Ann. Comb. 16 (2012) 57–75
DOI 10.1007/s00026-011-0121-6
Published online December 22, 2011

Annals of Combinatorics

Polyominoes Determined by Permutations: Enumeration via Bijections

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Received November 11, 2009

Mathematics Subject Classification: 05A15, 05A19

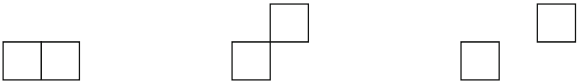
Abstract. A *permutominide* is a set of cells in the plane satisfying special connectivity constraints and uniquely defined by a pair of permutations. It naturally generalizes the concept of *permutomino*, recently investigated by several authors and from different points of view [1, 2, 4, 6, 7]. In this paper, using bijective methods, we determine the enumeration of various classes of convex permutominides, including, parallelogram, directed convex, convex, and row convex permutominides. As a corollary we have a bijective proof for the number of convex permutominoes, which was still an open problem.

Keywords: polyominoes enumeration, permutations, permutominoes

1. Polyominides and Permutominides

We assume that the reader is confident with the concept of *polyomino*, and with the most important classes of polyominoes, such as the *parallelogram*, the *convex*, the *directed convex*, the *row convex* polyominoes. For the main definitions concerning these objects we refer to [3].

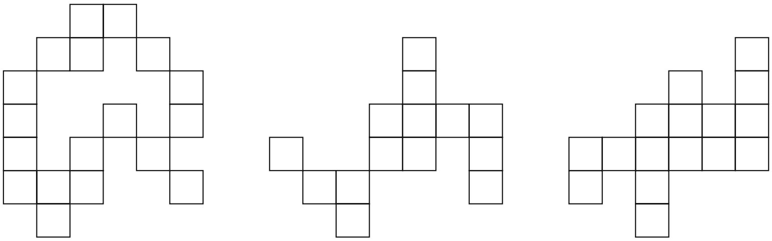
We start by giving a simple generalization of the concept of a polyomino admitting the connection of cells not only by edges but also by vertices. In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square having integer coordinates. Two cells are said to be *edge-connected* if they have a common edge (see Figure 1 (a)), and *vertex-connected* if they have a common vertex (see Figure 1 (b)).



(a) (b) (c)

Figure 1: (a) cells connected by edges; (b) cells connected by vertices; (c) disconnected cells.

A *polyominide* is a finite union of cells such that any two of them can be connected by means of a sequence of cells which are vertex-connected or edge-connected. It is obvious that, requiring the connection only by edges, we obtain the polyominoes. Polyominides are defined up to a translation. In this paper we will always consider those having no “holes”, i.e., polyominides where the boundary is made exactly of one component. A polyominide is said to be *row convex* (*column convex*, respectively) if, for any two cells with the same ordinate (abscissa, respectively), the row (column, respectively) containing them is connected, see Figure 2. Finally it is said to be *convex* if it is both row and column convex.



(a) (b) (c)

Figure 2: (a) A polyominide with a hole; (b) a column convex (not row convex) polyominide; (c) a column convex polyominide which is also a polyomino.

A cell of a polyominide is called an *internal* cell, if it is surrounded by 8 cells, otherwise it is called an *external* cell. To any vertex V of an external cell we assign a *multiplicity* $\mu(V)$, which is given by the number of cells to which V belongs. An *extremal vertex* of a polyominide is simply a vertex of one of its cells with multiplicity 1 or 3, see Figure 3. A *side* of a polyominide is any horizontal or vertical segment of its boundary joining two of its extremal vertices, as depicted in Figure 3.

Let us now define the combinatorial object which will be treated in this paper. A *permutominide* is a polyominide having exactly one side for every abscissa and exactly one side for every ordinate. It is easy to check that the minimal bounding rectangle containing a permutominide is always a square, and the side of such square is called the *size* of the permutominide.

The name *permutominide* is due to the fact that each of these objects is actually

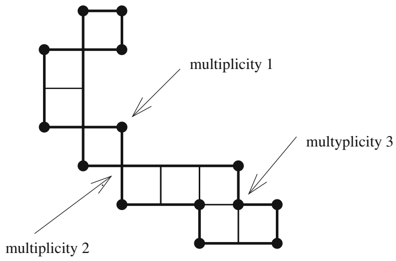


Figure 3: Vertices and sides of a polyominide. Extremal vertices are highlighted.

defined by a pair of permutations, as graphically depicted in Figure 4. The formal definition of this pair of permutations is quite laborious, and we prefer to omit it, since it goes beyond the scopes of the present paper. The interested reader can find all details on this subject in [4]. Permutominides which are also polyominoes are

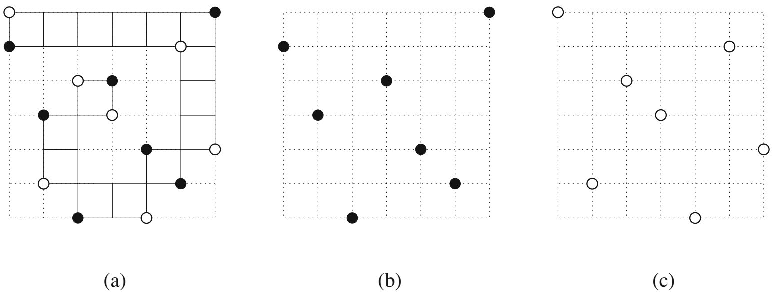


Figure 4: (a) a permutominide P of size 6; (b) its associated permutations $\pi_1(P) = (6,4,1,5,3,2,7)$, and (c) $\pi_2(P) = (7,2,5,4,1,6,3)$.

usually called *permutominoes*. Figure 5 depicts some important classes of permutominoes. Permutominoes were introduced in [7], and then considered by Incitti in [6] while studying the problem of determining the $\widetilde{R}$-polynomials associated with a pair (π_1, π_2) of permutations. Then permutominoes have been exhaustively studied regarding enumeration and related combinatorial problems in [1,2,4].

The main enumerative results about convex permutominoes are the following:

i. the number of *parallelogram permutominoes* of size n is equal to the n-th *Catalan number*,

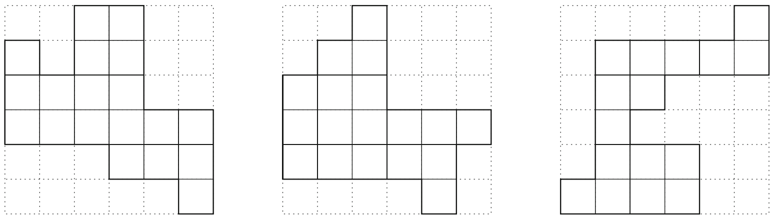
$$C_n = \frac{1}{n+1}\binom{2n}{n};$$

ii. the number of *directed convex permutominoes* of size n is equal to $\binom{2n-1}{n}$;

iii. the number of *convex permutominoes* of size n is

$$2(n+3)4^{n-2} - \frac{n}{2}\binom{2n}{n}, \quad n \geq 1. \tag{1.1}$$

We point out that formula (1.1) was proved using analytical techniques, and the problem of giving a bijective proof to (1.1) was pointed out independently in [2, 4, 5]. Moreover, the problem of enumerating row convex permutominoes is still open.



(a) (b) (c)

Figure 5: (a) a permutomino; (b) a convex permutomino; (c) a row convex permutomino.

In this paper we study some combinatorial and enumerative properties of various classes of convex permutominides. In the first part we give the enumeration of convex, directed convex, and parallelogram permutominides, using bijective techniques. Moreover, we show that both directed convex and convex permutominides have a rational generating function, while — as reported above — directed convex and convex permutominoes have an algebraic (non rational) generating function.

A relevant justification for our study of convex permutominides is presented in the second part of the paper. In fact, convex permutominoes can be obtained simply by subtracting, from the convex permutominides, those whose boundary crosses itself. As a consequence of our approach we are able to enumerate the latter class, thus we provide the first bijective proof for (1.1).

Finally, we obtain the enumeration of row convex permutominides according to their size. To do this we show a method to associate a permutation (the *base permutation*) to a given permutominide, and conversely, we show how to obtain a set of permutominides from a given permutation.

The table below reports the first terms of the number sequences treated in the paper, and the main enumeration results obtained here.

class of permutominides	1	2	3	4	5	6	7	**closed form**
parallelogram	1	3	10	35	126	426	...	$\binom{2n-1}{n}$
directed-convex	1	4	16	64	256	1024	...	4^{n-1}
convex	1	6	32	160	768	3584	...	$2(n+1)4^{n-2}$
row-convex	1	6	48	480	5768	80640	...	$(n+1)!\,2^{n-2}$

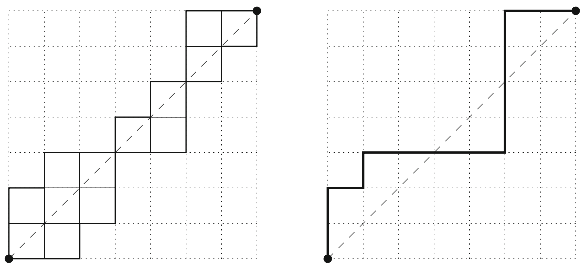
2. Enumeration of Convex Permutominides

In this section we provide the enumeration of some classes of convex permutominides using bijective techniques. To begin with, we agree without loss of generality that the lower leftmost vertex of the minimal bounding square of any permutominide is placed in the origin $(0,0)$. We recall from [1, 2] that, given a convex permutominide, in order to satisfy the convexity constraint, the points where its boundary crosses itself (if any) must lie all on the main diagonal or all on the antidiagonal (see, for instance, Figure 7).

Let us start by defining some classes of permutominides, which extend the classes of directed convex and of parallelogram polyominoes, respectively. A convex permutominide of size n is said to be *directed* if it contains a cell with lowest leftmost vertex in position $(0,0)$ (see Figure 7 (a)). A directed convex permutominide is said to be *parallelogram* if it contains a cell with upper rightmost vertex in position (n,n) (see Figure 6 (a)).

Proposition 2.1. *The number of parallelogram permutominides of size n is* $\binom{2n-1}{n}$, $n \geq 1$.

Proof. Let P be a parallelogram permutominide of size n; let $p(P)$ be the path made of the sequence of sides of P, starting from $(0,0)$ upwards, and following the boundary of P until it reaches (n,n) (see Figure 6). Clearly, a parallelogram permutominide P is uniquely determined by the path $p(P)$. Moreover, $p(P)$ is an unrestricted path of length $2n$ made of *north* and *east* unit steps (denoted by N and E, respectively) always starting with an N step. We remark that the ending step of $p(P)$ may be an E or an N step depending on the number of times the boundary crosses itself. So, the number of such paths is $\binom{2n-1}{n}$. ∎



(a) (b)

Figure 6: (a) a parallelogram permutominide P; (b) the associated path $p(P)$.

Proposition 2.2. *The number of directed convex permutominides of size n is* 4^{n-1}, $n \geq 1$.

Proof. In a directed convex permutominide P of size n let us identify the following points:

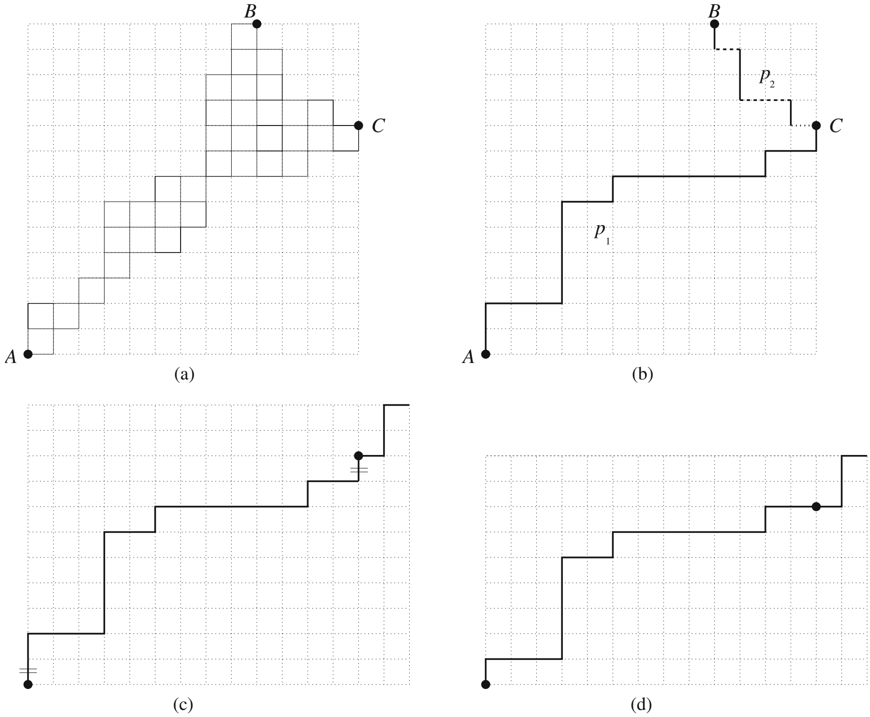


Figure 7: (a) Case (i): A directed convex permutominide P; (b) the associated paths p_1 and p_2, where the steps to be removed are dotted; (c) the path $p(P)$; (d) the path $\hat{p}(P)$ is obtained by removing from $p(P)$ the highlighted steps.

- A is the origin $(0,0)$;
- B is the rightmost point among those having ordinate n;
- C is the highest point among those having abscissa n.

Clearly, B and C coincide if and only if P is a parallelogram permutominide.

Our aim is to build a path $p(P)$ encoding P, similarly to what we did in Proposition 2.1. We start building a path denoted by $p_1(P)$ (briefly, p_1); such a path is constituted by the sequence of sides of P starting from A with an N step and following the boundary of P. The path p_1 may end

(i) with a north step, on the line $x = n$ (precisely, at C), if it crosses the main diagonal an odd number of times (see Figure 7 (a) (b));

(ii) with an east step, on the line $y = n$ (precisely, at B), if it crosses the main diagonal an even number of times (see Figure 8 (a) (b)).

Let us consider the two cases separately, and define the path $p_2(P)$ (briefly, p_2).

(i) We observe that p_1 starts and ends with a north step, hence we may write $p_1 = N\hat{p}_1N$. If B and C coincide, then p_2 is the empty path. Otherwise, $p_2(P)$ is made of the part of the boundary of P running from C to B counterclockwise (see Figure 7 (b)). For our purpose, p_2 can be uniquely decomposed as a sequence

$$p_2 = W^{r_1} N^{s_1} W^{r_2} N^{s_2} \cdots W^{r_{2\ell+1}} N^{s_{2\ell+1}} W^{r_{2\ell+2}} N^{s_{2\ell+2}} \cdots ,$$

where N (W, respectively) denotes a unit north (west, respectively) step and, for each i, $r_i, s_i \geq 1$, and $\ell \geq 0$.

The path $p_1 p_2$ uses north, east, and west unit steps, and one can easily see that it completely encodes the permutominide P. To conclude our proof, we would like now to encode the path $p_1 p_2$ as a word of length $2(n-1)$ in the alphabet $\{E, N\}$. We start by pointing out that the path p_1 determines the abscissas of the vertical sides in p_2, so $p_1 p_2$ can be obtained uniquely from the knowledge of p_1 and of the lengths of the vertical sides in p_2. So we map $p_1 p_2$ into the path $p(P)$, of length $2n$:

$$p(P) = N \hat{p}_1 N E^{s_1} N^{s_2} \cdots E^{s_{2\ell+1}} N^{s_{2\ell+2}} \cdots ,$$

as depicted in Figure 7 (c). In practice, the ith sequence of north steps in p_2, denoted by N^{s_i}, is mapped into E^{s_i}, if i is odd, and into N^{s_i} otherwise. Now we obtain the path $\hat{p}(P)$ from $p(P)$, by removing the first and the last steps from p_1 (as depicted in Figure 7 (d)):

$$\hat{p}(P) = \hat{p}_1 E^{s_1} N^{s_2} \cdots E^{s_{2\ell+1}} N^{s_{2\ell+2}} \cdots .$$

(ii) It is completely analogous to the previous one. In this case, p_1 starts with a north step and ends with an east step, hence $p_1 = N \hat{p}_1 E$. The path $p_2(P)$ is the empty path, if B and C coincide, otherwise it is made of the part of the boundary of P running from B to C, and then it can be written as

$$p_2 = S^{r_1} E^{s_1} S^{r_2} E^{s_2} \cdots S^{r_{2\ell+1}} E^{s_{2\ell+1}} S^{r_{2\ell+2}} E^{s_{2\ell+2}} \cdots ,$$

where S (E, respectively) denotes a unit south (east, respectively) step and, for each i, $r_i, s_i \geq 1$. Now, as before, we map $p_1 p_2$ into the path

$$p(P) = N \hat{p}_1 E N^{s_1} E^{s_2} \cdots N^{s_{2\ell+1}} E^{s_{2\ell+2}} \cdots ,$$

as depicted in Figure 8 (c). Now we obtain the path $\hat{p}(P)$ from $p(P)$, by removing the first and the last steps from p_1:

$$\hat{p}(P) = \hat{p}_1 N^{s_1} E^{s_2} \cdots N^{s_{2\ell+1}} E^{s_{2\ell+2}} \cdots .$$

An easy computation reveals that the path $\hat{p}(P)$ has length $2(n-1)$.

To prove that the correspondence between P and $\hat{p}(P)$ is a bijection, we briefly show how to determine p_1 and p_2 from $\hat{p}(P)$. If $\hat{p}(P)$ ends with abscissa greater than or equal to n, then we fall in Case (i), and the path $\hat{p}_1$ is given by the longest prefix of $\hat{p}(P)$ ending with abscissa equal to n. Otherwise, we fall in Case (ii), and then the path $\hat{p}_1$ is given by the longest prefix of $\hat{p}(P)$ ending with ordinate equal to $n-1$.

Finally, we have proved that every directed convex permutominide of size n can be encoded as a word of length $2(n-1)$ in the alphabet $\{E, N\}$. ∎

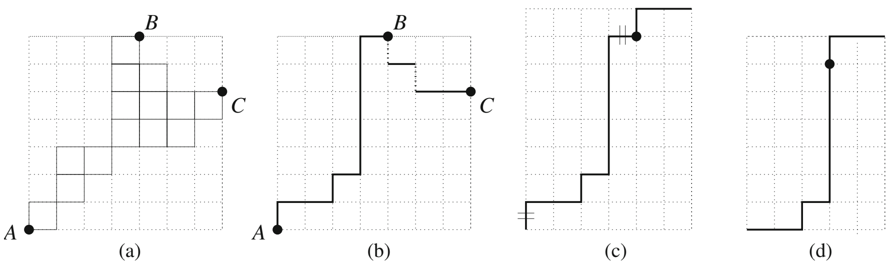


Figure 8: (a) Case (ii): A directed convex permutominide P; (b) the associated paths p_1 and p_2, where the steps to be removed are dotted; (c) the path $p(P)$; (d) the path $\hat{p}(P)$, obtained from $p(P)$ by removing the highlighted steps.

By extending the previous bijection we obtain the following remarkable result.

Proposition 2.3. *The number of convex permutominides of size n is* $2(n+1)4^{n-2}$, $n \geq 1$.

Proof. The number of directed convex permutominides was determined in Proposition 2.2, so it is sufficient to count the convex permutominides which are not directed convex. Proving that the number of these permutominides with size n is equal to $(n-1)2^{2n-3}$ leads to the thesis.

So, let P be a convex — but not directed convex — permutominide of size n. The main idea is to define a mapping $P \rightarrow (\hat{p}(P), h)$, where $\hat{p}(P)$ is an unrestricted path of length $2n-3$ using north and east unit steps, and h is a label belonging to the set $\{1, \ldots, n-1\}$.

Let us consider the following points on the boundary of P (see Figure 9 (a)):

- $A = (0, h)$ is the lower leftmost vertex of P; clearly, $1 \leq h \leq n-1$; in our construction, h is just the label of the path $\hat{p}(P)$ we are going to define;
- B is the rightmost point among those having ordinate n;
- C is the highest point among those having abscissa n;
- D is the leftmost point among those having ordinate 0.

We point out that B and C may coincide, while A and D are necessarily distinct, since the permutominide P is assumed not to be directed. Now, three distinct cases may occur:

(1) the boundary of P crosses itself on the main diagonal (Figure 9 (a));

(2) the boundary of P crosses itself on the anti-diagonal (Figure 11 (a));

(3) the boundary of P does not cross itself, i.e., P is a convex permutomino.

Due to the convexity constraint, a permutominide cannot satisfy both conditions (1) and (2), so we study cases (1) and (2) separately, and for each of the two cases we map P onto a pair $(\hat{p}(P), h)$, as previously explained.

(1) The construction resembles the one described in Proposition 2.2. We start by considering the path $p_1(P)$ (briefly, p_1), given by the sequence of sides of P starting from A with an N step and following the boundary of P (see Figure 9 (b)). Again, p_1 may terminate on $x = n$ or on $y = n$, so we consider two subcases separately:

(1.1) p_1 terminates with a north step, i.e., on the line $x = n$, if it crosses the main diagonal an odd number of times (see Figure 9);
(1.2) p_1 terminates with an east step, i.e., on the line $y = n$, if it crosses the main diagonal an even number of times (see Figure 10).

These two cases have to be studied separately, but as it happened in the proof of Proposition 2.2, they are completely analogous.

(1.1) The path p_1 starts and ends with a north step, hence we may write $p_1 = N\hat{p}_1N$. The path $p_2(P)$ (briefly, p_2) is empty if B and C coincide, otherwise it is made of the part of the boundary of P running counterclockwise from C to B (see Figure 9 (b)). Then p_2 can be uniquely decomposed as

$$p_2 = W^{r_1}N^{s_1}W^{r_2}N^{s_2}\cdots W^{r_{2\ell+1}}N^{s_{2\ell+1}}W^{r_{2\ell+2}}N^{s_{2\ell+2}}\cdots W^{r_t}N^{s_t},$$

where, for each i, $r_i, s_i \geq 1$, and $t \geq 1$. In practice, r_i (s_i, respectively) is the length of the ith sequence of west (north, respectively) steps in p_2.

Similarly, the path $p_3(P)$ (briefly, p_3) is made of the part of the boundary of P running from D to A (see Figure 9 (b)). Now, p_3 can be uniquely decomposed as

$$p_3 = N^{k_1}W^{j_1}\cdots N^{k_{q-(2\ell+1)}}W^{j_{q-(2\ell+1)}}N^{k_{q-2\ell}}W^{j_{q-2\ell}}\cdots N^{k_{q-1}}W^{j_{q-1}}N^{k_q}W^{j_q},$$

where, for each i, $k_i, j_i \geq 1$, $q \geq 1$, i.e., k_i (j_i, respectively) is the length of the ith sequence of north (west, respectively) steps in p_3.

One can easily observe that the path obtained by the concatenation of the previously defined paths, namely, $p_3p_1p_2$ completely encodes the permutominide P. To conclude our proof, we encode the path $p_3p_1p_2$ as a pair $(\hat{p}(P), h)$, where $\hat{p}(P)$ is a word of length $2n-3$ in the alphabet $\{N, E\}$, and h is just the label of $\hat{p}(P)$.

We start by pointing out that the path p_1 determines the abscissas of the vertical sides in p_2 and in p_3, so $p_3p_1p_2$ can be obtained uniquely from the knowledge of p_1, and of the lengths of the vertical sides in p_2 and in p_3. Thus we map $p_3p_1p_2$ into the path

$$p(P) = \left(X^{k_1}\cdots N^{k_{q-(2\ell+1)}}E^{k_{q-2\ell}}\cdots N^{k_{q-1}}E^{k_q}\right)N\hat{p}_1N$$
$$\left(E^{s_1}N^{s_2}\cdots E^{s_{2\ell+1}}N^{s_{2\ell+2}}\cdots Y^{s_t}\right),$$

as depicted in Figure 9 (c). We remark that $p(P)$ starts with a sequence of east steps (i.e., $X = E$), if and only if q is odd, otherwise it starts with a

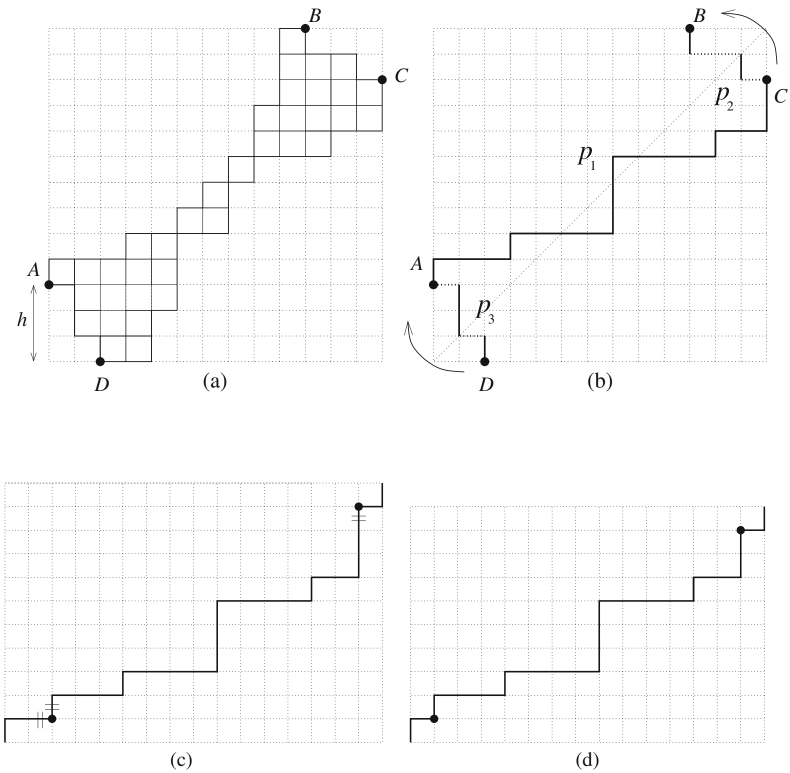


Figure 9: (a) A convex permutominide P satisfying condition (1.1); (b) the associated path $p_3 p_1 p_2$, where the steps to be removed are dotted; (c) the path $p(P)$; (d) the path $\hat{p}(P)$, obtained from $p(P)$ by removing the highlighted steps, with label h.

sequence of north steps (i.e., $X = N$). Similarly, $p(P)$ ends with a sequence of north steps (i.e., $Y = N$), if and only if t is even, otherwise it ends with a sequence of east steps (i.e., $Y = E$).

Now $p(P)$ is a path in $\{E, N\}$ of length $2n$, and we obtain the path $\hat{p}(P)$ from $p(P)$, by removing three steps: The first and the last steps from p_1, and the step preceding p_1, which is always an east step (see Figure 9 (d)). Then we have

$$\hat{p}(P) = \left(X^{k_1} \cdots N^{k_{q-(2\ell+1)}} E^{k_{q-2\ell}} \cdots N^{k_{q-1}} E^{k_q - 1} \right) \hat{p}_1$$

$$\left(E^{s_1} N^{s_2} \cdots E^{s_{2\ell+1}} N^{s_{2\ell+2}} \cdots Y^{s_t} \right).$$

(1.2) The construction is analogous to that explained in (1.1). Here p_1 starts with a north step and ends with an east step, hence we may write $p_1 = N \hat{p}_1 E$.

The path $p_2(P)$ (briefly, p_2) is empty if B and C coincide, otherwise it is made up of the part of the boundary of P running clockwise from B to C (see Figure 10 (b)). Then p_2 can be uniquely decomposed as

$$p_2 = S^{r_1}E^{s_1}S^{r_2}E^{s_2}\cdots S^{r_{2\ell+1}}E^{s_{2\ell+1}}S^{r_{2\ell+2}}E^{s_{2\ell+2}}\cdots S^{r_t}E^{s_t},$$

where, for each i, r_i, $s_i \geq 1$, i.e., r_i (s_i, respectively) is the length of the ith sequence of south (east, respectively) steps in p_2.
The path $p_3(P)$ (briefly, p_3) is defined in the same way as in (1.1), and similarly, we can write

$$p_3 = N^{k_1}W^{j_1}\cdots N^{k_{q-(2\ell+1)}}W^{j_{q-(2\ell+1)}}N^{k_{q-2\ell}}W^{j_{q-2\ell}}\cdots N^{k_{q-1}}W^{j_{q-1}}N^{k_q}W^{j_q}.$$

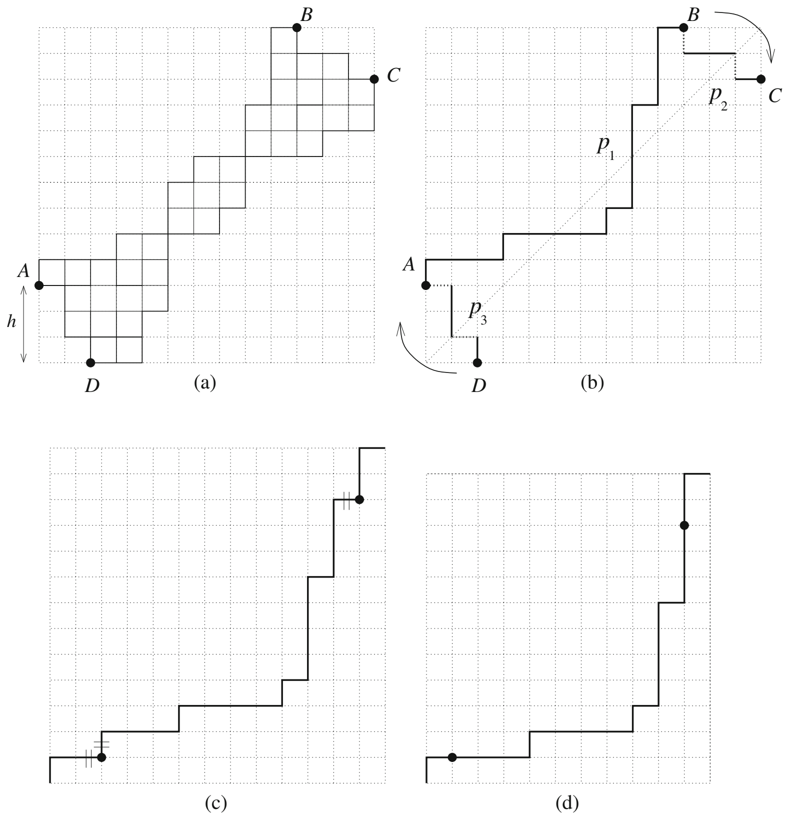


Figure 10: (a) A convex permutominide P satisfying condition (1.2); (b) the associated path $p_3p_1p_2$, where the steps to be removed are dotted; (c) the path $p(P)$; (d) the path $\hat{p}(P)$, obtained from $p(P)$ by removing the highlighted steps, with label h.

Again, we encode the path $p_3p_1p_2$ in terms of a pair $(\hat{p}(P), h)$. Here, the path p_1 determines the ordinates of the horizontal sides in p_2, and the abscissas of the vertical sides in p_3, so $p_3p_1p_2$ can be obtained uniquely from the knowledge of p_1, of the length of the horizontal sides in p_2, and of the length of the vertical sides in p_3. Thus we map $p_3p_1p_2$ into the path

$$p(P) = \left(X^{k_1} \cdots N^{k_{q-(2\ell+1)}} E^{k_{q-2\ell}} \cdots N^{k_{q-1}} E^{q_t}\right) N\hat{p}_1 E$$

$$\left(N^{s_1} E^{s_2} \cdots N^{s_{2\ell+1}} E^{s_{2\ell+2}} \cdots Y^{s_t}\right),$$

as depicted in Figure 10 (c). We remark that $p(P)$ starts with a sequence of east steps (i.e., $X = E$), if and only if q is odd, otherwise it starts with a sequence of north steps (i.e., $X = N$); moreover, $p(P)$ ends with a sequence of north steps (i.e., $Y = N$), if and only if t is odd, otherwise it ends with a sequence of east steps (i.e., $Y = E$).

Now the path $\hat{p}(P)$ is obtained from $p(P)$, by removing three steps: The first and the last steps from p_1, and the step preceding p_1, which is always an east step (see Figure 10 (d)). Then we have

$$\hat{p}(P) = \left(X^{k_1} \cdots N^{k_{q-(2\ell+1)}} E^{k_{q-2\ell}} \cdots N^{k_{q-1}} E^{k_q - 1}\right) \hat{p}_1$$

$$\left(N^{s_1} E^{s_2} \cdots N^{s_{2\ell+1}} E^{s_{2\ell+2}} \cdots Y^{s_t}\right).$$

(2) Now we consider the case where the boundary of P crosses itself on the anti-diagonal (Figure 11 (a)). Similarly to the previous cases, we build up three paths p_1, p_2, and p_3. The path $p_1(P)$ (briefly, p_1) is, as usual, given by the sequence of sides of P starting from A with an N step and following the boundary of P (see Figure 11 (b)). In this case, due to the convexity constraint, p_1 must terminate on the line $y = n$, then it starts with a north step, and ends with an east step, and we may write $p_1 = N\hat{p}_1 E$. Now we need to identify the subpaths p_2 and p_3.

So, let p_2 be the part of the boundary of P, running clockwise from B to C, and remaining weakly above the anti-diagonal. Such a path is made of south and east unit steps, and we may encode it as

$$p_2 = S^{r_1} E^{s_1} S^{r_2} E^{s_2} \cdots S^{r_{2\ell+1}} E^{s_{2\ell+1}} S^{r_{2\ell+2}} E^{s_{2\ell+2}} \cdots S^{r_t} E^{s_t},$$

where, for each i, $r_i, s_i \geq 1$, i.e., r_i (s_i, respectively) is the length of the ith sequence of south (east, respectively) steps in p_2. Similarly, the path p_3 is the part of the boundary of P, running clockwise from D to A, and remaining weakly below the anti-diagonal. Such a path is made of north and west unit steps, and we may encode it as

$$p_3 = N^{k_1} W^{j_1} \cdots N^{k_{q-(2\ell+1)}} W^{j_{q-(2\ell+1)}} N^{k_{q-2\ell}} W^{j_{q-2\ell}} \cdots N^{k_{q-1}} W^{j_{q-1}} N^{k_q} W^{j_q}.$$

Now, our representation follows the usual scheme, and leads us to encode the path $p_3p_1p_2$ as a pair $(\hat{p}(P), h)$. Clearly, the path $p_3p_1p_2$ completely encodes P. Moreover, $p_3p_1p_2$ can be obtained uniquely from the knowledge of p_1, of the length of the horizontal sides in p_2, and of the lengths of the vertical sides in p_3. Thus we map $p_3p_1p_2$ into the path

$$p(P) = \left(X^{k_1} \cdots N^{k_{q-(2\ell+1)}} E^{k_{q-2\ell}} \cdots N^{k_{q-1}} E^{q_t}\right) N\hat{p}_1 E$$

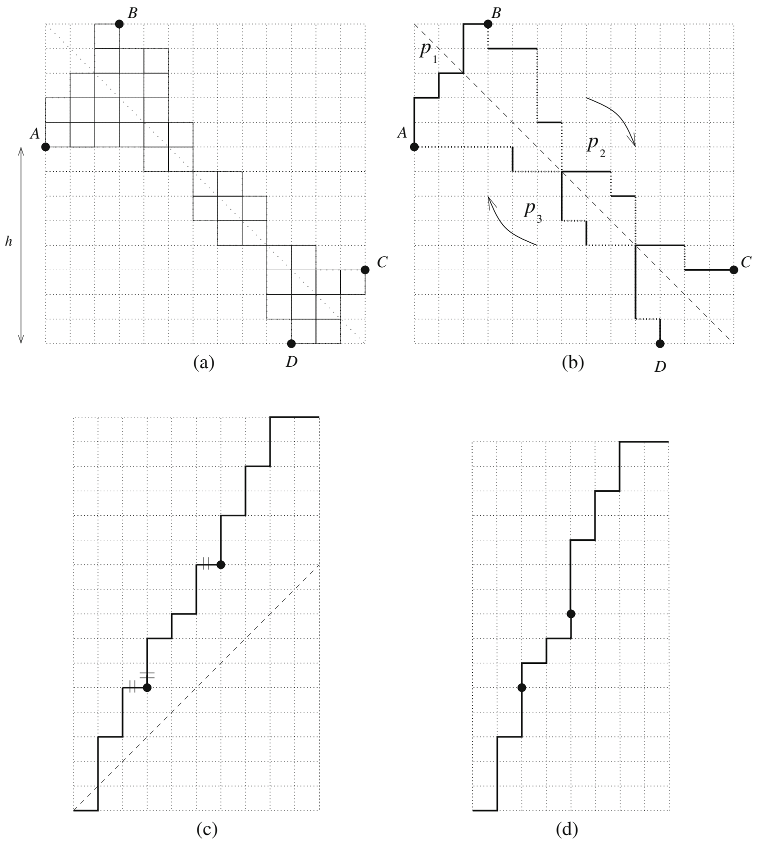


Figure 11: (a) A convex permutominide P satisfying condition (2); (b) the associated path $p(P) = p_1 p_2 p_3$; (c) the path $p(P)$, where the steps to be removed are emphasized; (d) the path $\hat{p}(P)$, labelled h.

$$\left(N^{s_1} E^{s_2} \cdots N^{s_{2\ell+1}} E^{s_{2\ell+2}} \cdots Y^{s_t}\right),$$

as depicted in Figure 11 (c). We remark that $p(P)$ starts with a sequence of east steps (i.e. $X = E$), if and only if q is odd, otherwise it starts with a sequence of north steps (i.e., $X = N$); moreover, $p(P)$ ends with a sequence of north steps (i.e., $Y = N$), if and only if t is odd, otherwise it ends with a sequence of east steps (i.e., $Y = E$).

Now the path $\hat{p}(P)$ is obtained from $p(P)$ by removing three steps: The first and the last steps from p_1, and the step preceding p_1, which is always an east step (see Figure 11 (d)). Then we have

$$\hat{p}(P) = \left(X^{k_1} \cdots N^{k_{q-(2\ell+1)}} E^{k_{q-2\ell}} \cdots N^{k_{q-1}} E^{k_q - 1}\right) \hat{p}_1$$

$$\left(N^{s_1} E^{s_2} \cdots N^{s_{2\ell+1}} E^{s_{2\ell+2}} \cdots Y^{s_t}\right).$$

(3) The mapping described in case (2) actually does not require that the boundary of the permutominide crosses itself. Then, if P is a permutomino, we apply the mapping of case (2), and obtain the pair $(\hat{p}(P), h)$.

We now prove that the above defined mapping $P \longrightarrow (\hat{p}(P), h)$ is indeed a bijection. So let us start from a given pair (p, h), where p is a path of length $2n-3$ using north and east unit steps, and $1 \leq h \leq n-1$, and we show how to uniquely determine a convex (non directed) permutominide P of size n, with lowest leftmost vertex in $A = (0, h)$, and such that $\hat{p}(P) = p$.

To begin with, we decompose the path p into three subpaths, called $\hat{p}_1$, $\hat{p}_2$, and $\hat{p}_3$. The path $\hat{p}_3$ is given by the prefix of P with length $h-1$. Let us denote by $\hat{p}_{12}$ the path obtained by removing $\hat{p}_3$ from p. To establish the point which divides $\hat{p}_{12}$ into the subpaths $\hat{p}_1$ and $\hat{p}_2$, we have to consider the following two factorizations of $\hat{p}_{12}$:

- $\hat{p}_{12} = \alpha\beta$, with $|\alpha|_N = n-h-1$, where $|\alpha|_N$ denotes the number of north steps in α;
- $\hat{p}_{12} = \alpha'\beta'$, with $|\alpha'|_E = n-1$, where $|\alpha'|_E$ denotes the number of east steps in α'.

Let us consider the following two cases separately.

(i) $|\alpha| \leq |\alpha'|$, it means that the path $p_1(P)$ of the permutominide P we are building up terminates on $y = n$. In this case, $\hat{p}_1$ is given by the longest prefix α'' of $\hat{p}_{12}$ such that $|\alpha''|_N = n-h-1$, and we set $p_3 = \hat{p}_3 E$, $p_1 = N\hat{p}_1 E$, and $p_2 = \hat{p}_2$. Then we are able to determine the coordinates of the point $B = (|p_1|_E, n)$. Now, the path p_1, running from A to B, may cross the main diagonal, and in this case we fall into case (1.2). Otherwise, we fall into case (2) or (3).

(ii) $|\alpha| > |\alpha'|$, it means that the path $p_1(P)$ of the permutominide P we are building up terminates on $x = n$ (then it satisfies condition (1.1)). In this case, $\hat{p}_1$ is given by the longest prefix α'' of $\hat{p}_{12}$ such that $|\alpha''|_E = n$, and we set $p_3 = \hat{p}_3 E$, $p_1 = N\hat{p}_1 N$, and $p_2 = \hat{p}_2$. Then we are able to determine the coordinates of the point $C = (n, h + |p_1|_N)$.

Now, the three paths p_1, p_2, and p_3 have been determined, the path p_1 begins at point A and terminates at B, in case (i) or in C, in case (ii). Then, inverting the construction described for cases (1) and (2) it is easy to construct the unique permutominide P such that $p_1 = p_1(P)$, $p_2 = p_2(P)$, and $p_3 = p_3(P)$. ∎

A consequence of the results in Propositions 2.2 and 2.3 is that the generating functions of convex and of directed convex permutominides are rational ones. This is a rather surprising result, since the generating functions of directed convex and convex permutominoes are instead algebraic ones. This fact will be further investigated in the next section, where we show the reason of the algebraicity of the generating function of convex permutominoes.

3. A Bijective Proof for the Number of Convex Permutominoes

In this section we will give a bijective proof of formula (1.1) for the number of convex permutominoes. Let us consider the following classes:

- $P(n)$ is the class of convex permutominides of size n;
- $C(n)$ is the class of convex permutominoes of size n;
- $T_1(n)$ is the class of convex permutominides of size n whose boundary crosses itself on the main diagonal;
- $T_2(n)$ is the class of convex permutominides of size n whose boundary crosses itself on the anti-diagonal.

Clearly, $P(n) = C(n) \cup T_1(n) \cup T_2(n)$. Moreover, $T_1(n)$ and $T_2(n)$ are disjoint sets, and they have the same cardinality. Since we know the cardinality of $P(n)$, it is sufficient to enumerate $T_1(n)$ to determine the number of convex permutominoes. More precisely,

$$|C(n)| = |P(n)| - 2\,|T_1(n)|\,. \tag{3.1}$$

Proposition 3.1. *The cardinality of $T_1(n)$ is equal to $\frac{n}{4}\binom{2n}{n} - \frac{4^{n-1}}{2}$.*

Proof. Let P be a convex permutominide in $T_1(n)$, and let $I = (i, i)$ be the first point on the main diagonal where the boundary of P crosses itself. We easily see that I uniquely decomposes P into two parts: On the south-west of I we have a (south-west) directed convex permutomino, while on the north-east of I we have a (north-east) directed convex permutominide. We remark that, since the boundary of P must cross itself, $0 < i < n$ (see Figure 12).

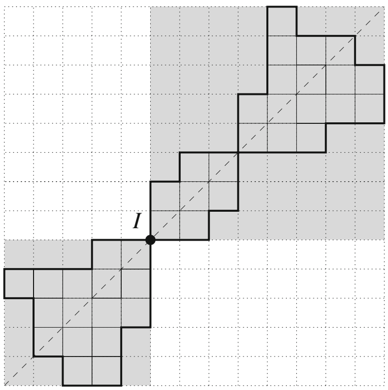


Figure 12: A convex permutominide in $T_1(n)$, and its unique decomposition.

We know from [4] that the number of directed convex permutominoes of size n is equal to $\binom{2n-1}{n}$, while from Proposition 2.2 we have that the number of directed convex permutominides of size n is equal to 4^{n-1}. Hence, according to our decomposition, we have that the cardinality of $T_1(n)$ is given by

$$\sum_{h=1}^{n-1} \binom{2h-1}{h} 4^{n-h-1} = \frac{n}{4}\binom{2n}{n} - \frac{4^{n-1}}{2}. \qquad \blacksquare$$

From the results of Propositions 2.3 and 3.1, and following Equation (3.1), we have a formula for the number of convex permutominoes of given size. We remark that it is the first purely bijective proof of such a result.

Corollary 3.2. *The number of convex permutominoes of size n is*

$$|C(n)| = 2(n+3)4^{n-2} - \frac{n}{2}\binom{2n}{n}, \quad n \geq 1.$$

4. Enumeration of Row Convex Permutominides

In this section we show an easy way to compute the number of row convex permutominides of given size. Here we use an approach different from that applied in Section 3. The main idea is to represent a row convex permutominide P by means of a permutation $\pi(P)$ (called the *base* permutation of P).

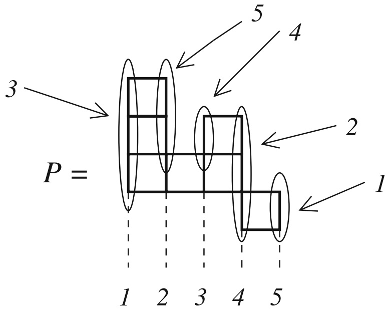
4.1. The Base of a Permutominide

Let P be a permutominide of size n. For simplicity of representation, we now assume that the minimal bounding square of P is placed in $(1,1)$. To each vertical side v in the boundary of P we assign the integer $x(v) \in \{1,\dots,n+1\}$, i.e., the abscissa common to all the points of v. Since there is exactly one vertical side for each abscissa, from now on we will identify the vertical side by means of its associated number. It is now possible to associate to P a permutation $\pi(P)$ of length $n+1$ by defining a total order on its vertical sides. First of all, to each vertical side v of P we uniquely associate a pair of integers $(y(v), l(v))$, where $y(v)$ is the ordinate of the lowest point of v and $l(v)$ is the length of v.

It is important to observe that in any permutominide there are no two vertical sides $v \neq v'$ such that $(y(v), l(v)) = (y(v'), l(v'))$. Now, if v and v' are two vertical sides of P, we say that $v \leq v'$ if and only if $(y(v), l(v)) \leq (y(v'), l(v'))$ in the lexicographical order. It is clear that this is a total order on the set of the vertical sides of P, with minimum $(1,1)$.

Now, the permutation $\pi(P)$ is defined as follows: $\pi(P)(i) = j$ if and only if the vertical side v such that $x(v) = i$ is the jth element in the previously defined order. Clearly, $\pi(P)$ is a permutation of length $n+1$, and it is called the *base permutation* of P (or simply the *base* of P).

For instance, the permutation associated with the permutominide of size 4 in the figure below is $\pi(P) = (3,5,4,2,1)$: Indeed its vertical sides are totally ordered as $5 < 4 < 1 < 3 < 2$, as it is shown in the table on the right.



$x(e)$	$(y(e), l(e))$
1	(2,3)
2	(3,2)
3	(3,1)
4	(1,3)
5	(1,1)

We would like to remark that the base permutation $\pi(P)$ of a permutominide P has nothing to do with the pair $(\pi_1(P), \pi_2(P))$ of permutations defining the permutominide, mentioned in the introductory section. For instance, it is easy to see that the same base permutation can be associated to different permutominides, as shown in Figure 13.

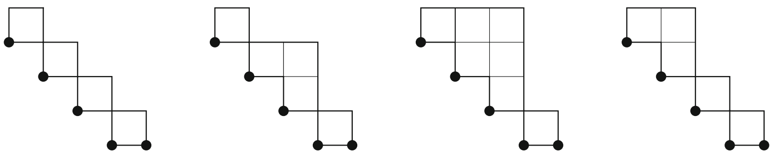


Figure 13: The four row convex permutominides with base permutation $(5,4,3,2,1)$.

4.2. The Number of Row-Convex Permutominides

Let us focus on the class of row convex permutominides. We characterize the set of row convex permutominides having the same base permutation, and then we determine the enumeration of this class. The following result is straightforward.

Proposition 4.1. *Let P be a permutominide; P is row convex if and only if, for any two different vertical sides v and v', $y(v) = y(v') = \ell$ implies $\ell = 1$.*

Concerning the class of row convex permutominides, the permutation $\pi(P)$ has a very simple interpretation. In practice, let P be a row convex permutominide and let $v_1, \ldots, v_{n+1}$ be its vertical sides, from left to right. The permutation $\pi(P)$ can be defined as

$$\pi(P)(i) = \begin{cases} y(v_i) + 1, & \text{if } y(v_i) \neq 1 \text{ or } l(v_i) > 1; \\ 1, & \text{otherwise.} \end{cases}$$

It seems now natural to study the set of row convex permutominides having the same base permutation, so for a given permutation π of length $n \geq 2$, let us consider the set

$$\mathcal{P}(\pi) = \{P \colon P \text{ is a row convex permutominide of size } n-1, \text{ and } \pi(P) = \pi\}.$$

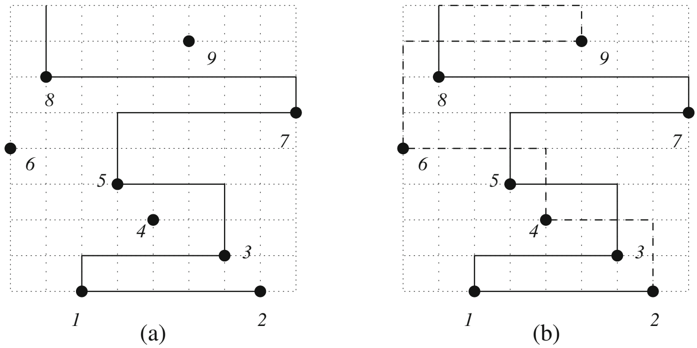


Figure 14: (a) the path defined by the permutation $(6, 8, 1, 5, 4, 9, 3, 2, 7)$ and the subset $S = \{5, 7, 8\}$; for simplicity, the vertical side v_i is represented by its index i; (b) the associated row convex permutominide.

Proposition 4.2. *Let π be a permutation of length n. Then the cardinality of the set $\mathcal{P}(\pi)$ is equal to 2^{n-3}.*

Proof. We give a constructive way to build the 2^{n-3} row convex permutominides with base permutation π. Let us consider the following points on the plane: $v_1 = \left(\pi^{-1}(1), 1\right)$, $v_i = \left(\pi(i)^{-1}, i-1\right)$, for $i > 1$ and let $S = \{i_1, \dots, i_k\}$, with $3 < i_1 < \cdots < i_k \leq n$.

Let us consider the path defined as follows (see Figure 14 (a))

- it starts from v_2 and connects it to v_1 by means of a sequence of horizontal unit steps (they can be east or west steps depending on the relative positions of v_1 and v_2);
- it connects v_1 to v_3 by means of a sequence of north unit steps followed by a sequence of horizontal unit steps;
- it connects v_3 to v_{i_1} by means of a sequence of north steps followed by a sequence of horizontal steps;
- it connects v_{i_r} to $v_{i_{r+1}}$ by means of a sequence of north steps followed by a sequence of horizontal steps;
- then it connects v_{i_k} to the line $y = n+1$ by means of a sequence of north steps.

It can be easily proved that there is a unique way to build a row convex permutominide $P(S)$ such that $\pi(P) = \pi$ with the defined path as a subpath of its boundary (see Figure 14 (b)).

Since to every subset S of the set $\{4, 5, \dots, n\}$ we can uniquely associate a row convex permutominide $P(S)$ in $\mathcal{P}(\pi)$, then the cardinality of the latter set is equal to 2^{n-3}. ∎

Figure 13 depicts the four row convex permutominides of size 4 with base $(5, 4, 3, 2, 1)$. The following result is a direct consequence of the previous statements.

Proposition 4.3. *The number of row convex permutominides of size n is equal to $2^{n-2}(n+1)!$.*

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