Quantum Chebyshev's Inequality and Applications

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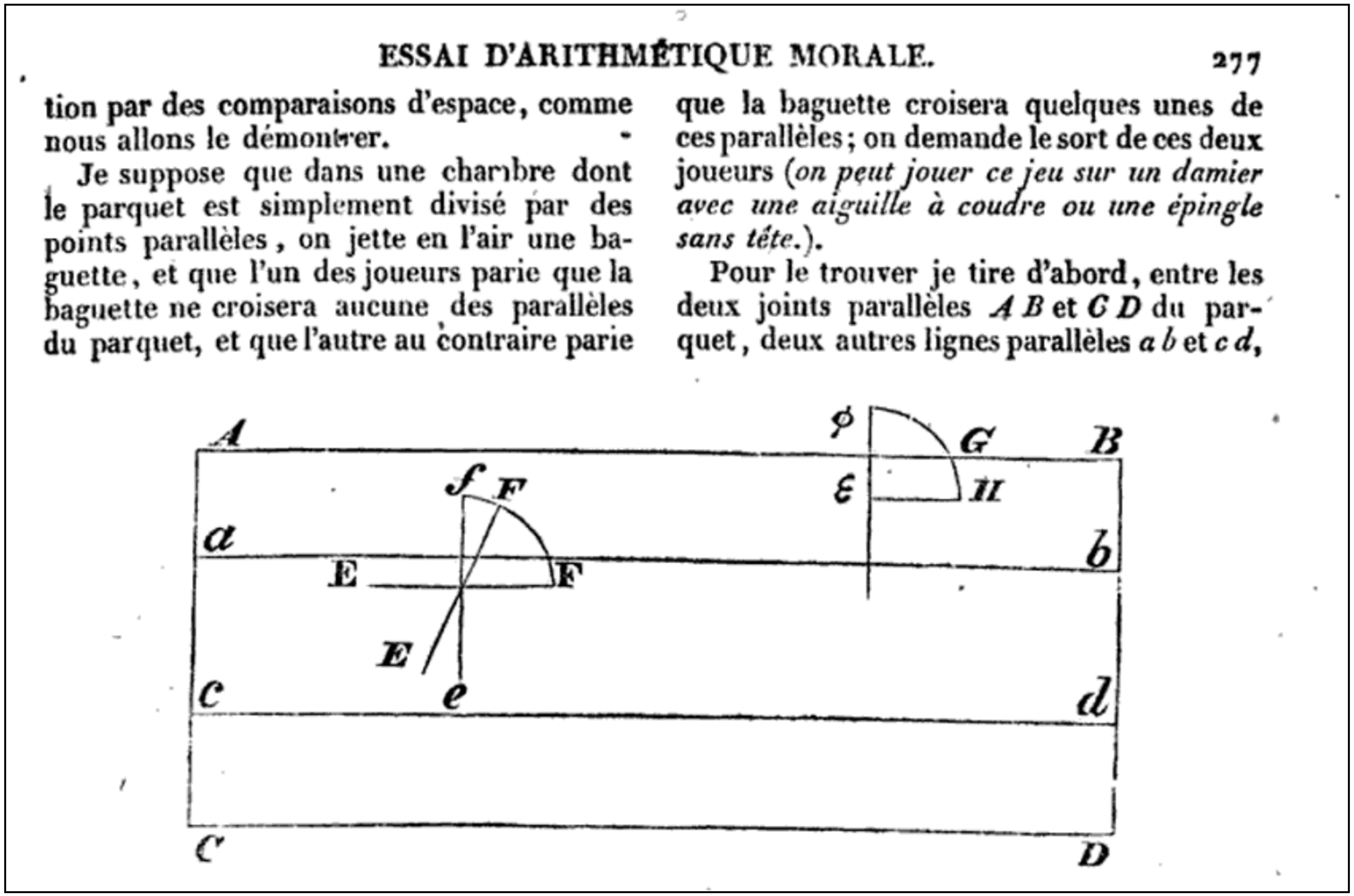
Buffon's needle

A needle dropped randomly on a floor with equally spaced parallel lines will cross one of the lines with probability **2/π**.



ESSAI D'ARITHMÉTIQUE MORALE. 277

tion par des comparaisons d'espace, comme nous allons le démontrer.

Je suppose que dans une chambre dont le parquet est simplement divisé par des joints parallèles, on jette en l'air une baguette, et que l'un des joueurs parie que la baguette ne croisera aucune des parallèles du parquet, et que l'autre au contraire parie que la baguette croisera quelques unes de ces parallèles; on demande le sort de ces deux joueurs (*on peut jouer ce jeu sur un damier avec une aiguille à coudre ou une épingle sans tête.*).

Pour le trouver je tire d'abord, entre les deux joints parallèles *A B* et *C D* du parquet, deux autres lignes parallèles *a b* et *c d*,

Buffon, G., *Essai d'arithmétique morale*, 1777.

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Law of large numbers: $$\frac{x_1 + \ldots + x_n}{n} \xrightarrow{n \to \infty} \mathbf{E}(X)$$

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multiplicative error 0 < ε < 1

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Number of samples needed: $O\left(\dfrac{\mathbf{E}(X^2)}{\epsilon^2 \mathbf{E}(X)^2}\right)$ (in fact $O\left(\frac{\mathbf{Var}(X)}{\epsilon^2 \mathbf{E}(X)^2}\right) = O\left(\frac{1}{\epsilon^2}\left(\frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} - 1\right)\right)$)

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In practice: given an upper-bound $\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$, take $n = \Omega\left(\frac{\Delta^2}{\epsilon^2}\right)$ samples

Counting with Markov chain Monte Carlo methods:

Counting vs. sampling [Jerrum, Sinclair'96] [Štefankovič et al.'09], Volume of convex bodies [Dyer, Frieze'91], Permanent [Jerrum, Sinclair, Vigoda'04]

Data stream model:

Frequency moments, Collision probability [Alon, Matias, Szegedy'99] [Monemizadeh, Woodruff'] [Andoni et al.'11] [Crouch et al.'16]

Testing properties of distributions:

Closeness [Goldreich, Ron'11] [Batu et al.'13] [Chan et al.'14], Conditional independence [Canonne et al.'18]

Estimating graph parameters:

Number of connected components, Minimum spanning tree weight [Chazelle, Rubinfeld, Trevisan'05], Average distance [Goldreich, Ron'08], Number of triangles [Eden et al. 17]

etc.

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Quantum sample: one (controlled-)**execution** of a **quantum sampler** S_X or S_X^{-1}, where

$$S_X|0\rangle = \sum_{x \in \Omega} \sqrt{p_x}\,|\psi_x\rangle\,|x\rangle$$

with $\psi_x =$ arbitrary unit vector

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[Montanaro'15]	$\frac{\Delta^2}{\epsilon}$	$\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$
[Li, Wu'17]	$\frac{\Delta}{\epsilon} \cdot \frac{H}{L}$	$\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$ $\quad L \leq E(X) \leq H$

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Our result	$\frac{\Delta}{\epsilon} \cdot \log^3\left(\frac{H}{\mathbf{E}(X)}\right)$	$\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$	$E(X) \leq H$

Our Approach

Amplitude Estimation Algorithm [Brassard et al.'02] [Brassard et al.'11] [Wocjan et al.'09] [Montanaro'15]

Input: Random variable **X** on sample space **Ω** ⊂ **[0,B]**

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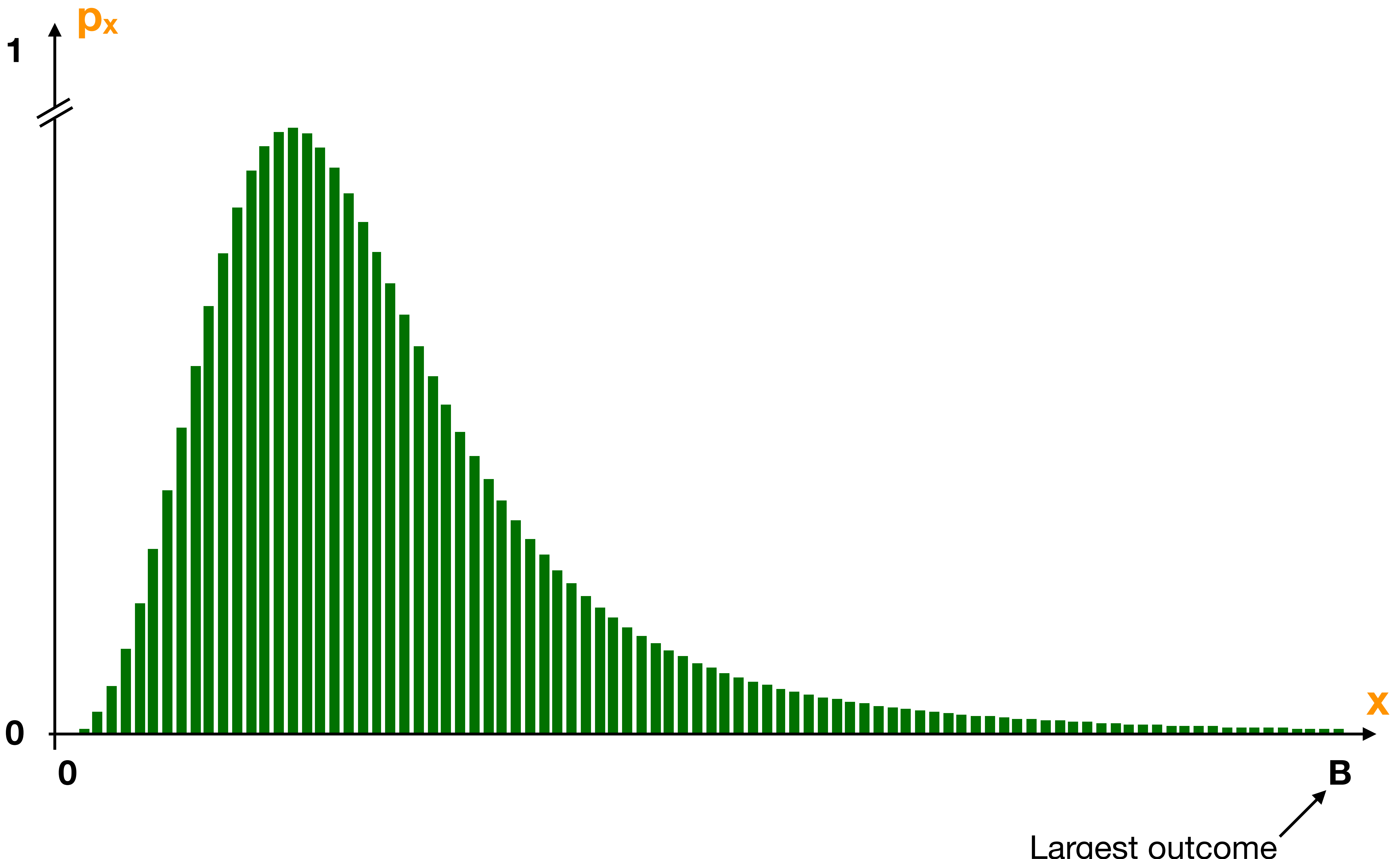
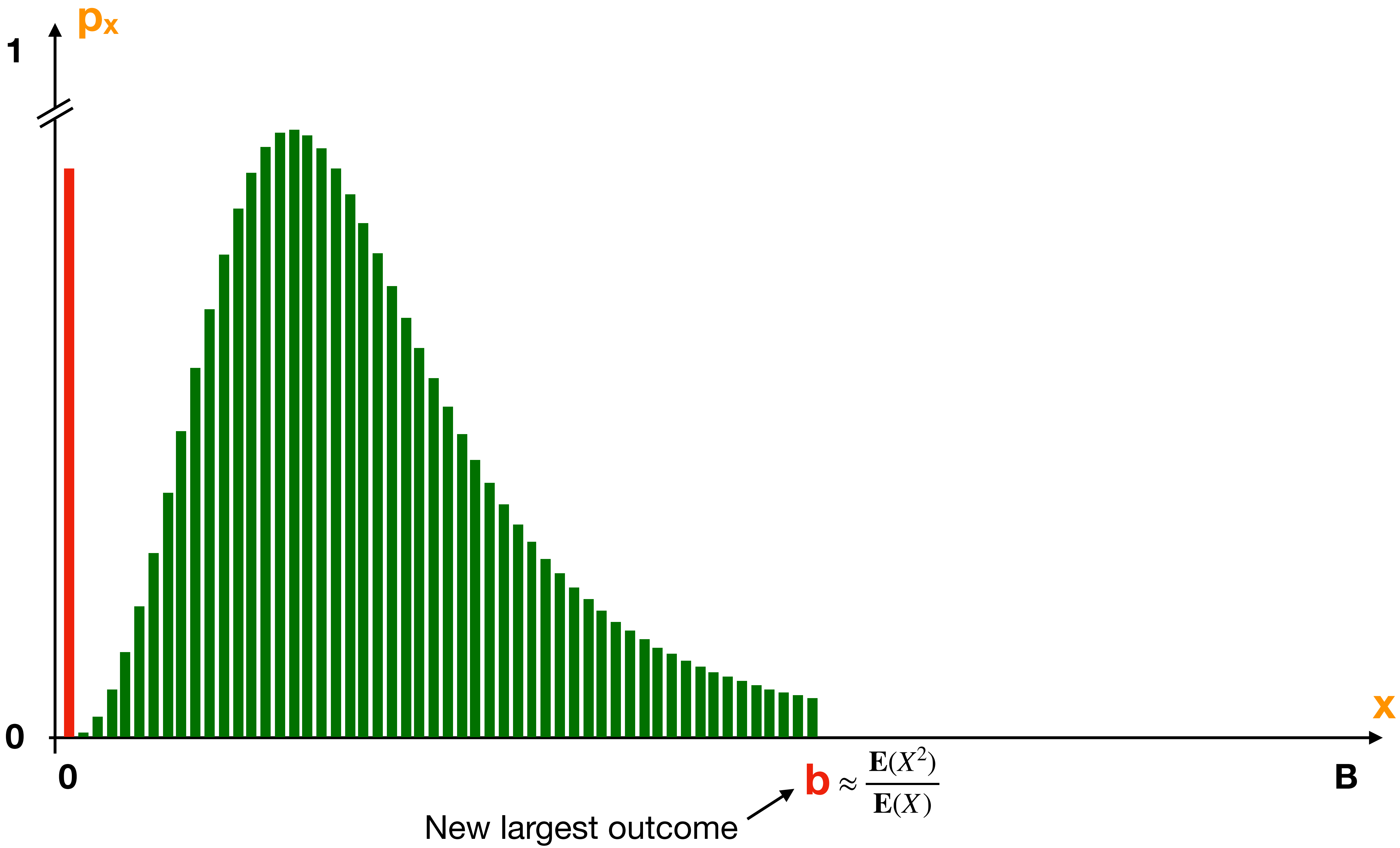


Random variable X_b



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Ingredient 3: **If** $b \approx \mathbf{E}(X) \cdot \Delta^2$ **then** $\frac{\mathbf{E}(X_b)}{b} \overset{\star}{\approx} \frac{\mathbf{E}(X)}{b} \approx \frac{1}{\Delta^2}$

Applications

Application 1: approximating graph parameters

Input: graph G=(V,E) with **n** vertices, **m** edges, **t** triangles

Query access: unitaries $O_{\text{deg}}|v\rangle|0\rangle = |v\rangle|\deg(v)\rangle$ *(degree query)*

$O_{\text{pair}}|v\rangle|w\rangle|0\rangle = |v\rangle|w\rangle|(v,w) \in E\ ?\rangle$ *(pair query)*

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[Goldreich, Ron'08] [Seshadhri'15]

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[Eden, Levi, Ron'15] [Eden, Levi, Ron, Seshadhri'17]

Application 2: frequency moments in the streaming model

Input: (finite) stream of updates $\mathbf{x_i} \leftarrow \mathbf{x_i} + \delta$ on **x** = (0,…,0) of **dimension n**

Output: (at the end of the stream) approximate of $F_k = \sum_{i=1}^{n} |x_i|^k$ (moment of order k ≥ 3)

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Result: $\mathbf{M} = \widetilde{\mathbf{O}}\left(\frac{\mathbf{n^{1-2/k}}}{\mathbf{P^2}}\right)$ qubits of memory

(vs. $M = \widetilde{\Theta}\left(\frac{n^{1-2/k}}{P}\right)$ classical bits of memory)

[Monemizadeh, Woodruff'10]
[Andoni, Krauthgamer, Onak'11]

Conclusion

The **mean** of a random variable X can be estimated with **multiplicative error ε** using $\widetilde{O}\left(\frac{\Delta}{\epsilon} \cdot \log^3\left(\frac{H}{E(X)}\right)\right)$ **quantum samples**, given $\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$ and $H \geq \mathbf{E}(X)$.

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Lower bound: $\boldsymbol{\Omega}\left(\frac{\boldsymbol{\Delta - 1}}{\epsilon}\right)$ quantum samples

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Lower bound: $\Omega\left(\frac{\Delta - 1}{\epsilon}\right)$ quantum samples

or $\Omega\left(\frac{\Delta^2 - 1}{\epsilon^2}\right)$ copies of the state $S_X|0\rangle = \sum_{x \in \Omega} \sqrt{p_x}\,|\psi_x\rangle|x\rangle$

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