), a_q)),\\
&\qquad\quad \Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, \widetilde{\Theta}_{qr}(a_q, a_r)), a_p)).
\end{aligned} \tag{E_p}$$

Let $c_q = \widetilde{\Theta}_{pq}(a_p, a_q)$ and $d_r = \widetilde{\Theta}_{qr}(a_q, a_r)$, hence $a_q = \Theta_{pq}(a_p, c_q)$ and $a_r = \Theta_{qr}(a_q, d_r)$. Equalities (E_r), (E_q), (E_p) are then equivalent respectively to

$$\widetilde{\Theta}_{qr}(c_q, \widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(\Theta_{pq}(a_p, c_q), d_r))) = \widetilde{\Theta}_{pr}(a_p, d_r), \qquad (E'_r)$$

$$\Theta_{rq}(\widetilde{\Theta}_{qr}(c_q, \widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(\Theta_{pq}(a_p, c_q), d_r))), c_q) \qquad (E'_q)$$
$$= \widetilde{\Theta}_{pq}(\Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, d_r), a_p), \Theta_{rq}(d_r, \Theta_{pq}(a_p, c_q))),$$

$$\Theta_{rp}(\widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(\Theta_{pq}(a_p, c_q), d_r)), \Theta_{qp}(c_q, a_p)) \qquad (E'_p)$$
$$= \Theta_{qp}(\widetilde{\Theta}_{pq}(\Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, d_r), a_p), \Theta_{rq}(d_r, \Theta_{pq}(a_p, c_q))), \Theta_{rp}(\widetilde{\Theta}_{pr}(a_p, d_r), a_p)).$$

Let $e_r = \widetilde{\Theta}_{pr}(a_p, d_r)$, hence $d_r = \Theta_{pr}(a_p, e_r)$. Equalities (E'_r), (E'_q), (E'_p) are then equivalent respectively to

$$\widetilde{\Theta}_{qr}(c_q, \widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(\Theta_{pq}(a_p, c_q), \Theta_{pr}(a_p, e_r)))) = e_r, \qquad (E''_r)$$

$$\Theta_{rq}(\widetilde{\Theta}_{qr}(c_q, \widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(\Theta_{pq}(a_p, c_q), \Theta_{pr}(a_p, e_r)))), c_q)$$
$$= \widetilde{\Theta}_{pq}(\Theta_{rp}(e_r, a_p), \Theta_{rq}(\Theta_{pr}(a_p, e_r), \Theta_{pq}(a_p, c_q))), \qquad (E''_q)$$

$$\Theta_{rp}(\widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(\Theta_{pq}(a_p, c_q), \Theta_{pr}(a_p, e_r))), \Theta_{qp}(c_q, a_p))$$
$$= \Theta_{qp}(\widetilde{\Theta}_{pq}(\Theta_{rp}(e_r, a_p), \Theta_{rq}(\Theta_{pr}(a_p, e_r), \Theta_{pq}(a_p, c_q))), \Theta_{rp}(e_r, a_p)). \qquad (E''_p)$$

By applying $\Theta_{qr}(c_q, .)$ and then $\Theta_{pr}(\Theta_{qp}(c_q, a_p), .)$ to (E''_r), we obtain

$$\Theta_{qr}(\Theta_{pq}(a_p, c_q), \Theta_{pr}(a_p, e_r)) = \Theta_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(c_q, e_r)), \qquad (E'''_r)$$

which is true by Condition (#3). Next, by applying $\Theta_{pq}(\Theta_{rp}(e_r, a_p), .)$ to (E''_q), we find

$$\Theta_{pq}(\Theta_{rp}(e_r, a_p), \Theta_{rq}(\widetilde{\Theta}_{qr}(c_q, \widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(\Theta_{pq}(a_p, c_q), \Theta_{pr}(a_p, e_r)))), c_q))$$
$$= \Theta_{rq}(\Theta_{pr}(a_p, e_r), \Theta_{pq}(a_p, c_q)). \qquad (E'''_q)$$

Now, by applying $\widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), .)$ to (E'''_r), we obtain

$$\widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(\Theta_{pq}(a_p, c_q), \Theta_{pr}(a_p, e_r))) = \Theta_{qr}(c_q, e_r), \qquad (E''''_r)$$

hence, by applying $\widetilde{\Theta}_{qr}(c_q, .)$,

$$\widetilde{\Theta}_{qr}(c_q, \widetilde{\Theta}_{pr}(\Theta_{qp}(c_q, a_p), \Theta_{qr}(\Theta_{pq}(a_p, c_q), \Theta_{pr}(a_p, e_r)))) = e_r.$$

According to the latter equality, (E'''_q) is equivalent to

$$\Theta_{pq}(\Theta_{rp}(e_r, a_p), \Theta_{rq}(e_r, c_q)) = \Theta_{rq}(\Theta_{pr}(a_p, e_r), \Theta_{pq}(a_p, c_q)),$$

which is true by Condition (#3). Finally, by using (E''''_r) in (E''_p), we obtain

$$\Theta_{rp}(\Theta_{qr}(c_q, e_r), \Theta_{qp}(c_q, a_p)) = \Theta_{qp}(\Theta_{rq}(e_r, c_q), \Theta_{rp}(e_r, a_p)),$$

which is true by Condition (#3). □

This completes the proof of Proposition 3.6. □

Corollary 3.7. *Assume that $M_1, \dots, M_n$ are small Gaussian monoids—or, more generally, cancellative atomic monoids with finitely many atoms. Then, for every family $\vec{\Theta}$ of functions satisfying Condition (#), $\bowtie_i^{\vec{\Theta}} M_i$ is atomic and cancellative.*

Proof. First, by Proposition 3.6, every element a in $\bowtie_i^{\vec{\theta}} M_i$ admits a unique decomposition as $a_1 \cdots a_n$ with $a_i \in M_i$ for $1 \le i \le n$, and, by Lemma 3.4, $\|a\|$ is the sum of the $\|a_i\|$'s. So, $\bowtie_i^{\vec{\theta}} M_i$ is an atomic monoid (see Lemma 1.3). Next, we have to show that, for a, b, c in $\bowtie_i^{\vec{\theta}} M_i$, $ac = bc$ implies $a = b$. As $\bowtie_i^{\vec{\theta}} M_i$ is atomic, we can use an induction on $\|c\|$. For $\|c\| = 0$, the result is trivial. Assume $ac = bc$ and $\|c\| > 0$. Then we have $c = xc'$ with $x \in A_r$ for some $1 \le r \le n$ and c' in $\bowtie_i^{\vec{\theta}} M_i$. By induction hypothesis, we have $ax = bx$. By Proposition 3.6, we can write $a = a_1 \cdots a_{r-1} a_{r+1} \cdots a_n a_r$ and $b = b_1 \cdots b_{r-1} b_{r+1} \cdots b_n b_r$ for some a_i, b_i in M_i, and, therefore, $a_1 \cdots a_{r-1} a_{r+1} \cdots a_n (a_r x) = b = b_1 \cdots b_{r-1} b_{r+1} \cdots b_n (b_r x)$. The uniqueness of decomposition implies $a_i = b_i$ for $i \ne r$ and $a_r x = b_r x$, hence $a = b$ by cancellativity of M_r. This completes the induction. The argument for left cancellativity is symmetric. □

Example 3.8. Let us consider the (isomorphic) small Gaussian monoids $\langle\, x_i, y_i : x_i y_i = y_i x_i \,\rangle^+$ for $i = 1, 2, 3$, and the family $\vec{\theta}$ formed by $\theta_{x_1} = \left(\begin{smallmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 \\ x_1 & y_1 & y_2 & x_2 & x_3 & y_3 \end{smallmatrix}\right)$, $\theta_{x_2} = \left(\begin{smallmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 \\ x_1 & y_1 & x_2 & y_2 & y_3 & x_3 \end{smallmatrix}\right)$ and $\theta_{y_1} = \theta_{y_2} = \theta_{x_3} = \theta_{y_3} = \left(\begin{smallmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 \\ x_1 & y_1 & x_2 & y_2 & x_3 & y_3 \end{smallmatrix}\right)$. Then $\vec{\theta}$ extends into a family of functions $\vec{\Theta}$ using (#1) and (#2), but $\vec{\Theta}$ does not satisfy Condition (#3). Let us observe that each of the three underlying bicrossed products is well-defined.

The results so far are valid for cancellative conical and/or atomic monoids with finitely many atoms. From now on, we shall concentrate on the specific case of small Gaussian monoids.

Lemma 3.9. *Assume that $M_1, \dots, M_n$ are small Gaussian monoids. Let A_i be the set of atoms in M_i for $1 \le i \le n$. Then, for every family $\vec{\theta}$ satisfying Condition (#), for $1 \le i \ne j \le n$, the function $(x, y) \mapsto (\theta_y(x), \theta_x(y))$ is a permutation of $A_i \times A_j$.*

Proof. Let us fix (i, j) with $1 \le i \ne j \le n$. Assume (x_1, x_2) in $A_i \times A_j$ such that there exist (y_1, y_2) and (z_1, z_2) in $A_i \times A_j$ satisfying

$$(\theta_{y_2}(y_1), \theta_{y_1}(y_2)) = (x_1, x_2) = (\theta_{z_2}(z_1), \theta_{z_1}(z_2)).$$

We obtain

$$\begin{cases} y_2 x_1 = y_1 x_2, \\ z_2 x_1 = z_1 x_2, \end{cases}$$

and, the monoid M_i being Gaussian,

$$\begin{cases} (z_1/y_1) y_2 x_1 = (z_1/y_1) y_1 x_2, \\ (y_1/z_1) z_2 x_1 = (y_1/z_1) z_1 x_2, \end{cases}$$

hence

$$(z_1/y_1)y_2x_1 = (y_1/z_1)z_2x_1.$$

Proposition 3.6 implies $z_1/y_1 = y_1/z_1$ and $y_2 = z_2$, but $z_1/y_1 = y_1/z_1$ leads to $z_1/y_1 = y_1/z_1 = 1$, hence $y_1 = z_1$, which proves injectivity of the function $(x, y) \mapsto (\theta_y(x), \theta_x(y))$. As the set $A_i \times A_j$ is finite, the function $(x, y) \mapsto (\theta_y(x), \theta_x(y))$ is a permutation of $A_i \times A_j$. $\square$

Example 3.10. Let us consider the (isomorphic) small Gaussian monoids $\langle\, x_1, y_1 : x_1y_1 = y_1x_1 \,\rangle^+$ and $\langle\, x_2, y_2 : x_2y_2 = y_2x_2 \,\rangle^+$, and the family $\vec{\theta}$ consisting of the permutations $\theta_{x_1} = \left(\begin{smallmatrix} x_1 & y_1 & x_2 & y_2 \\ x_1 & y_1 & x_2 & y_2 \end{smallmatrix}\right) = \theta_{x_2}$, $\theta_{y_1} = \left(\begin{smallmatrix} x_1 & y_1 & x_2 & y_2 \\ x_1 & y_1 & y_2 & x_2 \end{smallmatrix}\right)$ and $\theta_{y_2} = \left(\begin{smallmatrix} x_1 & y_1 & x_2 & y_2 \\ y_1 & x_1 & x_2 & y_2 \end{smallmatrix}\right)$. Then the function $(x, y) \mapsto (\theta_y(x), \theta_x(y))$ is not a permutation of $\{x_1, y_1\} \times \{x_2, y_2\}$. By Lemma 3.9, $\vec{\theta}$ is not a family satisfying Condition (#). Indeed, denoting by $\vec{\Theta}$ the family of functions associated with $\vec{\theta}$, we would find $\Theta_{21}(y_2, y_1x_1) = x_1^2$ and $\Theta_{21}(y_2, x_1y_1) = y_1x_1$.

Lemma 3.11. *Assume that $M_1, \ldots, M_n$ are small Gaussian monoids. Then, for every family $\vec{\Theta}$ of functions satisfying Condition (#), for a in M_i, for b in M_j with $1 \le i \ne j \le n$, $a\Theta_{ij}(a, b) = b\Theta_{ji}(b, a)$ is the right lcm of a and b in $\bowtie_i^{\vec{\Theta}} M_i$.*

Proof. By Lemma 3.5, $a\Theta_{ij}(a, b) = b\Theta_{ji}(b, a)$ is a right multiple of a and b in $\bowtie_i^{\vec{\Theta}} M_i$. Assume that $aa' = bb'$ is a right multiple of a and b in $\bowtie_i^{\vec{\Theta}} M_i$. By Proposition 3.6, we have $a' = a'_i a'_j \Pi_{1 \le k \le n, k \ne i, k \ne j} a'_k$ and $b' = b'_i b'_j \Pi_{1 \le k \le n, k \ne i, k \ne j} b'_k$ for some a'_k, b'_k in M_k with $1 \le k \le n$. By Proposition 3.6 again, $aa' = bb'$ implies both $aa'_i a'_j = bb'_i b'_j$ and $a'_k = b'_k$ for $1 \le k \le n, k \ne i, k \ne j$. By Proposition 3.6 always, $aa'_i a'_j = bb'_i b'_j$ implies $aa'_i = \widetilde{\Theta}_{ji}(b, b'_i)$, hence

$$b'_i = \Theta_{ji}(b, aa'_i) \overset{(3.2)}{=} \Theta_{ji}(b, a)\Theta_{ji}(\Theta_{ij}(a, b), a'_i).$$

Therefore, bb' is a right multiple of $b\Theta_{ji}(b, a)$. $\square$

Proposition 3.12. *Assume that $M_1, \ldots, M_n$ are small Gaussian monoids. Then, for every family $\vec{\theta}$ of permutations satisfying Condition (#), the monoid $\bowtie_i^{\vec{\theta}} M_i$ is a small Gaussian monoid, and the lattice of simple elements in $\bowtie_i^{\vec{\theta}} M_i$ is the product of the lattices of simple elements in $M_1, \ldots, M_n$.*

Proof. Let $\vec{\Theta}$ denote the family of functions associated with $\vec{\theta}$. Let A_i be the set of atoms in M_i for $1 \le i \le n$, and let A be the disjoint union $A_1 \sqcup \ldots \sqcup A_n$. First, by Corollary 3.7, $\bowtie_i^{\vec{\theta}} M_i$ is atomic and cancellative. Now, let us show by induction on $\| a \| \| b \|$ that any two elements a, b in $\bowtie_i^{\vec{\theta}} M_i$ admit a right lcm. For $\|a\|\|b\| = 0$, it is obviously true. Assume $\|a\|\|b\| > 0$. Then we have $a = a'x$

and $b = b'y$ for some $x \in A_r$, $y \in A_s$ with $1 \le r, s \le n$ and a', b' in $\bowtie_i^{\vec{\theta}} M_i$. By induction hypothesis, a' and b' admit a right lcm $a' \vee b'$. Let $a' \backslash b'$ and $b' \backslash a'$ denote those elements in $\bowtie_i^{\vec{\theta}} M_i$ satisfying $a' \vee b' = a'(a' \backslash b') = b'(b' \backslash a')$ (these two elements are unique by cancellativity and conicity). By Proposition 3.6, there exist c_i, d_i in M_i for $1 \le i \le n$ satisfying $a' \backslash b' = c_1 \cdots c_{r-1} c_{r+1} \cdots c_n c_r$ and $b' \backslash a' = d_1 \cdots d_{s-1} d_{s+1} \cdots d_n d_s$. Let

$$x' = \Theta_{nr}(c_n, .., \Theta_{(r+1)r}(c_{r+1}, \Theta_{(r-1)r}(c_{r-1}, \ldots \Theta_{2r}(c_2, \Theta_{1r}(c_1, x)) \ldots)) ..)$$

and

$$y' = \Theta_{ns}(d_n, .., \Theta_{(s+1)s}(d_{s+1}, \Theta_{(s-1)s}(d_{s-1}, \ldots \Theta_{2s}(d_2, \Theta_{1s}(d_1, y)) \ldots)) ..).$$

Lemma 3.2 implies $x' \in A_r$ and $y' \in A_s$. By Lemma 3.11, the element $c_r \backslash x'$ in M_r and the element $d_s \backslash y'$ in M_s admit a unique right lcm g in $\bowtie_i^{\vec{\theta}} M_i$. We deduce that $(a' \vee b')g$ is the right lcm of a and b. Therefore, $\bowtie_i^{\vec{\theta}} M_i$ is a Gaussian monoid.

In order to prove that $\bowtie_i^{\vec{\theta}} M_i$ is small, we show that the closure of A under $\backslash$ is $P = P_1 \sqcup \ldots \sqcup P_n$, where P_i denotes the closure of A_i under $\backslash$ for $1 \le i \le n$. Let $P^{(0)} = A$ and $P^{(k)} = \{a \backslash b \ ; \ a, b \in P^{(k-1)}\}$ for $k > 0$. We show by induction on k that $P^{(k)}$ is included in $P_1 \sqcup \ldots \sqcup P_n$. For $k = 0$, we deduce $A_1 \sqcup \ldots \sqcup A_n \subseteq P_1 \sqcup \ldots \sqcup P_n$ from $A_i \subseteq P_i$ for $1 \le i \le n$. Assume $k > 0$. Let a, b belong to $P^{(k-1)}$. By induction hypothesis, we have $a \in P_r$ and $b \in P_s$ for some $1 \le r, s \le n$. For $r = s$, $a \backslash b \in P_s \subseteq P_1 \sqcup \ldots \sqcup P_n$ and we are done. Assume $r \neq s$. Let us show inductively on $\|a\|$ that, for every a in P_r, $a \backslash P_s$ is included in P_s. For $\|a\| = 0$, we have $1 \backslash b = b \in P_s$ for every b in P_s. Assume $\|a\| > 0$. We have $a = x(x \backslash a)$ for some $x \in A_r$. Using Lemma 1.2, we obtain

$$a \backslash b = (x(x \backslash a)) \backslash b \overset{(1.2)}{=} (x \backslash a) \backslash (x \backslash b),$$

for every b in P_s. We have to show that $x \backslash b$ belongs to P_s for every x in P_r and every b in P_s. Let $P_s^{(0)} = A_s$ and $P_s^{(j)} = \{a \backslash b \ ; \ a, b \in P_s^{(j-1)}\}$ for $j > 0$. We show by induction on j that, for every b in $P_s^{(j)}$, $x \backslash b$ belongs to $P_s^{(j)}$ for every x in A_r. For $j = 0$, we have $b \in A_s$, hence $x \backslash b = \theta_x(b) \in A_s$. Assume $j > 0$. Then we have $b = b_1 \backslash b_2$ for some b_1, b_2 in $P_s^{(j-1)}$. Following Lemma 3.2, we denote by x' the image of x under the inverse of the permutation $\Theta_{sr}(b_1, .)$ of A_r. Using Lemmas 3.5 and 1.2, we obtain

$$x \backslash b = (b_1 \backslash x') \backslash (b_1 \backslash b_2) \overset{(1.3)}{=} (x' \backslash b_1) \backslash (x' \backslash b_2).$$

By induction hypothesis, both $x' \backslash b_1$ and $x' \backslash b_2$ belong to $P_s^{(j-1)}$, and, therefore, $x \backslash b$ belongs to $P_s^{(j)}$. We deduce that $x \backslash b$ belongs to P_s for every $x \in P_r$ and $b \in P_s$. Now, having $\|x \backslash a\| < \|a\|$, the initial induction hypothesis implies that $(x \backslash a) \backslash (x \backslash b)$ belongs to P_s, which completes the induction. Finally, as $P_1 \sqcup \ldots \sqcup P_n$ is finite, the closure of A under $\backslash$ is finite, and $P = P_1 \sqcup \ldots \sqcup P_n$ follows. Therefore, $\bowtie_i^{\vec{\theta}} M_i$ is a small Gaussian monoid.

Recall that, if $(X_i, \wedge_i, \vee_i)$ with $1 \le i \le r$ are lattices, their (Cartesian) product is the lattice $(X_1 \times \ldots \times X_r, \wedge, \vee)$, where $\wedge$ and $\vee$ are defined by $(a_1, \ldots, a_r) \wedge (b_1, \ldots, b_r) = (a_1 \wedge_1 b_1, \ldots, a_r \wedge_r b_r)$ and $(a_1, \ldots, a_r) \vee (b_1, \ldots, b_r) = (a_1 \vee_1 b_1, \ldots, a_r \vee_r b_r)$ for a_i, b_i in X_i and $1 \le i \le r$, see [3]. Here, for $1 \le i \le n$, S_i denotes the closure of P_i under $\vee$—by definition, the set of simples in M_i—and S denotes the closure of P under $\vee$. We consider the lattice homomorphism φ from $S_1 \times \ldots \times S_n$ into S defined by $\varphi(1, \ldots, 1, a_j, 1, \ldots, 1) = a_j$ for $a_j \in S_j$ and $1 \le j \le n$. Observe that every element in S that can be expressed as the right lcm of elements in $S_1, \ldots, S_n$ can also be expressed as the product of elements in $S_1, \ldots, S_n$. Indeed, an easy induction on n gives

$$a_1 \vee \ldots \vee a_n = a'_1 \cdots a'_n$$

with $a'_1 = a_1$ and $a'_i = (a'_1 \cdots a'_{i-1}) \backslash a_i$ for $i > 1$. Now, $\varphi(a_1, \ldots, a_n) = \varphi(b_1, \ldots, b_n)$ implies $a'_1 \cdots a'_n = b'_1 \cdots b'_n$, hence, by Proposition 3.6, $a'_i = b'_i$ for $1 \le i \le n$. We show inductively on i that $a_i = b_i$ holds for $1 \le i \le n$. The result is obviously true for $i = 1$. Assume $i > 1$. By induction hypothesis, $a'_i = b'_i$ implies

$$(a'_1 \cdots a'_{i-1}) \backslash a_i = (a'_1 \cdots a'_{i-1}) \backslash b_i,$$

hence, by using Lemmas 1.2 and 3.5,

$$\begin{aligned} &\Theta_{(i-1)i}(a'_{i-1}, \Theta_{(i-2)i}(a'_{i-2}, \ldots (\Theta_{2i}(a'_2, \Theta_{1i}(a'_1, a_i)) \ldots)) \\ &\qquad = \Theta_{(i-1)i}(a'_{i-1}, \Theta_{(i-2)i}(a'_{i-2}, \ldots, (\Theta_{2i}(a'_2, \Theta_{1i}(a'_1, b_i)) \ldots)). \end{aligned}$$

Now, by successively using injectivity of $\Theta_{(i-1)i}(a'_{i-1}, .)$, ..., $\Theta_{(1i)}(a'_1, .)$, we find $a_i = b_i$, which completes the induction and shows the injectivity of φ. By definition, for every a in S, there exist an integer m and elements $b_1, \ldots, b_m$ in $P = P_1 \sqcup \ldots \sqcup P_n$ satisfying $a = b_1 \vee \ldots \vee b_m$, hence we have $a = \varphi(a_1, \ldots, a_n)$ with $a_i = \bigvee(P_i \cap \{b_1, \ldots, b_m\})$ for $1 \le i \le n$, which means that φ is surjective. $\square$

Example 3.13. Let us consider again the small Gaussian monoids M_1 and M_2 of Example 3.3. Let $\vec{\theta}'$ be defined by $\theta'_{x_1} = \theta'_{x_2} = \theta'_{x_3} = \left(\begin{smallmatrix} x_1 & x_2 & x_3 & y & z \\ x_1 & x_2 & x_3 & y & z \end{smallmatrix}\right)$, $\theta'_y = \theta_y$ and $\theta'_z = \left(\begin{smallmatrix} x_1 & x_2 & x_3 & y & z \\ x_2 & x_3 & x_1 & y & z \end{smallmatrix}\right)$, and let $\vec{\theta}''$ be defined by $\theta''_{x_1} = \theta''_{x_2} = \theta''_{x_3} = \theta_{x_1}$, $\theta''_y = \theta_z$ and $\theta''_z = \theta'_z$. The monoids $M_1 \bowtie_{\vec{\theta}} M_2$ and $M_1 \bowtie_{\vec{\theta}'} M_2$ are small Gaussian, while the monoid $M_1 \bowtie_{\vec{\theta}''} M_2$ is not, as, for instance, $(x_2 \backslash y) \backslash (x_2 \backslash x_1) \neq (y \backslash x_2) \backslash (y \backslash x_1)$ contradicts Lemma 1.2.

4. Decomposition of a small Gaussian monoid

In this section, we introduce the notion of a Δ-pure small Gaussian monoid, that extends the one of irreducible spherical Artin monoids. On the one hand, we prove that the result of Brieskorn, Saito [7] and Deligne [13] stated in the special case of spherical Artin groups extends to the case of arbitrary small Gaussian groups : the quasi-center and the center of every Δ-pure small Gaussian group are infinite cyclic subgroups. On the other hand, we prove that every small Gaussian monoid is an iterated crossed product of some Δ-pure small Gaussian monoids.

Definition. Assume that M is a small Gaussian monoid. For a, b in M, we write $a \overset{\Delta}{\sim} b$ whenever $\Delta_a = \Delta_b$ holds. We say that M is Δ-*pure* if its atoms are $\overset{\Delta}{\sim}$-equivalent.

Proposition 4.1. *Assume that M is a Δ-pure small Gaussian monoid, Δ is its Garside element, e is its exponent, and G is its group of fractions.*

(i) The quasi-center of M is the infinite cyclic submonoid generated by Δ.

(ii) The center of M (resp. of G) is the infinite cyclic submonoid (resp. subgroup) generated by Δ^e.

Proof. Let A be the set of the atoms in M. According to Lemma 1.8 and Proposition 2.8, it suffices to show $\{\Delta_x \ ; \ x \in A\} = \{\Delta\}$. Let δ be the unique element of $\{\Delta_x \ ; \ x \in A\}$. By Proposition 1.6, Δ is quasi-central, and, by Proposition 2.10, δ divides Δ. Let

$$D = \bigcup_{x \in A} M \backslash x.$$

By hypothesis, δ is the right lcm of D. Let $a, b \in D$. By definition, we have $a = c\backslash z$ for some c in M and some atom z. By using (1.2), we find $b\backslash a = b\backslash(c\backslash z) = (cb)\backslash z$, which proves that D is closed under $\backslash$. As D includes A, D includes the closure P of A under $\backslash$, and, therefore, Δ, which is the right lcm of P, divides δ. Cancellativity and conicity allow to conclude. $\square$

Our aim is now to show that every small Gaussian monoid is an iterated crossed product of Δ-pure small Gaussian submonoids.

Definition. Assume that M is a small Gaussian monoid. Let A be its set of atoms. A subset A_1 of A is said to be *full* if, for every atom x in A_1 and every atom y in A, $x \overset{\Delta}{\sim} y$ implies $y \in A_1$.

Proposition 4.2. *Assume that M is a small Gaussian monoid. Then, for every full subset A_1 of atoms in M, the submonoid of M generated by A_1 is a small Gaussian monoid.*

Proof. Let M_1 denote the submonoid generated by A_1. First, the submonoid M_1 inherits cancellativity and conicity from M. Next, we prove $M\backslash M_1 = M_1$ and $\bigvee M_1 = M_1 = \widetilde{\bigvee} M_1$. We show using induction on $\| a \|$ that, for every a in M_1, the set $M\backslash a$ is included in M_1. For $\| a \| = 0$, we have $M\backslash 1 = \{1\}$. Assume $\|a\| > 0$. Then we have $a = xa'$ for some atom x in A_1 and some a' in M_1. We claim that Δ_x belongs to M_1 : let y be an atom in A_1 dividing Δ_x, then, by Lemma 2.7, Δ_y divides Δ_x, and, by Lemma 2.9, we have $x \overset{\wedge}{\sim} y$, hence $y \in A_1$. Now, let $b \in M$. By Lemma 1.2, we have

$$b\backslash a = b\backslash(xa') = (b\backslash x)((x\backslash b)\backslash a').$$

The element $b\backslash x$ belongs to M_1 as it divides Δ_x, and, by induction hypothesis, the element $(x\backslash b)\backslash a'$ belongs to M_1. Therefore, $b\backslash a$ belongs to M_1. Let $c, d \in M_1$. Then, from $c \vee d = c(c\backslash d)$ and $M\backslash M_1 = M_1$, we deduce $c \vee d \in M_1$. Symmetrically, we have $c \mathbin{\widetilde{\vee}} d \in M_1$. So, every pair (c, d) of elements in M_1 admits right and left lcm's in M_1. Finally, M_1 is a small Gaussian monoid since the closure of A_1 under $\backslash$ is included in the closure of A under $\backslash$. $\square$

Now, we have to investigate the relations amongst those atoms x, y satisfying $x \not\overset{\wedge}{\sim} y$. Though very easy, the following lemma is technically crucial.

Lemma 4.3. *Assume that M is a small Gaussian monoid.*

(i) Distinct atoms x, y in M satisfy $y\backslash x \overset{\wedge}{\sim} x$.

(ii) For every atom x and every b in M, $b \overset{\wedge}{\sim} x$ implies $c \overset{\wedge}{\sim} x$ for every non trivial element c dividing b in M.

Proof. (i) As Δ_x is $\bigvee(M\backslash x)$, $y\backslash x$ is a left divisor of Δ_x, and, by Lemma 2.7, $\Delta_{y\backslash x}$ is a left divisor of Δ_x. As y is an atom distinct from x, $y\backslash x$ is not trivial, and Lemma 2.9 implies $\Delta_{y\backslash x} = \Delta_x$.

(ii) Let $c \neq 1$ be a divisor of b in M. As b divides Δ_b, which is Δ_x by hypothesis, c divides Δ_x. Lemmas 2.7 and 2.9 imply then $\Delta_c = \Delta_x$. $\square$

Lemma 4.4. *Assume that M is a small Gaussian monoid. Then, for all disjoint full sets A_1, A_2 of atoms, the application $x_1 \mapsto x_2\backslash x_1$ is a permutation of A_1 for every atom x_2 in A_2.*

Proof. It suffices to show the following assertions:

(i) for all atoms x, y in M satisfying $x \not\overset{\wedge}{\sim} y$, the elements $y\backslash x$ and $x\backslash y$ are atoms in M, and they satisfy $y\backslash x \overset{\wedge}{\sim} x$ and $x\backslash y \overset{\wedge}{\sim} y$;

(ii) for all atoms x, y_1, y_2 in M satisfying $x \overset{\triangle}{\not\sim} y_1$, $x \overset{\triangle}{\not\sim} y_2$ and $y_1 \neq y_2$, the atoms $x\backslash y_1$ and $x\backslash y_2$ are distinct.

First, let us show (i). Let x, y be atoms in M satisfying $x \overset{\triangle}{\not\sim} y$. We have so $x \neq y$, and atomicity implies $x\backslash y \neq 1$ and $y\backslash x \neq 1$. Therefore, there exist atoms x', y' and elements a, b in M satisfying $x \vee y = xay' = ybx'$, hence, by Lemma 1.4,

$$x \vee y = xay' = ybx' = (xa \wedge yb)(y' \,\widetilde{\vee}\, x').$$

Assume $xa \wedge yb \neq 1$. Let z be an atom in M dividing $xa \wedge yb$ on the left. As $y \vee (xa \wedge yb)$ is a left divisor of yb, we have $z \neq x$. Therefore, $x \vee z$ being a left divisor of $x \vee y$, $x\backslash z$ is a non trivial left divisor of $x\backslash y$, and Lemma 4.3 implies $z \overset{\triangle}{\sim} x\backslash z \overset{\triangle}{\sim} x\backslash y \overset{\triangle}{\sim} y$. Symmetrically, we find $z \overset{\triangle}{\sim} y\backslash z \overset{\triangle}{\sim} y\backslash x \overset{\triangle}{\sim} x$, which contradicts the hypothesis $x \overset{\triangle}{\not\sim} y$. We obtain then

$$x \vee y = xay' = ybx' = y' \,\widetilde{\vee}\, x'.$$

By Lemma 4.3(i), $x\backslash y = ay'$ and $y\backslash x = bx'$ imply

$$ay' \overset{\triangle}{\sim} y \quad \text{and} \quad bx' \overset{\triangle}{\sim} x,$$

whereas $x'/y' = xa$ and $y'/x' = yb$ imply $xa \overset{\triangle}{\sim} x'$ and $yb \overset{\triangle}{\sim} y'$, hence, by using Lemma 4.3(ii),

$$xa \overset{\triangle}{\sim} x \quad \text{and} \quad yb \overset{\triangle}{\sim} y.$$

By Lemma 4.3(ii) again, the conjunction of $ay' \overset{\triangle}{\sim} y$ and $xa \overset{\triangle}{\sim} x$ implies $a = 1$, while the conjunction of $bx' \overset{\triangle}{\sim} x$ and $yb \overset{\triangle}{\sim} y$ implies $b = 1$.

Now, we prove (ii). Assume $x\backslash y_1 = x\backslash y_2$. Then we have $y_1(y_1\backslash x) = y_2(y_2\backslash x)$. In particular, $y_1\backslash y_2$ is a left divisor of $y_1\backslash x$. Now, by (i), $y_1\backslash x$ is an atom. The atoms y_1 and y_2 being distinct, $y_1\backslash y_2$ is not 1, and we deduce $y_1\backslash y_2 = y_1\backslash x$, which, by Lemma 4.3(i), implies $y_2 \overset{\triangle}{\sim} x$, a contradiction. □

Proposition 4.5. *Every small Gaussian monoid is an iterated crossed product of Δ-pure small Gaussian submonoids.*

Proof. Assume that M is a small Gaussian monoid. Let A be its set of atoms, and $A = A_1 \sqcup \ldots \sqcup A_n$ be a partition such that, for $1 \leq i \leq n$, A_i is a minimal nonempty full subset of A. For $1 \leq i \leq n$, let M_i (*resp.* $\overline{M_i}$) denote the submonoid generated by A_i (*resp.* by $A \setminus A_i$). Then, for $1 \leq i \leq n$, by Lemma 4.2, M_i and $\overline{M_i}$ are small Gaussian monoids, and, by Lemma 4.4, there exists a family $\vec{\theta}^{(i)}$ satisfying Condition (#) for $M_i, \overline{M_i}$ satisfying $M = M_i \bowtie_{\vec{\theta}^{(i)}} \overline{M_i}$. We find

$$M = \bowtie_i^{\vec{\theta}} M_i \quad \text{with} \quad \vec{\theta} = \bigsqcup_i \{\theta_x^{(i)} \ ; \ x \in A_i\}.$$

Except for $n = 1$, the small Gaussian monoids M_i need not be Δ-pure—see Example 4.8. Now, an iteration of the previous process leads to a decomposition of M as an iterated crossed product of Δ-pure small Gaussian monoids. Indeed, as the number of atoms strictly decreases, such an iteration is necessarily finite. □

Corollary 4.6. *Every small Gaussian monoid with two atoms is Δ-pure, except the rank two free abelian monoid.*

Remark. Let us come back to Proposition 2.15. Assume that M is a small Gaussian monoid, and QZ is its quasi-center. We have mentionned that the function $a \mapsto \Delta_a$ need not be a semilattice homomorphism from $(M, \wedge)$ onto $(QZ, \wedge)$. In fact, $a \mapsto \Delta_a$ is a semilattice homomorphism from $(M, \wedge)$ onto $(QZ, \wedge)$ if and only if M is a free abelian monoid. Indeed, for distinct atoms x, y in M satisfying $x \mathrel{\hat{\approx}} y$, we have $\Delta_{x \wedge y} = \Delta_1 = 1$ and $\Delta_x \wedge \Delta_y = \Delta_x \neq 1$. Therefore, if $a \mapsto \Delta_a$ is a semilattice homomorphism from $(M, \wedge)$ onto $(QZ, \wedge)$, then we have $x \mathrel{\hat{\not\approx}} y$ for all distinct atoms x, y in M, and, following the proof of Proposition 4.5, we deduce that M is free abelian. The converse implication is trivial.

Let us consider the special case of spherical Artin groups and monoids. Assume that M is a Artin monoid with set of atoms X and with Coxeter matrix $(m_{xy})_{x,y \in X}$. The Coxeter graph of M is defined to be the unoriented graph whose vertices are the atoms, and there is an edge between the vertices x and y whenever $m_{xy} \geq 3$ holds, m_{xy} labelling the corresponding edge [5], [7], [13]. The monoid M is irreducible if its Coxeter graph is connected.

Proposition 4.7. *Assume that M is a spherical Artin monoid. Then M is irreducible if and only if M is Δ-pure.*

Proof. Let Γ be the Coxeter graph of M. First, we show that, for all atoms x, y in the same connected component of Γ, $x \mathrel{\hat{\approx}} y$ holds. We can suppose x and y distinct. Then there exist a positive integer n and distinct atoms $x = z_0, \ldots, z_n = y$ in M such that, for $0 \leq i \leq n$, z_i and z_{i+1} are connected in Γ. Use induction on n to prove that x and y satisfy $x \mathrel{\hat{\approx}} y$. Assume $n = 1$. Then there exists an integer $m_{xy} \geq 3$ satisfying

$$x \backslash y = \mathrm{prod}(yx, m_{xy} - 1), \quad \text{and} \quad y \backslash x = \mathrm{prod}(xy, m_{xy} - 1),$$

where $\mathrm{prod}(w, k)$ denotes the length k prefix of the word w^∞. In particular, x divides $x \backslash y$, and y divides $y \backslash x$. Therefore, by definition, x divides Δ_y, and y divides Δ_x. By Lemma 2.7, by cancellativity and conicity, we deduce $\Delta_x = \Delta_y$, *i.e.*, $x \mathrel{\hat{\approx}} y$. Assume now $n > 1$. Then we have $x \mathrel{\hat{\approx}} z_1$ and, by induction hypothesis, $z_1 \mathrel{\hat{\approx}} y$, hence $x \mathrel{\hat{\approx}} y$.

Conversely, assume that M is not irreducible. Then M is the direct product of two non trivial spherical Artin monoids. Therefore, the quasi-center of M is not cyclic, and, by Proposition 4.1, M is not Δ-pure. $\square$

We conclude with an example of the decomposition mentionned in Proposition 4.5.

Example 4.8. Let us consider the monoids C_n considered by Garside in [14]—which are not Artin monoids for $n > 2$. Let C_3 be the monoid admitting the presentation

$$\langle\ x_1, x_2, x_3 : x_1x_2 = x_2x_3\ ,\ x_1x_3 = x_3x_1\ ,\ x_2x_1 = x_3x_2\ \rangle.$$

Then C_3 is a small Gaussian monoid, and the lattice of its simple elements is displayed in Figure 2. We compute in C_3

$$\Delta_{x_1} = \Delta_{x_3} = x_1x_3, \qquad \text{and} \qquad \Delta_{x_2} = x_2,$$

and deduce that the quasi-center (*resp.* the center) of C_3 is generated by x_1x_3 and x_2 (*resp.* by x_1x_3 and x_2^2). In particular, the monoid C_3 is not Δ-pure : we have $C_3 = M_1 \bowtie_{\vec{\theta}} M_2$ with $M_1 = \langle\ x_1, x_3 : x_1x_3 = x_3x_1\ \rangle$, $M_2 = \langle\ x_2 :\quad \rangle$, and $\theta_{x_1} = \theta_{x_3} = \left(\begin{smallmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{smallmatrix}\right)$, $\theta_{x_2} = \left(\begin{smallmatrix} x_1 & x_2 & x_3 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right)$. Now, M_2 is Δ-pure, while M_1 is not : we have $M_1 = \langle\ x_1 :\quad \rangle \times \langle\ x_3 :\quad \rangle$. Hence, we obtain $C_3 = \bowtie^{\vec{\theta}}_{i=1,2,3} \langle\ x_i :\quad \rangle$. According to Proposition 3.12, the lattice of simples in C_3 is isomorphic to the lattice of simples in the rank 3 free abelian monoid $\mathbf{N}^3$.

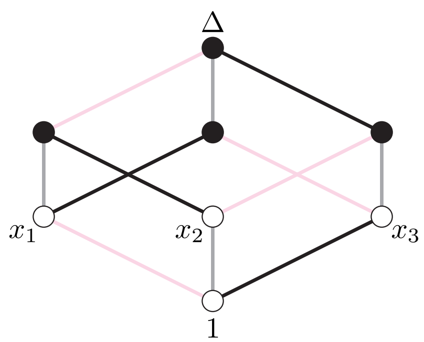

Figure 2. The lattice of simple elements in C_3.

Let us come back finally to the so-called parabolic submonoids of a small Gaussian monoid. A natural question is whether every submonoid of a small Gaussian monoid generated by atoms is a (small) Gaussian monoid as well—see Proposition 4.2. Considering the monoid C_3 of Example 4.8 again gives a negative answer. The submonoid M_∞ of C_3 generated by $\{x_1, x_2\}$ is not a Gaussian monoid. Indeed, the element x_2^2 is central in M_∞, but cannot be a multiple of x_1, and Proposition 2.8 does not work in this case. Actually, we can show that M_∞ admits the infinite presentation $\langle\ x_1, x_2 : x_1x_2x_1^kx_2 = x_2x_1^kx_2x_1,\ k \in\ \mathbf{N}\ \rangle$.

Remark. Except the property of effective computability of the function $a \mapsto \Delta_a$, most of the results in the previous sections extend to the most general framework of those monoids M where there exists an element Δ such that the left divisors of Δ coincide with its right divisors and they generate M, but whose we do not require the divisors of Δ to be finite in number. A typical example is the monoid presented by $\langle\ x, y : xyx = yx^2y\ \rangle$, whose group of fractions is isomorphic to the 3-strand braid group.

Acknowledgments

The author wishes to thank Patrick Dehornoy for some technical help, Luis Paris and François Digne for valuable comments and suggestions, and the referee for pointing out a gap in the proof of Proposition 3.6.

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