On q-series Identities Arising from Lecture Hall Partitions

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Abstract

In this paper, we highlight two q-series identities arising from the the "five guidelines" approach enumerating lecture hall partitions and give direct, q-series proofs. This requires two new finite corollaries of a q-analog of Gauss's second theorem. In fact, the method reveals stronger results about lecture hall partitions and anti-lecture hall compositions that are only partially explained combinatorially.

1 Introduction

The *lecture hall partitions*, L_n are sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying the constraints

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_{n-1}}{2} \geq \frac{\lambda_n}{1} \geq 0.$$

In [2], Bousquet-Mélou and Eriksson proved the following surprising result.

The Lecture Hall Theorem [2]:

$$L_n(q) \triangleq \sum_{\lambda \in L_n} q^{|\lambda|} = \frac{1}{(q;q^2)_n} \tag{1}$$

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with $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$. Continuing this work, the *anti-lecture hall compositions*, A_n, were defined as the sequences $\lambda = (\lambda_1, \ldots, \lambda_2)$ satisfying

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \geq \frac{\lambda_n}{n} \geq 0.$$

In contrast to the lecture hall partitions, these sequences need not be weakly decreasing.

The Anti-Lecture Hall Theorem [5]:

$$A_n(q) \triangleq \sum_{\lambda \in A_n} q^{|\lambda|} = \frac{(-q;q)_n}{(q^2;q)_n}. \tag{2}$$

Truncated versions of both of these families were introduced in [6] as $L_{n,k}$ and $A_{n,k}$. The *truncated lecture hall partitions*, $L_{n,k}$, are the sequences $(\lambda_1, \ldots, \lambda_k)$ satisfying

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_k}{n-k+1} > 0 \tag{3}$$

(note the strict inequality); the *truncated anti-lecture hall compositions*, $A_{n,k}$, are those sequences $(\lambda_1, \ldots, \lambda_k)$ satisfying

$$\frac{\lambda_1}{n-k+1} \geq \frac{\lambda_2}{n-k+2} \geq \ldots \geq \frac{\lambda_k}{n} \geq 0. \tag{4}$$

For fixed k, as $n \to \infty$, $L_{n,k}$ approaches the set of partitions into k distinct positive parts and $A_{n,k}$ approaches the set of ordinary partitions into at most k parts. The generating functions were computed in [6] as

$$L_{n,k}(q) = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1};q)_k}{(q^{2n-k+1};q)_k}, \tag{5}$$

$$A_{n,k}(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1};q)_k}{(q^{2(n-k+1)};q)_k}. \tag{6}$$

Three different approaches to the enumeration of $L_{n,k}$ and $A_{n,k}$ have led to three different pairs of recurrences satisfied by (5) and (6) [3, 4, 6]. The resulting q-series identities are interesting in themselves.

In Section 2, we derive two new corollaries of the q-analog of Gauss's second ${}_2F_1$ summation. In Sections 3 and 4, we highlight one pair of q-series identities, arising from the "five guidelines" approach to lecture hall partitions in [3], and give direct proofs using the corollaries in Section 2. In Section 5, we provide combinatorial proofs of the identities via sign-reversing involutions. Significantly, the q-Gauss approach reveals new results about lecture hall partitions and anti-lecture hall compositions that we have not been able to prove combinatorially.

2 The q-analog of Gauss's second theorem and implications

From [1], p. 526, eq. (1.8) (cf. [7], p. 355, eq. (II.11)):

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n q^{\binom{n+1}{2}}}{(q)_n (qab; q^2)_n} = \frac{(-q)_\infty (aq; q^2)_\infty (bq; q^2)_\infty}{(qab; q^2)_\infty}. \tag{7}$$

Corollary 1

$$\sum_{n=0}^{N} \frac{(q^{-N})_n (b)_n q^{\binom{n+1}{2}}}{(q)_n (q^{1-N} b; q^2)_n} = \begin{cases} 0 & \textit{if } N \textit{ is odd} \\ \frac{b^{-\nu}(q;q^2)_\nu}{(q/b;q^2)_\nu} & \textit{if } N = 2\nu. \end{cases}$$

Proof. Set $a = q^{-N}$ in (7) and recall Euler's identity $(-q)_\infty = 1/(q; q^2)_\infty$. Simplification yields the desired result. □

Corollary 2

$$\sum_{n=0}^{N} \frac{(q^{-N})_n (b)_n q^n}{(q)_n (q^{1-N} b; q^2)_n} = \begin{cases} 0 & \textit{if } N \textit{ is odd} \\ \frac{(q;q^2)_\nu}{(q/b;q^2)_\nu} & \textit{if } N = 2\nu. \end{cases}$$

Proof. Replace q by $1/q$ and b by $1/b$ in Corollary 1 and then simplify the result. □

3 The "truncated lecture hall" identity

Combining (5) with the recurrence for $L_{n,k}(q)$ derived in [3] gives Theorem 1 below. In this section we give a direct proof.

Theorem 1 *Given*

$$L_{n,k} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1}; q)_k}{(q^{2n-k+1}; q)_k} = \frac{q^{\binom{k+1}{2}} (q)_{2n-k}}{(q)_k (q^2; q^2)_{n-k} (q; q^2)_n}.$$

Then

$$L_{n,k} = \sum_{j \geq 1} (-1)^{j-1} \frac{q^j}{(q)_j} \cdot \frac{1 - q^{k(n-k+j) + \binom{k-j+1}{2}}}{1 - q^{\binom{n+1}{2} - \binom{n-k+1}{2}}} L_{n,k-j}. \tag{8}$$

Proof. In light of the fact that

$$k(n-k) + \binom{k+1}{2} = \binom{n+1}{2} - \binom{n-k+1}{2},$$

we see that we may rewrite the desired identity as

$$0 = \sum_{j\geq 0} (-1)^{j-1} \frac{q^j}{(q)_j} \cdot \frac{1-q^{k(n-k+j)+\binom{k-j+1}{2}}}{1-q^{\binom{n+1}{2}-\binom{n-k+1}{2}}} L_{n,k-j}.$$

Simplifying, we find that the result we wish to prove is equivalent to

$$\sum_{j\geq 0} \frac{(-1)^j q^j}{(q)_j} L_{n,k-j} = \sum_{j\geq 0} \frac{(-1)^j q^{k(n-k+j)+\binom{k-j+1}{2}+j} L_{n,k-j}}{(q)_j}. \tag{9}$$

We first evaluate the left-hand side of (9).

$$\begin{aligned}
\sum_{j\geq 0} \frac{(-1)^j q^j}{(q)_j} L_{n,k-j} &= \sum_{j\geq 0} \frac{(-1)^j q^j}{(q)_j} \frac{q^{\binom{j-k}{2}} (q)_{2n-k+j}}{(q)_{k-j}(q^2;q^2)_{n-k+j}(q;q^2)_n} \\
&= \frac{(q)_{2n-k}}{(q)_k (q^2;q^2)_{n-k}(q;q^2)_n} \sum_{j=0}^{k} \frac{(-1)^j q^{kj-\binom{j}{2}} (q^{-k})_j q^{j+\binom{j-k}{2}} (q^{2n-k+1})_j}{(q)_j (q^{2n-2k+2};q^2)_j} \\
&= \frac{(q)_{2n-k} q^{\binom{k+1}{2}}}{(q)_k (q^2;q^2)_{n-k}(q;q^2)_n} \sum_{j=0}^{k} \frac{(q^{-k})_j (q^{2n-k+1})_j q^j}{(q)_j (q^{2n-2k+2};q^2)_j} \\
&= \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{(q)_{2n-2\nu} q^{\binom{2\nu+1}{2}} (q;q^2)_\nu}{(q)_{2\nu}(q^2;q^2)_{n-2\nu}(q;q^2)_n (q^{2\nu-2n};q^2)_\nu} & \text{if } k = 2\nu \end{cases} \\
&\qquad \text{(by Corollary 2)} \\
&= \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{(q;q^2)_{n-\nu}(q^2;q^2)_{n-\nu}(-1)^\nu q^{\nu(2n-\nu+2)}(q;q^2)_\nu}{(q^2;q^2)_\nu (q;q^2)_\nu (q;q^2)_n (q^2;q^2)_{n-\nu}} & \text{if } k = 2\nu \end{cases} \\
&= \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{(-1)^\nu q^{\nu(2n-\nu+2)}}{(q^2;q^2)_\nu (q^{2n-2\nu+1};q^2)_\nu} & \text{if } k = 2\nu. \end{cases}
\end{aligned}$$

Second, we evaluate the right-hand side of (9).

$$\sum_{j\geq 0} \frac{(-1)^j q^{k(n-k+j)+\binom{k-j+1}{2}+j}}{(q)_j} L_{n,k-j}$$

$$= q^{k(n-k)+\binom{k+1}{2}} \sum_{j\geq 0} \frac{(-1)^j q^{\binom{j+1}{2}}}{(q)_j} \frac{q^{\binom{j-k}{2}}(q)_{2n-k+j}}{(q)_{k-j}(q^2;q^2)_{n-k+j}(q;q^2)_n}$$

$$= \frac{q^{k(n-k)+k(k+1)}}{(q)_k(q^2;q^2)_{n-k}(q;q^2)_n} \sum_{j\geq 0} \frac{(q^{-k})_j(q^{2n-k+1})_j q^{\binom{j+1}{2}}}{(q)_j(q^{2n-2k+2};q^2)_j}$$

(following the same steps as before)

$$= \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{q^{k(n-k)+k(k+1)-\nu(2n-2\nu+1)}(q)_{2n-2\nu}(q;q^2)_\nu}{(q)_{2\nu}(q^2;q^2)_{n-2\nu}(q;q^2)_n(q^{2\nu-2n};q^2)_\nu} & \text{if } k=2\nu \end{cases}$$

(by Corollary 1)

$$= \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{(q)_{2n-2\nu}q^{\binom{2\nu+1}{2}}(q;q^2)_\nu}{(q)_{2\nu}(q^2;q^2)_{n-2\nu}(q;q^2)_n(q^{2\nu-2n};q^2)_\nu} & \text{if } k=2\nu \end{cases}$$

$$= \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{(-1)^\nu q^{\nu(2n-\nu+2)}}{(q^2;q^2)_\nu(q^{2n-2\nu+1};q^2)_\nu} & \text{if } k=2\nu. \end{cases}$$

Thus both sides of (9) are equal to the same expression and consequently (9) is proved.

□

Question 1: Is the fact that both sides of (9) are equal to the last expression above of combinatorial significance or interest? This assertion is stronger than the "truncated lecture hall" identity which only requires that the two sides of (9) are equal.

4 The truncated anti-lecture hall identity

The "five guidelines" method used in [3] to derive a recurrence for $A_n(q)$ can be applied to $A_{n,k}(q)$. Combining the resulting recurrence with (6) gives rise to the identity of Theorem 2 below, which we now prove directly.

Theorem 2

$$A_{n,k} = \sum_{j\geq 1} \frac{(-1)^{j-1}}{(q;q)_j} \frac{q^{\binom{j}{2}} - q^{\binom{n+1}{2}-\binom{n-k+1}{2}}}{1 - q^{\binom{n+1}{2}-\binom{n-k+1}{2}}} A_{n-j,k-j}$$

where

$$A_{n,k} = \left[\begin{array}{c} n \\ k \end{array} \right] \frac{(-q^{n-k+1};q)_k}{(q^{2(n-k+1)};q)_k}.$$

Proof. We note that the desired identity may be rewritten as

$$\sum_{j\geq 0} \frac{(-1)^j}{(q;q)_j} \frac{q^{\binom{j}{2}} - q^{\binom{n+1}{2}-\binom{n-k+1}{2}}}{1 - q^{\binom{n+1}{2}-\binom{n-k+1}{2}}} A_{n-j,k-j} = 0.$$

We may assume $n \geq k \geq 0$; otherwise the identity becomes $0 = 0$. Hence our desired theorem reduces to the equivalent assertion that for $n \geq k \geq 0$,

$$\sum_{j\geq 0} \frac{(-1)^j q^{\binom{j}{2}}}{(q;q)_j} A_{n-j,k-j} = q^{\binom{n+1}{2}-\binom{n-k+1}{2}} \sum_{j\geq 0} \frac{(-1)^j A_{n-j,k-j}}{(q;q)_j}. \tag{10}$$

As we shall show, each side is equal to

$$\begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{q^{\nu(2n-2\nu+1)}}{(q^2;q^2)_\nu (q^{2n-4\nu+3};q^2)_\nu} & \text{if } k = 2\nu. \end{cases}$$

We begin with the left-hand side of (10).

$$\begin{aligned}
&\sum_{j\geq 0} \frac{(-1)^j q^{\binom{j}{2}}}{(q;q)_j} A_{n-j,k-j} \\
&\qquad = \sum_{j=0}^{k} \frac{(-1)^j q^{\binom{j}{2}} (q^2;q^2)_{n-j} (q;q)_{2n-2k+1}}{(q;q)_j (q;q)_{k-j} (q^2;q^2)_{n-k} (q;q)_{2n-k+1-j}} \\
&\qquad = \frac{(q;q)_{2n-2k+1}}{(q;q)_{2n-k+1}} \left[\begin{array}{c} n \\ k \end{array} \right]_{q^2} (-q;q)_k \sum_{j=0}^{k} \frac{(q^{-k};q)_j (q^{-2n+k-1};q)_j q^{\binom{j+1}{2}}}{(q;q)_j (q^{-2n};q^2)_j} \\
&\qquad = \frac{(q;q)_{2n-2k+1}}{(q;q)_{2n-k+1}} \left[\begin{array}{c} n \\ k \end{array} \right]_{q^2} (-q;q)_k \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{q^{(-\nu)(-2n+2\nu-1)}(q;q^2)_\nu}{(q^{2n-2\nu+2};q^2)_\nu} & \text{if } k = 2\nu \end{cases} \\
&\qquad\qquad \text{(by Corollary 1)} \\
&\qquad = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{q^{\nu(2n-2\nu+1)}}{(q^2;q^2)_\nu (q^{2n-4\nu+3};q^2)_\nu} & \text{if } k = 2\nu, \end{cases}
\end{aligned}$$

as desired.

Now we move to the right-hand side of (10).

$$
\begin{aligned}
&q^{\binom{n+1}{2}-\binom{n-k+1}{2}} \sum_{j\geq 0} \frac{(-1)^j A_{n-j,k-j}}{(q;q)_j} \\
&\quad = q^{k(2n-k+1)/2} \sum_{j=0}^{k} \frac{(-1)^j (q^2;q^2)_{n-j} (q;q)_{2n-2k+1}}{(q;q)_j (q;q)_{k-j} (q^2;q^2)_{n-k} (q;q)_{2n-k+1-j}} \\
&\quad = \frac{q^{k(2n-k+1)/2} (q;q)_{2n-2k+1}}{(q;q)_{2n-k+1}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} (-q;q)_k \sum_{j=0}^{k} \frac{q^{-k};q)_j (q^{-2n+k-1};q)_j q^j}{(q;q)_j (q^{-2n};q^2)_j} \\
&\quad = \frac{q^{k(2n-k+1)/2} (q;q)_{2n-2k+1}}{(q;q)_{2n-k+1}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} (-q;q)_k \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{(q;q^2)_\nu}{(q^{2n-2\nu+2};q^2)_\nu} & \text{if } k = 2\nu \end{cases} \\
&\qquad\qquad \text{(by Corollary 2)} \\
&\quad = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{q^{\nu(2n-2\nu+1)}}{(q^2;q^2)_\nu (q^{2n-4\nu+3};q^2)_\nu} & \text{if } k = 2\nu, \end{cases}
\end{aligned}
$$

as desired. Thus both sides of (10) are equal to the same expression and consequently (10) is proved. $\square$

Question 2: First of all, the questions raised about the other identity are completely relevant concerning the truncated anti-lecture hall identity. In addition, note that when k is odd,

$$A_{n,k} = \sum_{j\geq 1} \frac{(-1)^{j+1} q^{\binom{j}{2}}}{(q;q)_j} A_{n-j,n-j},$$

and when k is even, say $k = 2\nu$,

$$A_{n,k} = \sum_{j\geq 1} \frac{(-1)^{j+1} q^{\binom{j}{2}}}{(q;q)_j} A_{n-j,n-j} + \frac{q^{\nu(2n-2\nu+1)}}{(q^2;q^2)_\nu (q^{2n-4\nu+3};q^2)_\nu}.$$

Is there any direct combinatorial explanation of formulas like these?

What is most striking in the proofs of both identities is the fact that we use two instances of the same q-analog in each case. Furthermore, the two instances in question in each case are obtained from each other by replacing all the variables in question by their reciprocals. This would strongly suggest that there might well be straightforward elegant proofs of (9) and (10) combinatorially. We show in the next section that this is the case, although the combinatorial proofs do not explain the reciprocity.

5 Combinatorial proofs

In this section we give combinatorial proofs of eqs. (9) and (10). First note from the definition (4) of $A_{n,k}$ that since

$$(\lambda_1,\ldots,\lambda_k)\in A_{n,k}\leftrightarrow(\lambda_1+n-k+1,\ \lambda_2+n-k+2,\ \ldots,\ \lambda_k+n)\in A_{n,k},$$

$$\sum_{\{\lambda\in A_{n,k}|\ \lambda_k\geq n\}} q^{|\lambda|} \;=\; q^{\binom{n+1}{2}-\binom{n-k+1}{2}}A_{n,k}. \tag{11}$$

Similarly, from the definition (3) of $L_{n,k}$, $(\lambda_1,\ldots,\lambda_k)\in L_{n,k}$ if and only if $(\lambda_1+n,\lambda_2+n-1,\ldots,\lambda_k+n-k+1)\in L_{n,k}$ and $\lambda_k>0$. So,

$$\sum_{\{\lambda\in L_{n,k}|\ \lambda_k>n-k+1\}} q^{|\lambda|} \;=\; q^{\binom{n+1}{2}-\binom{n-k+1}{2}}L_{n,k}. \tag{12}$$

Lemma 1 *If $(\lambda_1,\ldots,\lambda_k)\in L_{n,k}$, then $(\lambda_1,\ldots,\lambda_k,a)\in L_{n,k+1}$, as long as*

$$0<a<n-k+1<\lambda_k \quad\text{or}\quad 0<a<\lambda_k\leq n-k+1.$$

Proof. Show that either condition implies that

$$\frac{\lambda_k}{n-k+1}\geq\frac{a}{n-k}>0.$$

The strict inequality follows since $a>0$. For the other inequality, if $0<a<n-k+1<\lambda_k$, then

$$\lambda_k(n-k)-a(n-k+1)>(n-k+1)(n-k-a)\geq 0.$$

If $0<a<\lambda_k\leq n-k+1$, then since $\lambda_k\geq a+1$ and $a\leq n-k$,

$$\lambda_k(n-k)-a(n-k+1)\geq(a+1)(n-k)-a(n-k+1)=n-k-a\geq 0.$$

$\square$

Lemma 2 *If $(\lambda_1,\ldots,\lambda_k)\in A_{n,k}$ and $\lambda_k<n$ then $\lambda_{k-1}\geq\lambda_k$.*

Proof. By definition of $A_{n,k}$, $\lambda_{k-1}/(n-1)\geq\lambda_k/n$. If $\lambda_{k-1}<\lambda_k$, we have $\lambda_k(n-1)\leq n\lambda_{k-1}\leq n(\lambda_k-1)$, and therefore $\lambda_k\geq n$. $\square$

Now, we interpret identity (9). The set $L_{n,k}$ is empty when $n < k$ and it contains only the empty partition when $k = 0$ and (9) holds in these cases. Thus, assume $n \geq k \geq 1$.

Let P_j be the set of partitions into j positive parts. Since P_j has generating function $q^j/(q)_j$, the left-hand side of (9) counts the elements of the set $\cup_{j\geq 0}(P_j \times L_{n,k-j})$, weighted by *sign*. The sign of a pair $(\mu, \lambda) \in P_j \times L_{n,k-j}$ is $(-1)^j$.

For $\lambda = (\lambda_1, \ldots, \lambda_k)$, denote by $s(\lambda)$ the last entry of the sequence, that is, $s(\lambda) = \lambda_k$. The right-hand side of (9) can be rewritten as

$$\sum_{j\geq 0} \frac{(-1)^j q^{k(n-k+j)+\binom{k-j+1}{2}+j} L_{n,k-j}}{(q)_j} = \sum_{j\geq 0}(-1)^j \frac{q^{j(n-k+j+1)}}{(q)_j} q^{\binom{n+1}{2}-\binom{n-(k-j)+1}{2}} L_{n,k-j}.$$

Note that the factor

$$\frac{q^{j(n-k+j+1)}}{(q)_j}$$

is the generating function for partitions $\mu \in P_j$ such that $s(\mu) \geq n - k + j + 1$. By (12), the factor

$$q^{\binom{n+1}{2}-\binom{n-(k-j)+1}{2}} L_{n,k-j}$$

is the generating function for partitions λ in $L_{n,k-j}$ such that $s(\lambda) > n - k + j + 1$. So, the right-hand side of (9) is counting only those pairs $(\mu, \lambda) \in \cup_{j\geq 0}(P_j \times L_{n,k-j})$ such that $s(\mu) \geq n - k + j + 1$ **and** $s(\lambda) > n - k + j + 1$, weighted by sign. Let

$$\mathcal{B}_j = \{(\mu, \lambda) \in P_j \times L_{n,k-j} \mid s(\mu) < n - k + j + 1 \;\; \textbf{or} \;\; s(\lambda) \leq n - k + j + 1\}.$$

(If $j > k$, $\mathcal{B}_j$ is empty.) To prove (9), it suffices to define a sign-reversing involution G on $\cup_j \mathcal{B}_j$ with no fixed points. To simplify the notation, define $s(\pi) = \infty$ if π is the empty partition.

The involution G:

- If $0 < s(\mu) < s(\lambda)$ then $G(\mu, \lambda) = ((\mu_1, \ldots, \mu_{j-1}), (\lambda_1, \ldots, \lambda_{k-j}, \mu_j))$;
- If $0 < s(\lambda) \leq s(\mu)$ $G(\mu, \lambda) = ((\mu_1, \ldots, \mu_j, \lambda_{k-j}), (\lambda_1, \ldots, \lambda_{k-j-1}))$.

Proposition 1 *G is a sign-reversing involution on $\cup_j \mathcal{B}_j$ proving* (9).

Proof. Let $0 \leq j \leq k$ and let $(\mu, \lambda) \in \mathcal{B}_j$. Let $(\mu', \lambda') = G((\mu, \lambda))$. We show that $(\mu', \lambda') \in \mathcal{B}_{j-1} \cup \mathcal{B}_{j+1}$ (reversing the sign) and that $G((\mu', \lambda')) = (\mu, \lambda)$.

Case 1.1: $0 < s(\mu) < s(\lambda)$. In this case, $s(\mu) \neq \infty$, so $j \geq 1$ and

$$(\mu', \lambda') = ((\mu_1, \ldots, \mu_{j-1}), (\lambda_1, \ldots, \lambda_{k-j}, \mu_j)).$$

If $k = j$, then since $(\mu, \lambda) \in \mathcal{B}_j$, $s(\mu) < n - k + j + 1 = n + 1$. Thus $s(\lambda') = \mu_j \leq n$ and $(\mu', \lambda') = ((\mu_1, \ldots, \mu_{j-1}), (\mu_j)) \in \mathcal{B}_{j-1}$.

Otherwise, $1 < j < k$ and $0 < \mu_j < \lambda_{k-j}$. If $\lambda_{k-j} > n - k + j + 1$, then $0 < \mu_j < n - k + j + 1$. Otherwise, $\mu_j < \lambda_{k-j} \leq n - k + j + 1$. In either case, by Lemma 1, $(\lambda_1, \ldots, \lambda_{k-j}, \mu_j) \in L_{n,k-j+1}$. Also, in either case, $s(\lambda') = \mu_j \leq n - k + j$ thus $(\mu', \lambda') \in \mathcal{B}_{j-1}$.

In Case 1.1, either $s(\mu') = \infty$ or $s(\mu') = \mu_{j-1}$. Either way, since $\mu \in P_j$, $s(\mu') \geq s(\lambda') = \mu_j$ and therefore $G((\mu', \lambda')) = (\mu, \lambda)$.

Case 1.2: $0 < s(\lambda) \leq s(\mu)$. In this case, since $k > 0$, it must be that $k - j > 0$ and

$$(\mu', \lambda') = ((\mu_1, \ldots, \mu_j, \lambda_{k-j}), (\lambda_1, \ldots, \lambda_{k-j-1})).$$

If $j = 0$, then since $(\mu, \lambda) \in \mathcal{B}_j$, it must be that $\lambda_k = \lambda_{k-j} \leq n - k + 1$ and therefore $(\mu', \lambda') = ((\lambda_k), (\lambda_1, \ldots, \lambda_{k-1}) \in \mathcal{B}_1$.

Otherwise, $0 < \lambda_{k-j} \leq \mu_j$. Then $(\mu_1, \ldots, \mu_j, \lambda_{k-j}) \in P_{j+1}$ and $(\lambda_1, \ldots, \lambda_{k-j-1}) \in L_{n,k-j-1}$. Since $(\mu, \lambda) \in \mathcal{B}_j$, either $\lambda_{k-j} \geq n - k + j + 1$ or $\lambda_{k-j} \leq \mu_j < n - k + j + 1$. In either case, $\lambda_{k-j} < n - k + j + 2$, so $(\mu', \lambda') \in \mathcal{B}_{j+1}$.

In Case 1.2, either $s(\lambda') = \infty$ or $s(\lambda') = \mu_{j-1}$. Either way, since $\lambda \in L_{n,k-j}$, $s(\lambda') \geq s(\mu') = \lambda_{k-j}$ and therefore $G((\mu', \lambda')) = (\mu, \lambda)$. $\square$

The combinatorial proof of (10) is similar. Again, we assume $n \geq k \geq 1$, since otherwise the identity is true. Let D_j be the set of partitions μ into j nonnegative distinct parts ($\mu_1 > \mu_2 > \ldots > \mu_{j-1} > \mu_j \geq 0$). Since D_j has generating function $q^{\binom{j}{2}}/(q)_j$, the left-hand side of (10) counts the elements of the set $\cup_{j\geq 0}(D_j \times A_{n-j,k-j})$, weighted by sign.

The right-hand side of (10) can be rewritten as

$$q^{\binom{n+1}{2} - \binom{n-k+1}{2}} \sum_j \frac{(-1)^j}{(q)_j} A_{n-j,k-j} \quad = \quad \sum_j (-1)^j \frac{q^{\binom{n+1}{2} - \binom{n-j+1}{2}}}{(q)_j} q^{\binom{n-j+1}{2} - \binom{n-k+1}{2}} A_{n-j,k-j}.$$

The factor

$$\frac{q^{\binom{n+1}{2} - \binom{n-j+1}{2}}}{(q)_j}$$

is the generating functions for partitions $\mu \in D_j$ such that $s(\mu) > n - j$. By (11), the factor

$$q^{\binom{n-j+1}{2} - \binom{n-k+1}{2}} A_{n-j,k-j}$$

is the generating function of compositions λ in $A_{n-j,k-j}$ such that $s(\lambda) \geq n-j$. So, the right-hand side of (10) is counting only those pairs $(\mu, \lambda) \in \cup_{j\geq 0}(D_j \times A_{n-j,k-j})$ such that $s(\mu) > n-k+j+1$ and $s(\lambda) \geq n-k+j+1$, weighted by $(-1)^j$.

Therefore to prove Equation (10), it suffices to define a sign-reversing involution F, with no fixed points, on the set $\cup_j \mathcal{A}_j$ where

$$\mathcal{A}_j = \{(\mu, \lambda) \in D_j \times A_{n-j,k-j} | \ s(\mu) \leq n-j \text{ or } s(\lambda) < n-j\}$$

and the sign of an element of $\mathcal{A}_j$ is $(-1)^j$.

The involution F:

- If $s(\mu) \leq s(\lambda)$ then $F(\mu, \lambda) = ((\mu_1, \ldots, \mu_{j-1}), (\lambda_1, \ldots, \lambda_{k-j}, \mu_j))$;
- If $s(\lambda) < s(\mu)$ $F(\mu, \lambda) = ((\mu_1, \ldots, \mu_j, \lambda_{k-j}), (\lambda_1, \ldots, \lambda_{k-j-1}))$.

Proposition 2 *F is a sign-reversing involution on $\cup_j \mathcal{A}_j$ proving* (10).

Proof. Let $0 \leq j \leq k$ and let $(\mu, \lambda) \in \mathcal{A}_j$. Let $(\mu', \lambda') = F((\mu, \lambda))$. We show that $(\mu', \lambda') \in \mathcal{A}_{j-1} \cup \mathcal{A}_{j+1}$ (reversing the sign) and that $F((\mu', \lambda')) = (\mu, \lambda)$.

Case 2.1: $s(\mu) \leq s(\lambda)$. Since $k \geq 1$, μ and λ cannot both be empty, so $s(\mu) \neq \infty$, and $j \geq 1$ and

$$(\mu', \lambda') = ((\mu_1, \ldots, \mu_{j-1}), (\lambda_1, \ldots, \lambda_{k-j}, \mu_j)).$$

Then $\mu' \in D_{j-1}$ and $\lambda' \in A_{n-j+1,k-j+1}$, since either λ is empty or $\lambda_{k-j} = s(\lambda) \geq s(\mu) = \mu_j$.

Since $(\mu, \lambda) \in \mathcal{A}_j$, either $s(\mu) \leq n-j$ or $s(\mu) \leq s(\lambda) < n-j$. In either case,

$$s(\lambda') = \mu_j = s(\mu) < n-j+1$$

and thus $(\mu', \lambda') \in \mathcal{A}_{j-1}$.

To show that $s(\lambda') < s(\mu')$, if $j = 1$, then $s(\mu') = \infty$, whereas $s(\lambda') = \mu_j$; otherwise $s(\lambda') = \mu_j < \mu_{j-1} = s(\mu')$, since $\mu \in D_j$. Thus, $F((\mu', \lambda')) = (\mu, \lambda)$.

Case 2.2: $s(\lambda) < s(\mu)$. In this case, λ cannot be empty so $0 \leq j < k$ and

$$(\mu', \lambda') = ((\mu_1, \ldots, \mu_j, \lambda_{k-j}), (\lambda_1, \ldots, \lambda_{k-j-1})).$$

Then $\lambda' \in A_{n-j-1,k-j-1}$ and $\mu' \in D_{j+1}$, since either μ is empty or $\mu_j = s(\mu) > s(\lambda) = \lambda_{k-j}$. Furthermore, since $(\mu, \lambda) \in \mathcal{B}_j$, we have $s(\lambda) < n-j$ or $s(\lambda) < s(\mu) \leq n-j$. In either case, $s(\mu') = \lambda_{k-j} = s(\lambda) \leq n-j-1$ and thus $(\mu', \lambda') \in \mathcal{A}_{j+1}$.

Finally, we need to show that $s(\mu') \leq s(\lambda')$ so that $F((\mu', \lambda')) = (\mu, \lambda)$. This is clearly true if $k-j = 1$. Otherwise, since $\lambda_{k-j} < n-j$, by Lemma 2, $s(\mu') = \lambda_{k-j} \leq \lambda_{k-j-1} = s(\lambda')$. □

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