Abstract numeration systems and tilings

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Abstract. An abstract numeration system is a triple $S = (L, \Sigma, <)$ where $(\Sigma, <)$ is a totally ordered alphabet and L a regular language over Σ; the associated numeration is defined as follows: by enumerating the words of the regular language L over Σ with respect to the induced genealogical ordering, one obtains a one-to-one correspondence between $\mathbb{N}$ and L. Furthermore, when the language L is assumed to be exponential, real numbers can also be expanded. The aim of the present paper is to associate with S a self-replicating multiple tiling of the space, under the following assumption: the adjacency matrix of the trimmed minimal automaton recognizing L is primitive with a dominant eigenvalue being a Pisot unit. This construction generalizes the classical constructions performed for Rauzy fractals associated with Pisot substitutions [16], and for central tiles associated with a Pisot beta-numeration [23].

1 Introduction

To any infinite regular language L over a totally ordered alphabet $(\Sigma, <)$, an *abstract numeration system* $S = (L, \Sigma, <)$ is associated in the following way [10]. Enumerating the words of L by increasing genealogical order gives a one-to-one correspondence between $\mathbb{N}$ and L, the non-negative integer n being represented by the $(n+1)$-th word of the ordered language L. Nonnegative integers as well as positive real numbers (under some natural assumptions on L) can thus be expanded in such a numeration system [10–12]. In this latter situation, a real number is represented by an infinite word which is the limit of a converging sequence of words in L. These systems generalize in a natural way classical positional systems like the k-ary numeration, the Fibonacci numeration, more generally, the numeration scales built on a sequence of integers satisfying a linear recurrence relation, including the beta-numeration when β is a Parry number [13], as well as the Dumont-Thomas numeration associated with a substitution [6, 7]. Many classical properties of such numerations extend in a natural way to abstract numeration systems: see for instance [3, 9, 11, 12, 17–19].

The aim of this paper is to introduce a self-replicating multiple tiling of the space that can be associated with an abstract numeration system with some prescribed algebraic properties: these systems are built upon an exponential regular language such that the adjacency matrix of the trimmed minimal automaton recognizing L is primitive with a dominant eigenvalue being a Pisot unit. We recall that a *Pisot number* is an algebraic integer whose other conjugates have modulus smaller than 1; a Pisot number is a *unit* if its norm is equal to 1, that is, the constant term in its minimal polynomial equals ± 1. The basic tiles are compact sets that are the closure of their interior, that have non-zero measure and a fractal boundary; they are attractors of some graph-directed Iterated Function System. By tiling, we mean here tilings by translation having finitely many tiles up to translation (a tile is assumed to be the closure of its interior); we assume furthermore that each compact set intersects a finite number of tiles. By *multiple tiling*, we mean arrangements of tiles such that almost all points are covered exactly p times for some positive integer p.

The multiple tiling we propose here is directly inspired by the tilings of the space that can be associated with beta-numeration [23] and with substitutions (see e.g., Chap. 7 in [15]). It is conjectured that the corresponding multiple tiling is indeed a tiling in the Pisot case. This conjecture is known as the *Pisot conjecture* and can also be reformulated in spectral terms: the associated dynamical systems have pure discrete spectrum. Notice that the existence of such tilings has applications in Diophantine approximation, or in the study of mathematical quasicrystals, for instance.

Our main motivation for this work is the following. The central tiles in the beta-numeration framework are defined in a natural way [1, 2, 23]; one can consider the formalism introduced in the substitutive case as a first generalization of the beta-numeration formalism [5]. Indeed the underlying substitutions and automata have a very particular shape in the beta-numeration case. We develop here a further generalization by working directly on the automaton. In particular final acceptance states play a crucial rôle in our study. We thus wish to put to the test the Pisot conjecture in a more general context.

This paper is organized as follows. We first recall in Section 2 a few basic definitions and properties. We focus on the representation of real numbers in abstract number systems in Section 3. We respectively introduce in Section 4 and in Section 5 the central tile and our multiple tiling. We illustrate these notions in Section 6 with two examples. We conclude this paper by mentioning a few natural prospects concerning this work in Section 7.

2 Definitions

An *abstract numeration system* is a triple $S = (L, \Sigma, <)$, where L is an infinite regular language over the totally ordered alphabet $(\Sigma, <)$.

Let $\Sigma = \{s_0 < s_1 < \cdots < s_k\}$ be a finite and totally ordered alphabet. Since Σ is totally ordered, we can order the words of Σ^* using the *genealogical ordering*. Let $u, v \in \Sigma^*$. We say that $u < v$ if $|u| < |v|$ or if $|u| = |v|$ and there exist $p, u', v' \in \Sigma^*$, $s, t \in \Sigma$, $s < t$ such that $u = psu'$ and $v = ptv'$.

The trimmed minimal automaton of L is denoted $\mathcal{M}_L = (Q, q_0, \Sigma, \delta, F)$ where Q is the set of states, q_0 is the initial state, $F \subseteq Q$ is the set of final states and $\delta : Q \times \Sigma \to Q$ is the (partial) transition function. As usual, δ can be extended to $Q \times \Sigma^*$. In this paper $L \subset \Sigma^*$ will always denote an infinite regular language having the property that $\mathcal{M}_L$ is such that

$$\delta(q_0, s_0) = q_0. \tag{1}$$

In other words, $\mathcal{M}_L$ has a loop of label s_0 in the initial state q_0. In particular, this implies that L has the following property: $s_0^* L \subseteq L$.

The entry of index $(p,q) \in Q^2$ of the *adjacency matrix* $\mathbf{M}_L$ of the automaton $\mathcal{M}_L$ is given by the cardinality of the set of letters $s \in \Sigma$ such that $\delta(p,s) = q$. An abstract numeration system is said *primitive* if the matrix $\mathbf{M}_L$ is primitive, that is, there exists a nonnegative integer n such that $\mathbf{M}_L^n$ has only positive entries. According to Perron-Frobenius theorem, the adjacency matrix of a primitive abstract numeration system admits a simple dominating eigenvalue $\beta > 0$.

For any state $q \in Q$, we denote by L_q the regular language accepted by $\mathcal{M}_L$ from state q, by $\mathbf{u}_q(n)$ the number of words of length n in L_q, and by $\mathbf{v}(n)$ the number of words of length at most n in L. In particular, $L = L_{q_0}$ and $\mathbf{u}_q(n) = e_q \mathbf{M}_L^n e_F$ for appropriate row (resp. column) vector e_q (resp. e_F).

Let us introduce several sets of right-sided and left-sided infinite words built upon the abstract numeration system S. We use here the topology induced by the infinite product topology on $\Sigma^{\mathbb{N}}$, $\Sigma^{\mathbb{N}^*}$ and ${}^{\mathbb{N}}\Sigma$ respectively, where $\mathbb{N}^*$ denotes the set of positive integers, and ${}^{\mathbb{N}}\Sigma$ the set of left-infinite words over Σ. We use the following notation for elements of ${}^{\mathbb{N}}\Sigma$: $v = \cdots v_2 v_1 v_0$.

We first define $\mathcal{L}^\omega \subset \Sigma^{\mathbb{N}^*}$ as the set of right-infinite words $w = (w_i)_{i \in \mathbb{N}^*}$ for which there exists a sequence of words $(W_n)_{n \in \mathbb{N}}$ in L converging to w, that is, for all ℓ, there exists N_ℓ such that for all $n \geq N_\ell$, a prefix of length at least ℓ of W_n is a prefix of w. Notice that a main difference with the set $\overrightarrow{L}$ classically encountered in the literature (we refer for instance to [22]) is that if w belongs to $\mathcal{L}^\omega$ then it does not necessarily imply that infinitely many prefixes of w belongs to L (see [11]).

Definition 1. *We define the set $\mathcal{K}^\omega \subset (\Sigma \times Q)^{\mathbb{N}}$ by $(w,r) = (w_0 w_1 \cdots, r_0 r_1 \cdots)$ belongs to $\mathcal{K}^\omega$ if and only if the following conditions hold*

1. *there exists a sequence of words $(W_n)_{n \in \mathbb{N}}$ in $\cup_{q \in Q} L_q$ converging to $w_1 w_2 \cdots$,*
2. *for all $i \geq 0$, $\delta(r_i, w_{i+1}) = r_{i+1}$.*

For a given $q \in Q$, the subset $\mathcal{K}_q^\omega \subset \mathcal{K}^\omega$ is defined as the set of elements $(w,r) \in \mathcal{K}^\omega$ such that $r_0 = q$. One has $\mathcal{K}^\omega = \cup_{q \in Q} \mathcal{K}_q^\omega$.

Definition 2. *We similarly define the set ${}^\omega\mathcal{K} \subset {}^{\mathbb{N}}(\Sigma \times Q)$. A pair $(v,p) = (\cdots v_2 v_1 v_0, \cdots p_2 p_1 p_0)$ belongs to ${}^\omega\mathcal{K}$ if and only if the following conditions hold*

1. *there exists a sequence $(V_n)_{n \in \mathbb{N}}$ of words in L converging to v, i.e., $v_0 v_1 v_2 \cdots$ is the limit of the sequence of words $(\widetilde{V_n})_{n \in \mathbb{N}}$, where $\widetilde{W}$ denotes the mirror image of the word W,*
2. *for all $i \geq 0$, $\delta(p_{i+1}, v_i) = p_i$.*

Definition 3. *Finally, ${}^\omega\mathcal{K}^\omega \subset (\Sigma \times Q)^{\mathbb{Z}}$ is defined as the set of two-sided sequences $((\cdots v_2 v_1 v_0 \cdot w_1 w_2 \cdots), (\cdots p_2 p_1 p_0 \cdot r_1 r_2 \cdots))$ (denoted $((v,w);(p,r))$) that satisfy*

1. $(\cdots v_2 v_1 v_0, \cdots p_2 p_1 p_0)$ *belongs to* ${}^{\omega}\mathcal{K}$,
2. $(v_0 w_1 w_2 \cdots, p_0 r_1 r_2 \cdots)$ *belongs to* $\mathcal{K}^{\omega}$.

These three sets are easily shown to be nonempty, by a classical compactness argument; they have the rôle played by the beta-shift in the beta-numeration case [13, Chap. 7].

3 Expansions of real numbers

The abstract numeration system $S = (L, \Sigma, <)$ gives a one-to-one correspondence between $\mathbb{N}$ and L [10]: the representation of the integer n is defined as the $(n+1)$-th word w of L. We conversely define $\mathrm{val} : L \to \mathbb{N}$, which maps the $(n+1)$-th word of L onto n.

We want now to expand real numbers. Let us assume that S is a primitive abstract numeration system. Let $\beta > 1$ denote its dominating eigenvalue. Consequently, L is an exponential regular language (i.e., $\mathbf{u}_{q_0}(n) \geq C\beta^n$, for infinitely many n and some $C > 0$) and thanks to [11, Prop. 3], we deduce that the set $\mathcal{L}^{\omega}$ is uncountable.

We assume moreover that L is a language for which there exist $P \in \mathbb{R}[X]$, and some nonnegative real numbers a_q, $q \in Q$, which are not simultaneously equal to 0, such that for all state $q \in Q$

$$\lim_{n\to\infty} \frac{\mathbf{u}_q(n)}{P(n)\beta^n} = a_q. \tag{2}$$

The coefficients a_q are defined up to a scaling constant; in fact, the vector $(a_q)_{q\in Q}$ is an eigenvector of $\mathbf{M}_L$ [12]; by Perron-Frobenius theorem, all its entries a_q are positive; we normalize it so that $a_{q_0} = 1 - 1/\beta$, according to [19].

For $q \in Q$ and $s \in \Sigma$, set

$$\alpha_q(s) := \sum_{q'\in Q} a_{q'} \cdot \mathrm{Card}\{t < s \mid \delta(q,t) = q'\} = \sum_{\substack{t<s \\ (q,t)\in\mathrm{dom}(\delta)}} a_{\delta(q,t)}.$$

One has for all $q \in Q$, $0 \leq \alpha_q(s) \leq \beta a_q$, since $(a_q)_{q\in Q}$ is a positive eigenvector of $\mathbf{M}_L$. Notice also that if $s < t$, $s, t \in \Sigma$, then $\alpha_q(s) \leq \alpha_q(t)$.

For any sequence of words $(W_k)_{k\in\mathbb{N}}$ converging to a word $w \in \mathcal{L}^{\omega}$, let us recall that the limit

$$\lim_{k\to\infty} \frac{\mathrm{val}(W_k)}{\mathbf{v}(|W_k|)}$$

only depends on w, belongs to $[1/\beta, 1]$, and is equal to

$$(1 + \alpha_{q_0}(w_1))\beta^{-1} + \sum_{j=2}^{\infty} \alpha_{\delta(q_0, w_1\cdots w_{j-1})}(w_j)\beta^{-j},$$

according to [11, 19]. Hence it is natural to introduce the map

$$\varphi^{\omega} : \mathcal{K}^{\omega} \to [0, \max(a_q)],\ (w, r) = (w_0 w_1 \cdots, r_0 r_1 \cdots) \mapsto \sum_{j=1}^{\infty} \alpha_{r_{j-1}}(w_j)\beta^{-j}.$$

Conversely, let us expand real numbers by introducing a suitable dynamical system analogous to the β-transformation $T_\beta : \ x \in [0,1] \mapsto \{\beta x\}$, where $\{z\}$ denotes the fractional part of z. The corresponding transformation for abstract dynamical systems has been introduced in [19]. The underlying dynamics depends on each interval $[0, a_q)$, and is defined as follows: we first set for $y \in \mathbb{R}^+$,

$$\lfloor y \rfloor_q = \max\{\alpha_q(s) \mid s \in \Sigma,\ \alpha_q(s) \le y\};$$

let us recall that $(a_q)_{q \in Q}$ is an eigenvector of $\mathbf{M}_L$ of eigenvalue β, hence

$$\beta a_q = \sum_{r \in Q} a_r \cdot \mathrm{Card}\{s \in \Sigma \mid \delta(q,s) = r\},$$

and one checks that for $y \in [0, a_q)$, then $\beta y - \lfloor \beta y \rfloor_q \in [0, a_{q'})$, with $\lfloor \beta y \rfloor_q = \alpha_q(s)$ and $\delta(q,s) = q'$. Furthermore, s may be not uniquely determined since α_q is nondecreasing. We define

$$\begin{aligned} T_S : (\cup_{q\in Q}[0,a_q]) \times Q &\to (\cup_{q\in Q}[0,a_q]) \times Q, \\ (x,q) &\mapsto (\beta x - \lfloor \beta x \rfloor_q, q') \end{aligned}$$

where q' is determined as follows: let s be the largest letter such that $\alpha_q(s) = \lfloor \beta x \rfloor_q$; then $q' = \delta(q,s)$. To retrieve this information given by the largest letter s, we thus set

$$\begin{aligned} \rho_S : (\cup_{q\in Q}[0,a_q]) \times Q &\to \Sigma, \\ (x,q) &\mapsto s. \end{aligned}$$

We thus can expand any real number $x \in [0, a_{q_0}) = [0, 1 - 1/\beta)$ as follows. Let $(x_i, r_i)_{i \ge 1} := (T_S^i(x, q_0))_{i \ge 1} \in ((\cup_{q \in Q}[0, a_q]) \times Q)^{\mathbb{N}^*}$. Moreover, set $(w_0, r_0) := (s_0, q_0)$ and for every $i \ge 1$, set $w_i := \rho_S(x_{i-1}, r_{i-1})$ where it is assumed that $x_0 := x$. According to [11], one has $x = \sum_{j=1}^{\infty} \alpha_{r_{j-1}}(w_j)\beta^{-j}$. So we have the following definition.

Definition 4. *Let S be a primitive abstract numeration system satisfying (1) and (2). Every real number $x \in [0, 1 - 1/\beta)$ can be expanded as*

$$x = \sum_{i \ge 1} \alpha_{r_{i-1}}(w_i)\beta^{-i},$$

where (w,r) belongs to $\mathcal{K}^\omega$ and satisfies for every $i \ge 1$, $w_i = \rho_S(x_{i-1}, r_{i-1})$, with $(x_i, r_i)_{i \ge 1} = (T_S^i(x, q_0))_{i \ge 1}$, $(w_0, r_0) = (s_0, q_0)$ and $x_0 = x$. We call $(w,r) = (w_i, r_i)_{i \in \mathbb{N}} \in \mathcal{K}^\omega$ the S-expansion *of x and denote it $d_S(x)$.*

Similarly, one can expand every real positive number by rescaling. Indeed let $x \ge a_{q_0}$. Let k be the smallest positive integer such that $\beta^{-k}x \in [0, a_{q_0})$ and let us set $(w_i, r_i)_{i \in \mathbb{N}} := d_S(\beta^{-k}x)$. One has $\beta^{-k}x = \sum_{j \ge 1} \alpha_{r_{j-1}}(w_j)\beta^{-j}$. Let $(v,p) \in {}^\omega\mathcal{K}$, with $v = (\cdots s_0 \cdots s_0 v_{k-1} v_{k-2} \cdots v_0)$, $p = (\cdots q_0 \cdots q_0 p_{k-1} \cdots p_0)$, and $v_{k-1} \cdots v_0 = w_1 \cdots w_k$, $p_{k-1} \cdots p_0 = r_1 \cdots r_k$. (Notice that we have explicitly used (1) to define (v,p).) One thus gets

$$\begin{aligned} x = {} & \alpha_{q_0}(v_{k-1})\beta^{k-1} + \alpha_{p_{k-1}}(v_{k-2})\beta^{k-2} + \cdots + \alpha_{p_1}(v_0) \\ & + \alpha_{r_k}(w_{k+1})\tfrac{1}{\beta} + \cdots + \alpha_{r_{k+1}}(w_{k+2})\tfrac{1}{\beta^2} + \cdots \end{aligned}$$

with $(v \cdot w_{k+1}w_{k+2}\cdots, p \cdot r_{k+1}r_{k+2}\cdots) \in {}^{\omega}\mathcal{K}^{\omega}$.

Definition 5. *Let S be a primitive abstract numeration system and x be a positive real number. If $x \in [0, a_{q_0}) = [0, 1-1/\beta)$, then the S-*fractional part *of x is simply $d_S(x)$. Otherwise, let k be the smallest positive integer such that $\beta^{-k}x \in [0, a_{q_0})$. Using the same notation as above, the S-*fractional part *of x is $(w_k w_{k+1} w_{k+2}\cdots, r_k r_{k+1} r_{k+2}\cdots) \in \mathcal{K}^{\omega}$. We denote it* $\mathrm{Frac}_S(x)$.

4 The central tile

We have given in Section 3 a geometric representation of the set $\mathcal{K}^{\omega}$ thanks to the map φ^{ω}. The aim of the present section is to provide a similar representation for the set ${}^{\omega}\mathcal{K}$. We follow here the formalism of [1, 2, 5].

We assume now that S is a primitive abstract numeration system satisfying (1) and (2), whose dominant eigenvalue β is a Pisot unit. Let $\beta^{(2)}, \ldots, \beta^{(r)}$ denote the real conjugates of β, and let $\beta^{(r+1)}, \overline{\beta^{(r+1)}}, \ldots, \beta^{(r+s)}, \overline{\beta^{(r+s)}}$ be its complex conjugates. If d denotes the degree of β, then $d = r + 2s$. We set $\beta^{(1)} = \beta$. Let $\mathbb{K}^{(k)}$ be equal to $\mathbb{R}$ if $1 \le k \le r$, and to $\mathbb{C}$, if $k > r$. We furthermore denote by $\mathbb{K}_\beta$ the *representation space*

$$\mathbb{K}_\beta := \mathbb{R}^{r-1} \times \mathbb{C}^s \simeq \mathbb{R}^{d-1}.$$

Let us note that, according to [19, Lemma 4.1], a_q belongs to $\mathbb{Q}(\beta)$ for all $q \in Q$. Let us consider now the following algebraic embeddings:

- The *canonical embedding* on $\mathbb{Q}(\beta)$ maps a polynomial to all its conjugates

$$\Phi_\beta : \mathbb{Q}(\beta) \to \mathbb{K}_\beta,\ P(\beta) \mapsto (P(\beta^{(2)}), \ldots, P(\beta^{(r)}), P(\beta^{(r+1)}), \ldots, P(\beta^{(r+s)})).$$

- For any $(v, p) \in {}^{\omega}\mathcal{K}$, the series

$$\lim_{n\to\infty} \Phi_\beta\left(\sum_{i=0}^{n} \alpha_{p_{i+1}}(v_i)\beta^i\right) = \sum_{i\ge 0} \Phi_\beta(\alpha_{p_{i+1}}(v_i))\Phi_\beta(\beta^i)$$

  are convergent in $\mathbb{K}_\beta$. The *representation map* of ${}^{\omega}\mathcal{K}$ is then defined as

$${}^{\omega}\varphi :\ {}^{\omega}\mathcal{K} \to \mathbb{K}_\beta,\ (v, p) \mapsto \lim_{n\to+\infty} \Phi_\beta\left(\sum_{i=0}^{n} \alpha_{p_{i+1}}(v_i)\beta^i\right).$$

Definition 6. *Let S be a primitive abstract numeration system satisfying (1) and (2) whose dominant eigenvalue β is a Pisot number. We define the* central tile *$\mathcal{T}_S$ as*

$$\mathcal{T}_S := {}^{\omega}\varphi({}^{\omega}\mathcal{K}).$$

The central tile can be naturally divided into $\mathrm{Card}(Q)$ pieces, called *basic tiles*, as follows:

$$\text{for } q \in Q,\ \mathcal{T}_S(q) := {}^{\omega}\varphi\Big(\{(v, p) \in {}^{\omega}\mathcal{K} \mid\ p_0 = q\}\Big).$$

5 A self-replicating multiple tiling

We introduce the following countable set

$$\mathcal{F}_S := \mathrm{Frac}_S(\mathbb{Z}[\beta]_{>0}) \subset \mathcal{K}^\omega.$$

Let $(w, r) = (w_i, r_i)_{i \in \mathbb{N}} \in \mathcal{F}_S$. By definition of $\mathcal{F}_S$, we can apply Φ_β to $\varphi^\omega(w, r)$ which belongs to $\mathbb{Q}(\beta)$. We define the tile $\mathcal{T}_{(w,r)}$ as

$$\mathcal{T}_{(w,r)} = \Phi_\beta \circ \varphi^\omega(w, r) + {}^\omega\varphi\Big(\{(v, p) \in {}^\omega\mathcal{K} \mid ((v, w); (p, r)) \in {}^\omega\mathcal{K}^\omega\}\Big).$$

One checks that the tiles $\mathcal{T}_{(w,r)}$ are finite unions of translates of the basic tiles $\mathcal{T}_S(q)$ for $q \in Q$ by considering the minimal automaton $\mathcal{M}_L$; furthermore, one proves similarly as in [2] that there are finitely many such tiles.

Definition 7. *The primitive abstract numeration system S for which* (1) *and* (2) *hold is said to satisfy the* strong coincidence condition *if for any pair of states $(q, q') \in Q$, there exist a state $q'' \in Q$, a positive integer n and two words $w_1 \cdots w_n$, $w'_1 \cdots w'_n \in \Sigma^n$ such that*

$$\begin{cases} \sum_{1 \le i \le n} \alpha_{\delta(q, w_1 \cdots w_{i-1})}(w_i)\beta^{n-i} = \sum_{1 \le i \le n} \alpha_{\delta(q', w'_1 \cdots w'_{i-1})}(w'_i)\beta^{n-i} \\ \delta(q, w_1 \cdots w_n) = \delta(q', w'_1 \cdots w'_n) = q''. \end{cases}$$

We have now gathered all the required tools to be able to state and prove the main theorem of the present paper. This theorem and its proof are directly inspired by the corresponding statements in the beta-numeration case [1, 2, 5], and in the substitutive case [21].

Theorem 1. *Let S be a primitive abstract Pisot numeration system for which* (1) *and* (2) *hold and whose dominant eigenvalue β is a Pisot number. The finite (up to translation) set of tiles $\mathcal{T}_{(w,r)}$, for $(w, r) \in \mathcal{F}_S$, covers $\mathbb{K}_\beta$, that is,*

$$\mathbb{K}_\beta = \bigcup_{(w,r) \in \mathcal{F}_S} \mathcal{T}_{(w,r)}. \tag{3}$$

For each (w, r), the tile $\mathcal{T}_{(w,r)}$ has non-empty interior. Hence it has non-zero measure.

*We denote by $h_\beta : \mathbb{K}_\beta \to \mathbb{K}_\beta$ the β-*multiplication map *that multiplies the coordinate of index i by $\beta^{(i)}$, for $2 \le i \le d$. The basic tiles of the central tile $\mathcal{T}_S$ are solutions of the following graph-directed self-affine Iterated Function System:*

$$\forall q \in Q, \ \mathcal{T}_S(q) = \bigcup_{\substack{p \in Q, \ s \in \Sigma, \\ \delta(p,s) = q}} h_\beta(\mathcal{T}_S(p)) + \Phi_\beta(\alpha_p(s)). \tag{4}$$

If S satisfies the strong coincidence condition, then the basic tiles have disjoint interiors and they are the closure of their interior.

Furthermore, there exists an integer $k \ge 1$ such that the covering (3) is almost everywhere k-to-one.

Proof. We first notice that there exists $C > 0$ such that if

$$\Phi_\beta \circ \varphi^\omega(\mathrm{Frac}_S(P(\beta))) \neq \Phi_\beta \circ \varphi^\omega(\mathrm{Frac}_S(P'(\beta))),$$

with $\mathrm{Frac}_S(P(\beta)) \neq \mathrm{Frac}_S(P(\beta))$, then

$$\|\Phi_\beta \circ \varphi^\omega(\mathrm{Frac}_S((P(\beta)))) - \Phi_\beta \circ \varphi^\omega(\mathrm{Frac}_S((P'(\beta))))\| > C, \tag{5}$$

where $\|\cdot\|$ denotes a given norm in $\mathbb{K}_\beta$. Indeed $\Phi_\beta \circ \varphi^\omega(\mathrm{Frac}_S(P - P')(\beta))$ is an algebraic integer: this a direct consequence of the fact that β is a unit and that $a_q \in \mathbb{Q}(\beta)$, for all q. We now conclude by using the fact that for any $C' > 0$, there exist only finitely many algebraic integers x in $\mathbb{Q}(\beta)$ such that $|x| < \max(a_q)$ and $\|\Phi_\beta(x)\| < C'$.

Let us prove now (3). From β being a Pisot number, we first deduce that $\Phi_\beta(\mathbb{Z}[\beta]_{\geq 0})$ is dense in $\mathbb{K}_\beta$, according to [1, Prop. 1]. Let $x \in \mathbb{K}_\beta$. There thus exists a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials in $\mathbb{Z}[X]$ with $P_n(\beta) \geq 0$, for all n, such that $(\Phi_\beta(P_n(\beta)))_{n\in\mathbb{N}}$ tends towards x. For all n, $\Phi_\beta(P_n(\beta)) \in \mathcal{T}_{(w,r)^{(n)}}$, with $(w,r)^{(n)} = \mathrm{Frac}_S(P_n(\beta))$. We deduce from (5) that there exist infinitely many n such that $\Phi_\beta \circ \varphi^\omega((w,r)^{(n)})$ take the same value, say, $\Phi_\beta \circ \varphi^\omega(w,r)$. Since the tiles are closed, $x \in \mathcal{T}_{(w,r)}$. We now deduce from Baire's theorem that the tiles have non-empty interior.

Let $q \in Q$ be given. Let $(v,p) \in {}^\omega\mathcal{K}$ with $p_0 = q$. One has:

$$\begin{aligned}{}^\omega\varphi(v,p)\ {}^\omega\varphi((v_k,p_k)_{k\geq 1}) + \Phi_\beta(\alpha_{p_1}(v_0))\\ = h_\beta \circ {}^\omega\varphi\,((v_{k-1},p_{k-1})_{k\geq 1}) + \Phi_\beta(\alpha_{p_1}(v_0)).\end{aligned}$$

One deduces (4) by noticing that $(v_{k-1},p_{k-1})_{k\geq 1}$ belongs to ${}^\omega\mathcal{K}$.

We deduce from the uniqueness of the solution of the IFS [14], that the basic tiles are the closure of their interior, since the interiors of the pieces are similarly shown to satisfy the same IFS equation (4).

We assume that S satisfies the strong coincidence condition. We deduce from this strong coincidence condition that there exist $q'' \in Q$, $w_1 \cdots w_n$, $w'_1 \cdots w'_n \in \Sigma^n$ such that $\mathcal{T}_S(q)$ contains

$$h_\beta^n(\mathcal{T}_S(q)) + \Phi_\beta\Bigg(\sum_{1\leq i\leq n} \alpha_{\delta(q,w_1\cdots w_{i-1})}(w_i)\beta^{n-i}\Bigg)$$

and

$$h_\beta^n(\mathcal{T}_S(q')) + \Phi_\beta\Bigg(\sum_{1\leq i\leq n} \alpha_{\delta(q',w'_1\cdots w'_{i-1})}(w'_i)\beta^{n-i}\Bigg)$$

with

$$\sum_{1\leq i\leq n} \alpha_{\delta(q',w_1\cdots w_{i-1})}(w_i)\beta^{n-i} = \sum_{1\leq i\leq n} \alpha_{\delta(q',w'_1\cdots w'_{i-1})}(w'_i)\beta^{n-i},$$

according to (4), when iterated n times. Indeed $\mathcal{T}_S(q'')$ is equal to the union on the states $p \in Q$ for which there exists a path $a_1 \cdots a_n$ of length n in $\mathcal{M}_L$ from p to q'', of

$$h_\beta^n(\mathcal{T}_S(p)) + \Phi_\beta(\sum_{1\leq i\leq n} \alpha_{\delta(p,a_1\cdots a_{i-1})}(a_i)\beta^{n-i}).$$

We denote by μ the Lebesgue measure of $\mathbb{K}_\beta$: for every Borelian set B of $\mathbb{K}_\beta$, one has $\mu(h_\beta(B)) = \frac{1}{\beta}\mu(B)$, according to [20]: we have used here the fact that β is a Pisot unit. One has for a given $q \in Q$ according to (4)

$$\begin{aligned} \mu(\mathcal{T}_S(q)) &\leq \textstyle\sum_{p:\delta(p,a)=q} \mu(h_\beta(\mathcal{T}_S(p))) \\ &\leq \tfrac{1}{\beta} \textstyle\sum_{p:\delta(p,a)=q} \mu(\mathcal{T}_S(p)). \end{aligned} \tag{6}$$

Let $\mathbf{m} = (\mu(\mathcal{T}_S(q)))_{q\in Q}$ denotes the vector in $\mathbb{R}^d$ of measures in $\mathbb{K}_\beta$ of the basic tiles; we have proved above that $\mathbf{m}$ is a non-zero vector. Since $\mathbf{m}$ has furthermore nonnegative entries, according to Perron-Frobenius theorem the previous inequality implies that $\mathbf{m}$ is an eigenvector of the primitive matrix $\mathbf{M}_L$, and thus of $\mathbf{M}_L^n$. In particular

$$\mu(\mathcal{T}_S(q)) = \sum_{p\in Q} \mathbf{M}_L^n[p,q] \cdot \mu(\mathcal{T}_S(p)),$$

which implies that $\mathcal{T}_S(q)$ and $\mathcal{T}_S(q')$ have disjoint interiors. We thus have proved that the Card(Q) basic tiles are disjoint up to sets of zero measure.

Finally, one deduces from the statement below ([5], Lemma 1) that there exists an integer k such that this covering is almost everywhere k-to-one:

Let $(\Omega_i)_{i\in I}$ be a collection of open sets in $\mathbb{R}^k$ such that $\cup_{i\in I}\overline{\Omega_i} = \mathbb{R}^k$ and for any compact set K, $I_k := \{i \in I;\ \overline{\Omega_i} \cap K \neq \emptyset\}$ is finite. For $x \in \mathbb{R}^k$, let $f(x) := \mathrm{Card}\{i \in I;\ x \in \overline{\Omega_i}\}$. Let $\Omega = \mathbb{R}^k \setminus \cup_{i\in I}\delta(\omega_i)$, where $\delta(\Omega_i)$ denotes the boundary of Ω_i. Then f is locally constant on Ω.

6 Some examples

Example 1. Let us consider the automaton depicted in Fig. 1. It defines an ab-

Fig. 1. A trimmed minimal automaton.

stract numeration system over $\Sigma = \{0, 1, 2\}$. Its adjacency matrix is

$$\mathbf{M}_L = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

It is primitive, its characteristic polynomial is $X^3 - X^2 - 2X - 1$. The unique real root of the characteristic polynomial is $\beta \simeq 2,148$ and one of its complex root is $\beta^{(2)} \simeq -0,573 + 0,369\,i$, with $|\beta^{(2)}| < 1$, hence β is a Pisot unit. We have $\mathbb{K}_\beta = \mathbb{C}$. We assume that q_1 and q_2 are final states. One can check that

$$\begin{cases} \alpha_{q_0}(0) = 0,\ \alpha_{q_1}(1) = a_{q_0}, \\ \alpha_{q_1}(0) = 0,\ \alpha_{q_1}(1) = a_{q_0}, \\ \alpha_{q_2}(0) = 0,\ \alpha_{q_2}(1) = a_{q_0},\ \alpha_{q_2}(2) = 2a_{q_0}. \end{cases}$$

Let us note that on this particular example, the value taken by $\alpha_{q_i}(j)$ does only depend on the letter $j \in \{0,1,2\}$. Let us recall that $a_{q_0} = 1 - 1/\beta$, hence $\Phi_\beta(a_{q_0}) = 1 - \frac{1}{\beta^{(2)}}$. We represent the basic tiles (the one associated to q_0, q_1 and q_2 is coloured in red, green and blue respectively) in Fig. 2.

Fig. 2. The basic tiles.

Let $\widetilde{\mathcal{M}_L}$ denote the automaton obtained by reversing in $\mathcal{M}_L$ the direction of the arrows. The basic tiles satisfy for $i = 0,1,2$:

$$\mathcal{T}_S(q_i) = \Big\{(1 - \tfrac{1}{\beta^{(2)}}) \cdot \Big(\textstyle\sum_{i \geq 0} v_i \cdot (\beta^{(2)})^i\Big) \mid (v_0 v_1 \cdots) \text{ being the label of an infinite path in the automaton } \widetilde{\mathcal{M}_L} \text{ starting from state } q_i\Big\}.$$

The strong coincidence condition is satisfied: for any pair of states (q_i, q_j), the transitions $\delta(q_i, 0) = q_0$, and $\delta(q_j, 0) = q_0$ are suitable. The graph-directed IFS equation satisfies

$$\begin{cases} \mathcal{T}_S(q_0) = (\beta^{(2)} \cdot \mathcal{T}_S(q_0)) \cup (\beta^{(2)} \cdot \mathcal{T}_S(q_1)) \cup (\beta^{(2)} \cdot \mathcal{T}_S(q_2)) \\ \qquad\qquad \cup (\beta^{(2)} \cdot \mathcal{T}_S(q_2) + (1 - \frac{1}{\beta^{(2)}})) \\ \mathcal{T}_S(q_1) = (\beta^{(2)} \cdot \mathcal{T}_S(q_0)) \cup (\beta^{(2)} \cdot \mathcal{T}_S(q_2) + (2 - \frac{2}{\beta^{(2)}})) \\ \mathcal{T}_S(q_2) = \beta^{(2)} \cdot \mathcal{T}_S(q_1)) + (1 - \frac{1}{\beta^{(2)}}). \end{cases}$$

The multiple tiling of Theorem 1 is indeed a tiling (up to a set of zero measure). This result can be proved by using the same ideas as in [1, 2]: it can be checked in an effective way that every element of $\mathbb{Z}[\beta]$ admits a finite fractional part; this classical property for beta-numeration is called *Finiteness Property* [8], and implies that the covering (3) is a tiling.

Example 2. Let us consider another example given by the automaton depicted in Fig 3. Again we fulfill the Pisot type assumption. Here $\beta \simeq 2,324$ and $\beta^{(2)} \simeq$

Fig. 3. Another minimal automaton and the corresponding basic tiles.

$0,338 + 0,526i$. When considering q_2 as the unique final state, all the α_q's are vanishing except for

$$\alpha_{q_0}(1) = \alpha_{q_2}(2) = 1 - \frac{1}{\beta},\ \alpha_{q_0}(2) = 1 - \frac{1}{\beta} + \frac{1}{\beta^2}.$$

The corresponding three basic tiles are represented in Fig. 3.

7 Conclusion

All the theory developed in the substitutive and in the beta-numeration case can now be extended in the present framework; mutual insight will by no doubt be brought by handling this more general situation. To mention just but a few prospects, we are planning to study the topological properties of the tiles (connectedness, disklike connectedness), to work out some sufficient tiling conditions in the flavour of the finiteness properties, to define p-adic tiles according to [20], to characterize purely periodic expansions in abstract numeration systems thanks to the central tile as in [4], and to study the geometric representation of the underlying dynamical systems, such as the odometer introduced in [3]: for instance, an exchange of pieces can be performed on the central tile by exchanging the basic tiles; the action of this exchange of pieces can be factorized into a rotation of the torus when the covering (3) is a tiling.

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