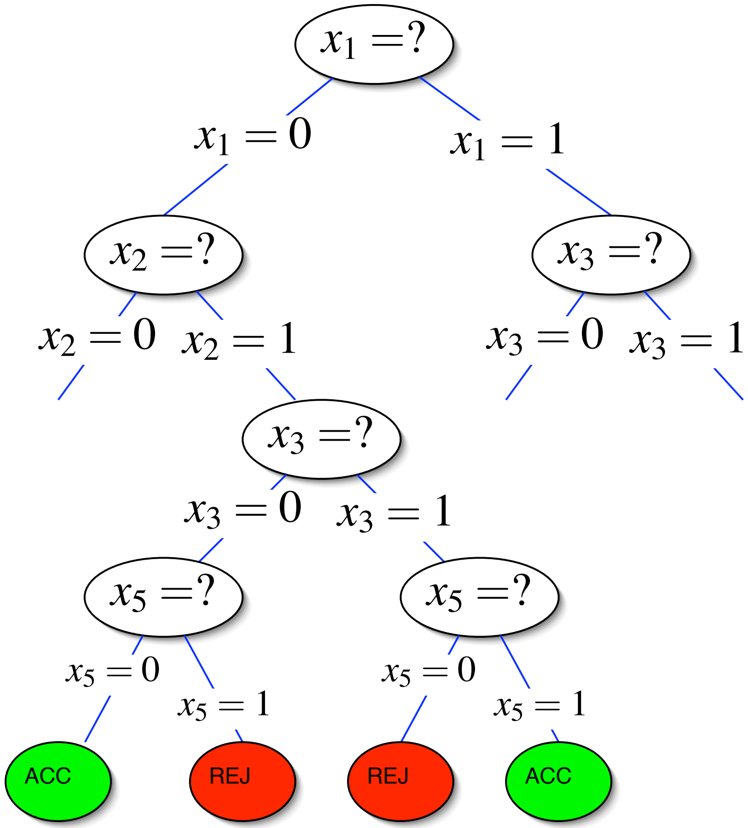
Quantum and randomized query complexity lower bounds using Kolmogorov adversary arguments

Sophie Laplante - *LRI*
Frédéric Magniez - *CNRS LRI*
Université Paris-Sud, Orsay

UNIVERSITÉ PARIS-SUD 11

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Query complexity: classical model

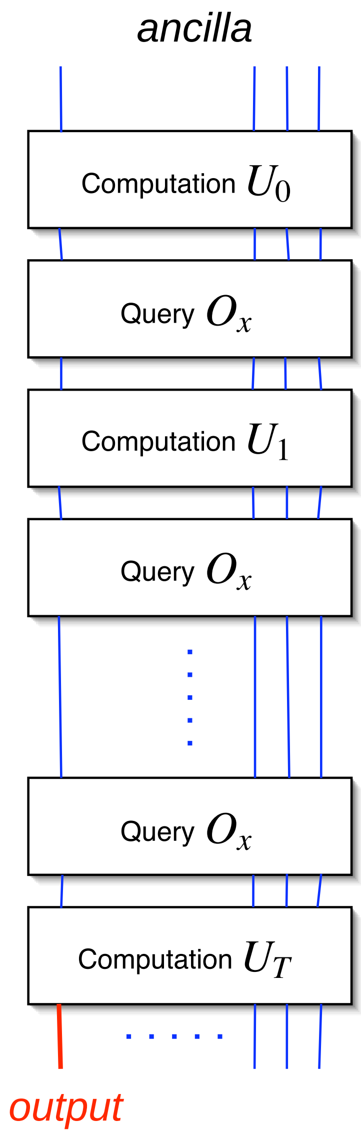


For any boolean function *f*:

- Goal compute *f(x)*
- Cost worst case number of queries to bits of *x*

DT(f) = depth of the shallowest decision tree that computes *f*

Query complexity: quantum model

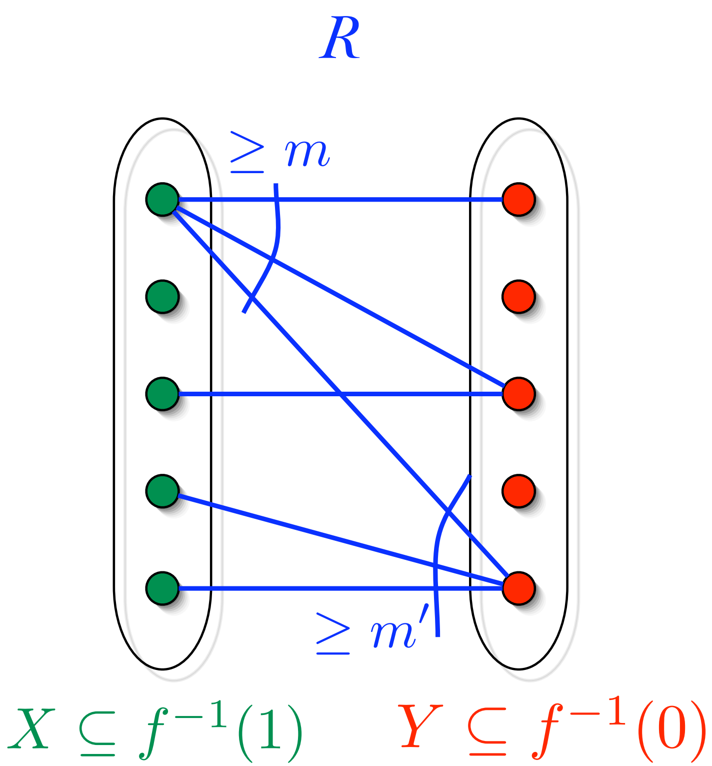


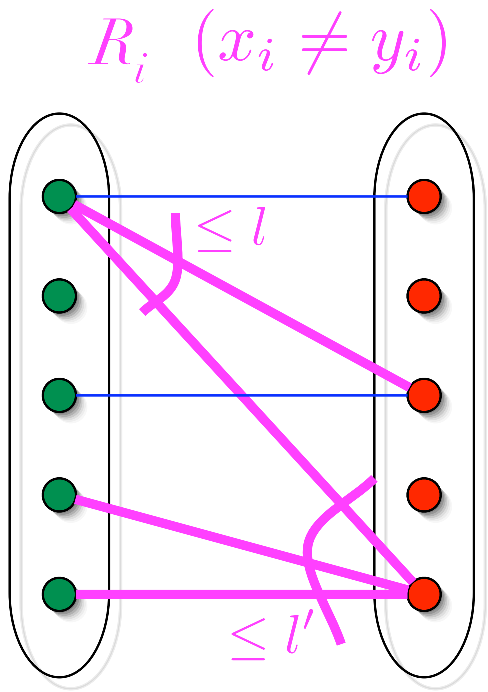
- Queries : unitary transformation O_x that maps $|i,b\rangle$ to $|i, b \oplus x_i\rangle$ (identity on remaining qubits)
- Computation :

$$|\psi_T\rangle = U_T O_x \cdots O_x U_0 |0\rangle$$

- Output : Measure first qubit of $|\psi_T\rangle$
- Error probability bounded by 1/3
- Same model with stochastic matrices for randomized query complexity

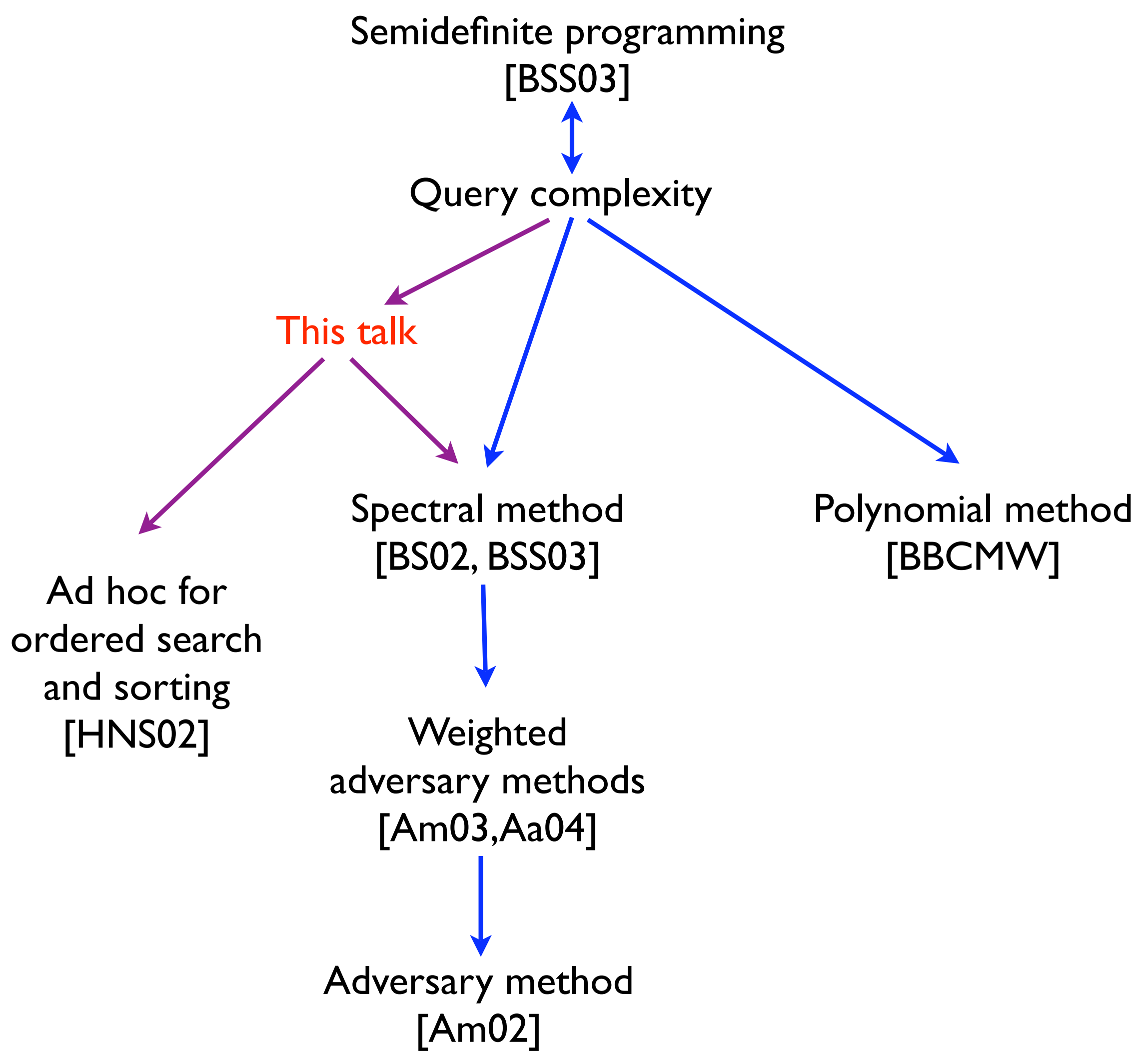
Ambainis' unweighted adversary method





$$\boxed{\mathrm{QQC} \geq \Omega\left(\sqrt{\frac{mm'}{ll'}}\right)}$$

Quantum lower bound techniques



Kolmogorov complexity

Defn $K(x|y)$ is the length of the shortest program that prints x, given as input the string y

For any finite set of strings A,

- $\forall x \in A \;\; K(x|A) \leq \log(|A|)$
- $\exists x \in A \;\; K(x|A) \geq \log(|A|)$
  - Such a string x is called *incompressible* with respect to A.
  - Incompressible strings are "typical", behave like strings taken at random from A.

Why use Kolmogorov complexity?

- Captures intuition of "not enough information to carry out computational task".
- Similar to, but often easier to apply than:
  - Information theoretic techniques,
  - Probabilistic method.

Simple case: decision tree complexity

For any function $f : \{0,1\}^n \to D$ and any inputs x, y such that $f(x) \neq f(y)$, if *T* decides *f*, its deterministic decision tree complexity is

$$DT(f) \geq \min_{i : x_i \neq y_i} \{\max\{2^{K(i|x,T)}, 2^{K(i|y,T)}\}\}$$

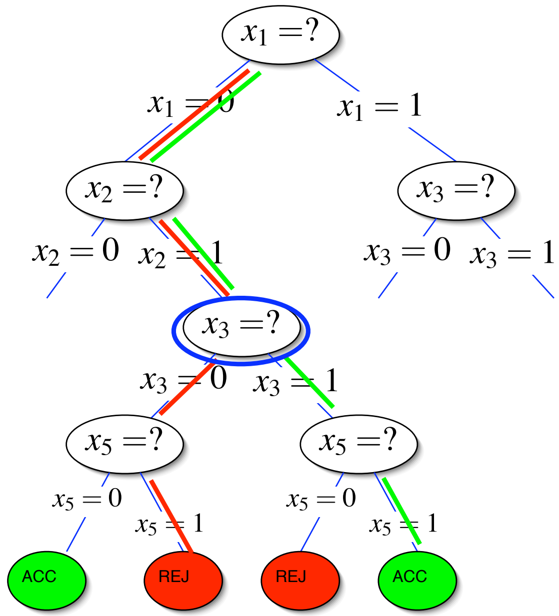
Claim:

$\exists i : x_i \neq y_i \quad (i = 3)$

$$K(i|x,T) \leq \log(DT(f))$$

$$K(i|y,T) \leq \log(DT(f))$$

$$x = 01011$$
$$y = 01101$$



Main Theorem

For any function $f : \{0,1\}^n \rightarrow D$

and any inputs x, y such that $f(x) \neq f(y)$

if A decides *f*,

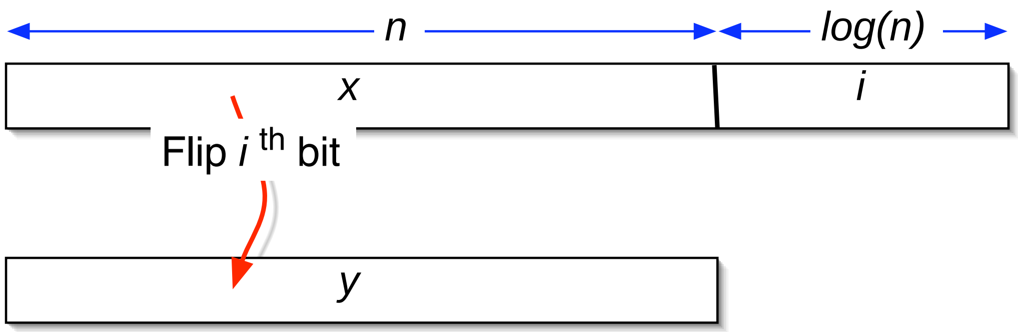
- The quantum query complexity is

$$\mathrm{QQC} \geq \Omega\left(\frac{1}{\sum_{i:x_i \neq y_i} \sqrt{2^{-K(i|x,A)-K(i|y,A)}}}\right)$$

- The randomized query complexity is

$$\mathrm{RQC} \geq \Omega\left(\frac{1}{\sum_{i:x_i \neq y_i} \min\{2^{-K(i|x,A)}, 2^{-K(i|y,A)}\}}\right)$$

Example: Lower bound for parity



Pick an incompressible string of length $n + \log(n)$

- $K(i|x) \geq log(n)$
- $n + \log(n) \leq K(i,x) \leq K(i,y)$, so
- $K(i|y) \geq \log(n)$

Therefore

$$\text{QQC(Parity)} \geq \Omega\left(\sqrt{2^{2\log(n)}}\right) = \Omega\left(n\right)$$

Sketch of proof (1/2)

1. Model-dependent part

- Quantum case:

$$2T \sum_{i:x_i \neq y_i} \sqrt{\overline{p}^x(i)\overline{p}^y(i)} \geq \Omega(1)$$

- Randomized case:

$$2T \sum_{i:x_i \neq y_i} \min\left(\overline{p}^x(i), \overline{p}^y(i)\right) \geq \Omega(1)$$

$p_t^x(i)$ = *probability of querying* i *at step* t *on input* x.

$$\overline{p}^x(i) = \frac{1}{T}\sum_t p_t^x(i)$$

Sketch of proof (2/2)

2. Model-independent part

Using the Shannon-Fano code for the probability distribution on queries,

$$K(i|x, A) \leq \log\left(\frac{1}{p^x(i)}\right)$$

Therefore,

- *Quantum case:*

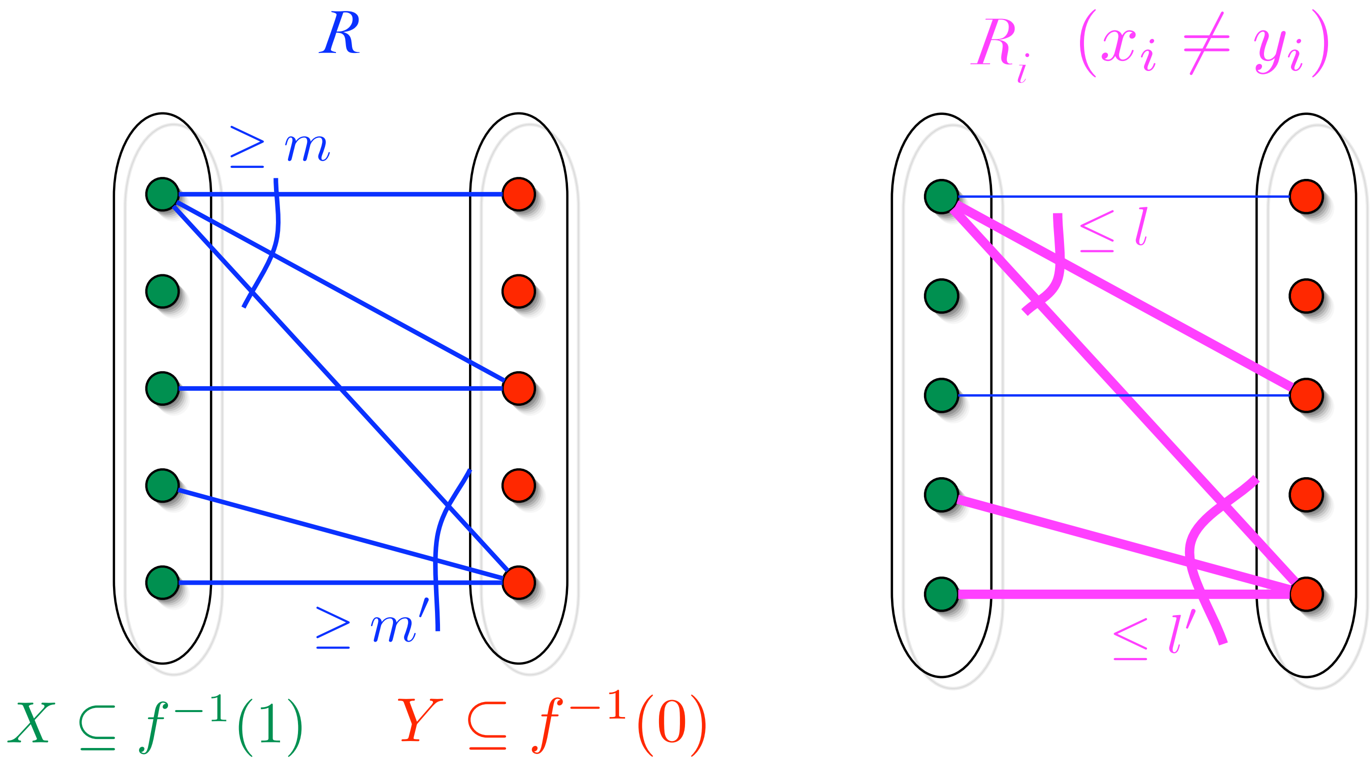
$$2T \sum_{i:x_i \neq y_i} \sqrt{2^{-K(i|x,A)-K(i|x,A)}} \geq \Omega(1)$$

- *Randomized case:*

$$2T \sum_{i:x_i \neq y_i} \min\{2^{-K(i|x,A)}, 2^{-K(i|y,A)}\} \geq \Omega(1)$$

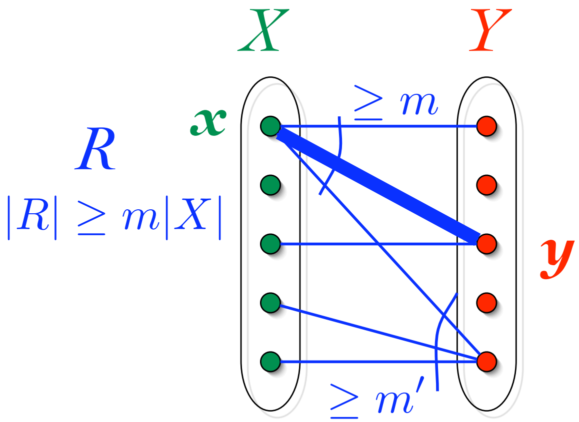
QED

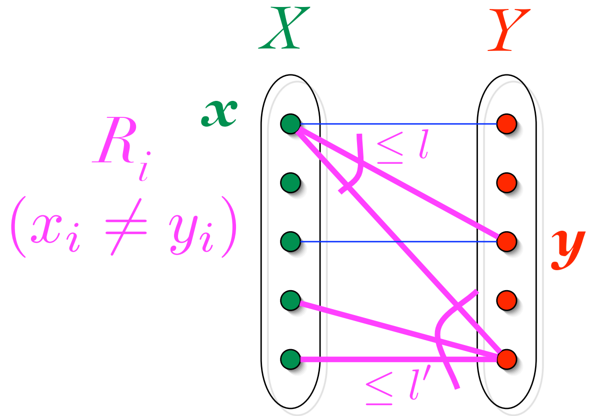
Ambainis' unweighted adversary method



$$\boxed{\mathrm{QQC} \geq \Omega\left(\sqrt{\frac{mm'}{ll'}}\right)}$$

Main theorem implies unweighted adversary method





$$\boxed{K(i|x) \geq K(x,y) - K(x) - K(y|i,x) + K(i|x,y,K(x,y))}$$

$$\exists x,y\; K(x,y) \geq \log|R| \geq \log m + \log|X|$$

$$\forall x\;\; K(x) \leq \log|X|$$

$$\forall x,y,i\; K(y|x,i) \leq \log l$$

$$\geq \log\left(\frac{m}{l}\right) + K(i|x,y,K(x,y))$$

Main theorem implies unweighted adversary method

We have shown $\exists x, y\ \forall i \text{ s.t. } x_i \neq y_i$

$$
\begin{aligned}
K(i|x) &\geq \log(\frac{m}{l}) + K(i|x,y,K(x,y)) \\
K(i|y) &\geq \log(\frac{m'}{l'}) + K(i|x,y,K(x,y))
\end{aligned}
$$

Recall the general theorem:

$$\text{QQC} \geq \Omega\left(\frac{1}{\sum_{i:x_i\neq y_i}\sqrt{2^{-K(i|x)-K(i|y)}}}\right)$$

and apply Kraft's inequality:

$$\boxed{\text{QQC} \geq \Omega\left(\sqrt{\frac{mm'}{ll'}}\right)}$$

Certificate complexity

- A *b-certificate of size m for f* is a partial assignment of *m* bits of the input, which forces the value of a function *f* to *b* (*b=0,1*).

  0-certificate: $f(*0*11**0*0***) = 0$

  1-certificate: $f(*0*1***1*0*0*) = 1$

- The *b-certificate complexity, $C_b(f)$,* is the size of the largest minimal *b*-certificate for *f*.
  - Bipartiteness: an odd cycle is a 0-certificate.
  - Connectivity: a spanning tree is a 1-certificate.

Limits of adversary methods

[Troy Lee] Consider *x,y* with *f(x)=0* and *f(y)=1*.

0-certificate consistent with x:

$$f(*0*11**0*0***) = 0$$

$$f(y) = 1$$

So there exists *i* with $x_i \neq y_i$, such that

$$K(i|x) \leq \log(C_0(f))$$

Similarly, there exists *j* with $x_j \neq y_j$

$$K(j|y) \leq \log(C_1(f))$$

$$\frac{1}{\sum_{i:x_i \neq y_i} \sqrt{2^{-K(i|x,A)-K(i|y,A)}}} \leq \sqrt{nC_0(f)}, \sqrt{nC_1(f)}$$

(Indep. S. Zhang, for weighted method, ICALP 2004)

Limits of adversary methods

R. Špalek

[Troy Lee] For any total function *f*, and inputs *x,y* with *f(x)=0* and *f(y)=1*, there exist *i*, ~~*j*~~ with $x_i = y_i$, ~~$x_j = y_j$~~

$$K(i|x) \leq \log(C_0(f))$$

$$K(\cancel{j}^{\,i}|y) \leq \log(C_1(f))$$

$$\cancel{\frac{1}{\sum_{i:x_i \neq y_i} \sqrt{2^{-K(i|x,A)-K(i|y,A)}}} \leq \sqrt{nC_0(f)}, \sqrt{nC_1(f)}}$$

$$\frac{1}{\sum_{i:x_i \neq y_i} \sqrt{2^{-K(i|x,A)-K(i|y,A)}}} \leq \sqrt{C_0(f)C_1(f)}$$

Summary of results

- New framework to prove lower bounds in query complexity.
- Unified proofs for quantum and randomized lower bounds.
- Generalizes previous adversary methods.
- Applies to boolean as well as non-boolean functions.
- Easy-to-prove limits of adversary methods in terms of certificate complexity.

Directions for future work

- Lower bounds for bounded rounds (adaptive vs nonadaptive queries).
- Similar techniques involving $K(i|x)$ may apply to other models, such as communication complexity, time/space tradeoffs.
- Quantum Kolmogorov complexity might be necessary to handle these models.