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Recursive complexity of the Carnap first order modal logic C

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We consider first order modal logic **C** firstly defined by Carnap in "Meaning and Necessity" [1]. We prove elimination of nested modalities for this logic, which gives additionally the Skolem-Löwenheim theorem for **C**. We also evaluate the degree of unsolvability for **C**, by showing that it is exactly $\mathbf{0}'$. We compare this logic with the logics of Henkin quantifiers, Σ^1_1 logic, and SO. We also shortly discuss properties of the logic **C** in finite models.

1 Introduction

This work was inspired by the discussion of the paper by Hintikka [5], who claims that the tautology problem for "alethic" modal logic is recursively equivalent to that of the second order logic. The "alethic" semantics considered by him are slightly ad hoc. Moreover, he gives neither a complete definition nor any reference to such definition. His intuitive description of "alethic" modal logic conforms to the logic originally considered by Carnap in [1].[1)] Hintikka uses his claim about recursive complexity of "alethic" modal logic as an argument against its plausibility. We show that – taking the logic **C** as a good explication of "alethic" logic – its recursive complexity is much lower than Hintikka claims. Nevertheless the logic **C** seems to be interesting for many other reasons. For the discussion see [4].

2 C-modal logic

In this section we define basic notions related to syntax and semantics of the first order modal logic **C**. Let σ be a purely relational vocabulary and $\mathcal{V} = \{x_0, x_1, x_2, \ldots\}$ the set of all first order variables.

Definition 2.1 Firstly we define *the set* AFrm_σ *of atomic formulae of vocabulary* σ:

$$\mathrm{AFrm}_\sigma = \{x_i = x_j \mid i, j \in \omega\} \cup \{P(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \mid i_1, i_2, \ldots, i_n \in \omega, \; P \text{ a predicate of arity } n \text{ in } \sigma\}.$$

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1) The general name for this logic used by Carnap is L. The modal logic is determined by the lanuage equivalent to that of first order logic with modal operators. It is called S_2 *system*. Possible worlds are state descriptions which – under some additional assumptions – are equivalent to models in our terminology. Its sentential fragment is considered by Gottlob in [4], where the character C is used as its name. Here we follow Gottlob's terminology because the symbols L, S_2 are used in many other meanings.

LFrm_σ is the smallest set X containing AFrm_σ and such that, if $\varphi, \psi \in X$, then

1. $\neg\varphi \in X$;
2. $(\varphi \Rightarrow \psi) \in X$;
3. $\Box\varphi \in X$;
4. $\forall x_i\, \varphi \in X$, for each $i \in \omega$.

Definition 2.2 Let M be a σ-model. Any function $\bar{a} : \mathcal{V} \longrightarrow |M|$ is called a *valuation in* M. We define *the satisfaction relation of a formula* φ *in a model* M *under a valuation* $\bar{a}$ in the following way:

1. For $\varphi \in \mathrm{AFrm}_\sigma$, $M \models \varphi[\bar{a}]$ is defined in the same way as for the first order case.
2. For arbitrary $\varphi, \psi \in \mathrm{LFrm}_\sigma$, we have:
   (i) $M \models \neg\varphi[\bar{a}] \Leftrightarrow_{\mathrm{Df}} M \not\models \varphi[\bar{a}]$.
   (ii) $M \models (\varphi \Rightarrow \psi)[\bar{a}] \Leftrightarrow_{\mathrm{Df}}$ if $M \models \varphi[\bar{a}]$, then $M \models \psi[\bar{a}]$.
   (iii) $M \models \forall x_i\, \varphi[\bar{a}] \Leftrightarrow_{\mathrm{Df}} M \models \varphi[\bar{a}(x_i/b)]$, for each $b \in |M|$.
   (iv) $M \models \Box\varphi[\bar{a}] \Leftrightarrow_{\mathrm{Df}}$ for each σ-model M' such that $|M| = |M'|$, $M' \models \varphi[\bar{a}]$.

3 Kripke semantics versus Carnap semantics

In this section we compare the modal logic **C** with **S5** interpreted according to possible worlds semantics. In the Kripke semantics the accessibility relation of **S5**-models is an equivalence relation. Consequently, one can consider these models simply as couples (M, S), where S is a family of models of the same vocabulary as M, $M \in S$ and for each $M' \in S$ we have $|M| = |M'|$. We then define the satisfiability relation of a formula φ in a model (M, S) under the valuation $\bar{a}$. In the case when φ contains no modalities, we have

$$(M, S) \models \varphi[\bar{a}] \Leftrightarrow M \models \varphi[\bar{a}].$$

Additionally we have

$$(M, S) \models \Box\varphi[\bar{a}] \Leftrightarrow \text{for each } M' \in S \text{ we have } M' \models \varphi[\bar{a}],$$

and all other inductive conditions are similar to that of negation

$$(M, S) \models \neg\varphi[\bar{a}] \Leftrightarrow (M, S) \not\models \varphi[\bar{a}].$$

C-models are **S5**-models of the form (M, S) such that S is the class of all models with the same universe as M. We define

$$\models_{\mathbf{C}} \varphi \Leftrightarrow_{\mathrm{Df}} \varphi \text{ is true in all } \mathbf{C}\text{-models}, \quad \models_{\mathbf{S5}} \varphi \Leftrightarrow_{\mathrm{Df}} \varphi \text{ is true in all } \mathbf{S5}\text{-models}.$$

Fact 3.1 *If* $\models_{\mathbf{S5}} \varphi$, *then* $\models_{\mathbf{C}} \varphi$. *Nevertheless it happens that* $\models_{\mathbf{C}} \varphi$ *and* $\not\models_{\mathbf{S5}} \varphi$.

Let us consider two examples. We have $\models_{\mathbf{C}} \Diamond\forall x\, P(x)$, whereas $\not\models_{\mathbf{S5}} \Diamond\forall x P(x)$. In general for every first order formula φ, if φ has models of arbitrary cardinality, then we have $\models_{\mathbf{C}} \Diamond\varphi$.

Additionally we have

$$\begin{aligned}&\models_{\mathbf{C}}\Box(\forall x\forall y\,(f(x) = f(y) \Rightarrow x = y) \Rightarrow \forall z\exists x\,(f(x) = z))\\ &\quad\Rightarrow \Box(\forall x\forall y\forall z\,(P(x,y) \wedge P(y,z) \Rightarrow P(x,z)) \wedge \forall x\,(\neg P(x,x)) \Rightarrow \exists x\forall y\,(\neg P(x,y))).\end{aligned}$$

According to **C**-interpretation the antecedent of this formula says that necessarily each injective function is surjective.[2)] So it says that the universe is Dedekind finite. Its succedent says that each transitive and antireflexive relation has a maximal element, what is another way of saying that the universe is finite.[3)] On the other hand it is easy to show **S5** models in which the antecedent holds, but the succedent is false. Let $M = (\omega, <, \mathrm{id})$, where id is the identity on ω. In the model $(M, \{M\})$ our formula is false according to **S5** semantics.

2) Let us observe that in our langugage we have no function symbols. Moreover, the standard method of eliminating them in favour of predicates does not work in a scope of modal operators in the logic **C**. This is so because modal operators forget all the assumptions outside of its scope. Nevertheless, function symbols can be eliminated also in this case replacing all formulae of the form $\Box\varphi(f)$ by $\Box(F \text{ is functional} \Rightarrow \varphi(F))$, where $\varphi(F)$ is obtained by replacement in $\varphi(f)$ of atomic formulae of the form $f(t) = t'$ by $F(t, t')$.

3) The equivalence of these two characteristics of finiteness requires the assumption of the Axiom of Choice.

4 Elimination of nested modalities

In this section we prove that all nested modalities can be eliminated in **C**. Firstly, we need some auxiliary notions. For the sake of readability we will use special variables $\alpha_1, \alpha_2, \ldots$ for truth values. It is obvious that it does not extend semantical power of the language.

Definition 4.1 Let $\bar{a}$ be a valuation in a structure M of a finite relational vocabulary σ. By a *σ-type T of a valuation $\bar{a}$ with respect to variables $x_1, \ldots, x_k$* we mean the conjunction of all the atomic formulae and their negations which hold between $x_1, \ldots, x_k$ under $\bar{a}$ in M.

If σ is empty, then the type T is a conjunction of equalities and inequalities. In such a case, for n variables $\bar{x} = x_1, \ldots, x_n$ we have $\frac{n(n-1)}{2}$ conjuncts in each type over $\bar{x}$. The only type over the empty sequence of variables is $\top$. The type of $\bar{a}$ is therefore a first-order quantifier free formula.

Lemma 4.2 *Let $\vartheta(\bar{x})$ be a type over $\bar{x}$. Let $\bar{a}$ and $\bar{b}$ be valuations in M and let φ be a formula in the empty vocabulary in which the free variables are among the x_i's, namely* $\mathrm{fv}(\varphi) \subseteq \{\bar{x}\}$. *If we have $M \models \vartheta(\bar{x})[\bar{a}]$ and $M \models \vartheta(\bar{x})[\bar{b}]$, then the following equivalence holds:*

$$M \models \varphi[\bar{a}] \Leftrightarrow M \models \varphi[\bar{b}].$$

By simple **C**-formulae we mean formulae of the form $\Box\psi$, where ψ has no modalities.

Corollary 4.3 *Each relation definable in M by a simple* **C***-formula φ is definable by a formula of the form*

$$\vartheta_1(\bar{x}) \vee \vartheta_2(\bar{x}) \vee \cdots \vee \vartheta_s(\bar{x}),$$

where $\vartheta_i(\bar{x})$ is a type over $\bar{x}$ for all $i \in \{1, 2, \ldots, s\}$ and $\mathrm{fv}(\varphi) = \{\bar{x}\}$.

Definition 4.4 A formula φ *has no nested modalities* if and only if for each of its sub-formulae of the form $\Box\xi$, the formula ξ has no modalities.

Theorem 4.5 *There exists a recursive procedure constructing a formula $\psi \in \mathrm{LFrm}_\sigma$, for any $\varphi \in \mathrm{LFrm}_\sigma$, such that ψ has no nested modalities, and for each σ-model M,*

$$M \models \forall\bar{x}\,(\varphi \equiv \psi),$$

namely, φ and ψ are **C***-equivalent.*

Proof. We proceed by induction on the number of nested modalities.

Let us assume that the thesis holds for all formulae having less than n nested modalities. If φ has a sub-formula $\Box\xi$ such that ξ contains a modal operator, then ξ contains a sub-formula of the form $\Box\vartheta$, where ϑ has no modal operator. We represent φ as $\varphi'(\Box\vartheta)$, where $\varphi'(P(x_1, x_2, \ldots, x_n))$ is obtained by replacing $\Box\vartheta$ in φ by $P(x_1, x_2, \ldots, x_n)$, where $\bar{x} = x_1, x_2, \ldots, x_n$ are all free variables of $\Box\vartheta$. We eliminate this nested modality replacing φ by the following

$$\forall P\,(\forall\bar{x}\,(P(\bar{x}) \equiv \Box\vartheta) \Rightarrow \varphi'(P(\bar{x}))).$$

However, the so obtained formula has a second order quantifier. Fortunately, by Corollary 4.3, we can replace this second order quantification by considering types over $\bar{x}$ as follows. Let $\vartheta_1(\bar{x}), \vartheta_2(\bar{x}), \ldots, \vartheta_s(\bar{x})$ be all types over $\bar{x}$ in empty vocabulary (with the identity predicate only). Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be Boolean variables. We define $\vartheta'(\bar{x})$ as the disjunction

$$(\alpha_1 \wedge \vartheta_1(\bar{x})) \vee (\alpha_2 \wedge \vartheta_2(\bar{x})) \vee \cdots \vee (\alpha_s \wedge \vartheta_s(\bar{x}))$$

and the formula φ^* as

$$\forall\alpha_1 \cdots \forall\alpha_s\,([(\alpha_1 \equiv \forall\bar{x}\,(\vartheta_1(\bar{x}) \Rightarrow \Box\vartheta)) \wedge \cdots \wedge (\alpha_s \equiv \forall\bar{x}\,(\vartheta_s(\bar{x}) \Rightarrow \Box\vartheta))] \Rightarrow \varphi'(\vartheta'(\bar{x})))$$

or equivalently as

$$\exists\alpha_1 \cdots \exists\alpha_n\,([(\alpha_1 \equiv \forall\bar{x}\,(\vartheta_1(\bar{x}) \Rightarrow \Box\vartheta)) \wedge \cdots \wedge (\alpha_s \equiv \forall\bar{x}\,(\vartheta_s(\bar{x}) \Rightarrow \Box\vartheta))] \wedge \varphi'(\vartheta'(\bar{x}))).$$

The so obtained formula is equivalent to φ and it has a smaller number of nested modalities than φ. □

Let us illustrate the proof on a simple example. We start with the following formula φ:

$$\Box(Q(x) \Rightarrow \Box R(x,y)).$$

We apply our procedure to this formula. The first idea given in the proof in order to eliminate the nested modality was the following:

$$\forall P\,(\forall\bar{x}\,(P(\bar{x}) \equiv \Box\vartheta) \Rightarrow \varphi'(P(\bar{x}))).$$

Which gives in our case:

$$\forall P\,((\forall x \forall y\, P(x,y) \equiv \Box R(x,y)) \Rightarrow \Box(Q(x) \Rightarrow P(x,y))).$$

We then define $\vartheta'(\bar{x})$:

$$(\alpha_1 \wedge x = y) \vee (\alpha_2 \wedge x \neq y).$$

And finally we obtain φ^*:

$$\begin{aligned}\forall\alpha_1\forall\alpha_2\,([(\alpha_1 \equiv \forall x\forall y\,(x = y \Rightarrow \Box R(x,y))) \wedge (\alpha_2 \equiv \forall x\forall y\,(x \neq y \Rightarrow \Box R(x,y)))]\\ \Rightarrow \Box(Q(x) \Rightarrow ((\alpha_1 \wedge x = y) \vee (\alpha_2 \wedge x \neq y)))),\end{aligned}$$

which contains no nested modalities.

Let us observe that the elimination of modalities can enlarge formulae at the worst case exponentially.[4)] This is so because the number of needed types is exponential in the number of free variables in a simple formula in which we eliminate the modal operator.

Because we have the following:

Lemma 4.6 *Let φ be a Π^1_1 sentence in empty vocabulary (with identity only), or equivalently a simple* **C***-sentence. For all infinite models Γ, Γ':*

$$\Gamma \models \varphi \Leftrightarrow \Gamma' \models \varphi.$$

Then as a byproduct of our proof of Theorem 4.5 we obtain the following:

Theorem 4.7 (Skolem-Löwenheim theorem for **C**) *For each* **C***-sentence φ, φ has an infinite model if and only if φ has a countable model.*

5 Recursive complexity of first order C-modal logic

In this section we evaluate the degree of unsolvability of the logic **C**. The method applied here is similar to that of [10].

Lemma 5.1 *The following sets are recursively enumerable: the set of first order sentences true in all infinite models and the set of* **C***-formulae consistent in finite models.*

We use this lemma in the proof of the following:

Theorem 5.2 *Let T be $\mathrm{Taut}_\sigma(\mathbf{C})$ – the set of all* **C***-tautologies in a vocabulary σ with at least one binary predicate. T is recursive with a recursively enumerable oracle. Moreover T is exactly of degree $\mathbf{0}'$.*

P r o o f. We describe a deciding algorithm for T using two recursively enumerable sets as oracles. They are the following:

1. X_σ, the complement of the set of **C**-formulae true in in all finite models;
2. Y_σ, the set of first order formulae true in in all infinite models.

4) This remark was suggested by the anonymous referee.

Let φ be a **C**-sentence. For checking whether $\varphi \in T$ we proceed as follows:

1. Check whether φ is true in all finite models. This is an oracle step and it is carried out by checking whether $\varphi \in X_\sigma$. If not, then answer "No", otherwise continue.

2. Check whether φ is true in all infinite models.

We first take $\Box\psi_1, \dots, \Box\psi_n$, all the sub-formulae of φ such that ψ_i has no modalities for $i = 1, \dots, n$.

We proceed as follows: Take $\bar{x} = x_1, \dots, x_m$, all the free variables of $\Box\psi_i$. Let $\vartheta_j(\bar{x})$ be an $\bar{x}$-type in empty vocabulary over $\bar{x}$, for $j = 1, \dots, s$. There are exactly $s = \frac{m(m-1)}{2}$ $\bar{x}$-types.

Compute values for Boolean variables $\alpha_1, \dots, \alpha_s$ such that

$$\alpha_j = \top \Leftrightarrow \models_{\text{infinite models}} \Box\forall\bar{x}\,(\vartheta_j \to \psi_i).$$

This is carried out by checking whether $\forall\bar{x}\,(\vartheta_j \Rightarrow \psi_i) \in Y_\sigma$, for $j = 1, \dots, s$.

Then we replace each formula $\Box\psi_i$ by the following $\vartheta'(\bar{x}, \bar{\alpha})$:

$$(\alpha_1 \wedge \vartheta_1) \vee (\alpha_2 \wedge \vartheta_2) \vee \dots \vee (\alpha_s \wedge \vartheta_s).$$

Repeat this until φ contains no modalities. We finally obtain φ which is a first order modality free formula. Again, ask the oracle whether φ is true in all infinite models, by checking whether $\varphi \in Y_\sigma$. Return the answer.

This shows that the degree of T is $\leq_{\mathrm{T}} \mathbf{0}'$. It is exactly $\mathbf{0}'$ because the set of all first order tautologies of vocabulary σ is Σ^0_1-complete. □

6 Necessity on the basis of meaning postulates

Let us add something about necessity on the basis of meaning postulates. Carnap in [1] had considered also another definition of "necessity operator". According to it necessity means "follows from the meaning postulates". Meaning postulates are just some sentences. Let A be a set of **C**-sentences. Following Carnap we can consider the $\Box_A$ operator, necessity on the basis of A. Semantics for this operator is given by the following condition:

$$M \models \Box_A\varphi[\bar{a}] \Leftrightarrow \text{for each model } M' \text{ such that } M \models A \text{ and } |M| = |M'| \text{ we have } M' \models \varphi[\bar{a}].$$

In the natural case, when A is finite, this operator can be easily eliminated, because $\Box_A\varphi$ is equivalent to

$$\Box(\bigwedge A \Rightarrow \varphi),$$

where $\bigwedge A$ is the conjunction of all sentences from A. In this case Theorems 4.5 and 5.2 remain valid for the logic with operator $\Box_A$. For an infinite set A, the only assumption needed for Theorem 5.2 is that A is recursively enumerable.

7 Comparison with some other logics

In this section we compare the logic **C** with some other logics. Particularly, we consider logics with Henkin quantifiers L^*, second order logic SO, and its sublogic Σ^1_1, which is known to be equivalent to IF-logic (see e. g. [6]). For comparing these logics we use the following standard notions[5)].

Definition 7.1 Let L_1 and L_2 be logics. We say that L_1 *is a sublogic of* L_2 if for each L_1-sentence φ there is an L_2-sentence ψ in the vocabulary of φ true in the same models as φ. In this case we write $L_1 \leq L_2$. If $L_1 \leq L_2$ but not $L_2 \leq L_1$, then we say that L_1 *is a proper sublogic of* L_2 and we write $L_1 < L_2$. We say that logics L_1 and L_2 are *equivalent* ($L_1 \equiv L_2$) if $L_1 \leq L_2$ and $L_2 \leq L_1$.

The following theorem summarizes the known results about relations between the logics considered.

Theorem 7.2 $\Sigma^1_1 < L^* <$ SO.

The first inequality is the Enderton-Walkoe theorem. The second one follows from the Enderton theorem saying that $L^* \leq \Delta^1_2$. For the detailed references see [8].

5) For all the logics considered we use here standard framework as defined in [8].

From Theorem 4.5 we obtain additionally the following:

Theorem 7.3 $\mathbf{C} < L^*$.

P r o o f. We know that each **C**-formula is effectively equivalent to the first order closure of basic formulae, which are atomic first order formulae and Σ^1_1-formulae in empty vocabulary. However L^*-formulae are also closed on first order constructions and all the basic formulae are effectively equivalent to some L^*-formulae. This proves that $\mathbf{C} \leq L^*$.

The inequality is strict because **C**-sentences cannot differentiate infinite cardinalities. □

Now let us consider the recursive complexity of the sets of tautologies of the considered logics. By $\mathrm{Taut}_\sigma(L)$ we mean the set of all L-tautogies in vocabulary σ. By $\mathrm{Taut}(L)$ we mean the set of all L-tautogies in recursive vocabularies. For a set of expressions A, by $\deg(A)$ we denote the Turing degree of A. The relative recursivity of Turing degrees we denote by $\leq_{\mathrm{T}}$.

Then all considered inequalities are effective in the sense that if we have $L_1 \leq L_2$, then there is a recursive function f such that for each L_1-sentence φ, $f(\varphi)$ is L_2-sentence equivalent to φ. Then we obtain the following.

Proposition 7.4 $\deg(\mathrm{Taut}(\mathbf{C})), \deg(\mathrm{Taut}_\emptyset(L^*)), \deg(\mathrm{Taut}(\Sigma^1_1)) \leq_{\mathrm{T}} \deg(\mathrm{Taut}(L^*))$ *and*

$$\deg(\mathrm{Taut}(L^*)) \leq_{\mathrm{T}} \deg(\mathrm{Taut}(\mathrm{SO})).$$

In what follows we evaluate the degrees considered in a more precise way.

Theorem 7.5 (Krynicki-Lachlan, see [7]) $\deg(\mathrm{Taut}_\sigma(L^*)) = \deg(\mathrm{Taut}_\sigma(\mathrm{SO}))$*, for each recursive vocabulary σ with at least one binary predicate.*

We can improve this theorem by the following:

Theorem 7.6 $\deg(\mathrm{Taut}_\sigma(\Sigma^1_1)) = \deg(\mathrm{Taut}_\sigma(\mathrm{SO}))$*, for each vocabulary σ with at least one binary predicate.*

P r o o f. We know that $\Sigma^1_1 < L^* < \Delta^1_2$. So it suffices to prove that $\deg(\mathrm{Taut}(\Pi^1_2)) \leq_{\mathrm{T}} \deg(\mathrm{Taut}(\Sigma^1_1))$. We will show an effective reduction of the set $\mathrm{Taut}(\Pi^1_2)$ to the set $\mathrm{Taut}(\Sigma^1_1)$.

Let us consider a Π^1_2-formula ψ in a vocabulary σ of the form

$$\forall P_1 \cdots \forall P_n \, \varphi,$$

where φ is Σ^1_1-formula in a vocabulary σ. We consider φ as a formula in a richer vocabulary σ' obtained by adding to σ new predicates $P_1, \ldots, P_n$. Obviously the following two statements are equivalent:

1. ψ is true in all σ-models.
2. φ is true in all σ'-models.

The requirement that the vocabulary σ contains at least one binary predicate is sufficient by the argument similar to that of [7]. □

The following characterization of the logic L^* in empty vocabulary is known.

Theorem 7.7 (Mostowski-Zdanowski, see [10]) $\deg(\mathrm{Taut}_\emptyset(L^*)) = \mathbf{0}'$.

Now let us summarize all the results discussed.

Theorem 7.8 *The following sets are of degree $\mathbf{0}'$:*

1. $\mathrm{Taut}_\emptyset(L^*)$*, the set of all tautologies with Henkin quantifiers in empty vocabulary;*
2. $\mathrm{Taut}(\mathbf{C})$*, the set of all tautologies of the Carnap modal logic in any recursive vocabulary.*

Theorem 7.9 *The following sets are of the same Turing degree:*

1. $\mathrm{Taut}(\Sigma^1_1)$*, or equivalently the set of* IF*-tautologies in any recursive vocabulary with at least one binary predicate;*
2. $\mathrm{Taut}(L^*)$*, the set of all tautologies with Henkin quantifiers in any recursive vocabulary with at least one binary predicate;*
3. $\mathrm{Taut}(\mathrm{SO})$*, the set of all second order tautologies in any recursive vocabulary.*

It is known that the degree of $\mathrm{Taut}(\mathrm{SO})$ falls far beyond all arithmetical degrees and $\mathbf{0}'$ is one of the lower arithmetical degrees.

8 C in finite models

Gottlob in [4] gives a survey of the results related to propositional modal logic **C**. In one of his earlier papers [2] he proves that the tautology problem for it is $\mathrm{LOGSPACE}^{\mathrm{NP}}$-complete[6)]. In his paper [3] Gottlob proves that on ordered finite models logic L^* captures $\mathrm{LOGSPACE}^{\mathrm{NP}}$.

In what follows we consider shortly descriptive complexity of first order modal logic **C**. From Theorem 7.3 and Gottlob's paper [3] it follows:

Proposition 8.1 *All classes of finite models definable in* **C** *are in* $\mathrm{LOGSPACE}^{\mathrm{NP}}$.

On the other hand first order modal logic **C** does not cover (also in finite models) Σ^1_1, which – by the Fagin theorem – captures exactly NP. A simple example of a Σ^1_1-property not expressible in **C** is the class of finite models of the form $M = (U, A)$, where $A \subseteq U$ and $\mathrm{card}(A) > \mathrm{card}(U - A)$.

Let us finish these short remarks about finite model theoretic properties of **C** with the following question: Which natural complexity class is captured by first order modal logic **C**?

9 Some conclusions

We know that first order modal logic **C** is much simpler than SO. However it is not axiomatizable. This is an essential difference between **C** and modal logics defined axiomatically or by some Kripke semantics, which are recursively enumerable, or equivalently: axiomatizable. The corresponding modal logics can be faithfully embedded into first order theories of their Kripke structures. Standard Kripke semantics are defined by first order conditions imposed on a relative accessibility relation. The corresponding modal logics can be faithfully embedded into first order theories of their Kripke structures.[7)] Therefore all so defined logics should be recursively enumerable.

On the other hand logic **C** is not recursively enumerable. It can be shown by interpreting the first order consistency problem in the tautology problem for **C**. We translate a first order formula φ into φ' of the form

$$(\neg\Box(\forall x\forall y\,(f(x) = f(y) \Rightarrow x = y) \Rightarrow \forall z\exists x\,(f(x) = z)) \wedge \exists x\, U(x) \Rightarrow \Diamond\varphi^U),$$

where φ^U is the relativization of φ to a unary predicate U. The formula φ is consistent if and only if it has at most countable model, which is equivalent to truth of $\Diamond\varphi^U$ in each infinite model. Nevertheless the set of tautologies of **C** is of the same degree as that of first order logic. It follows that pure "alethic" modal logic – which according to Hintikka is the logic **C** – is not recursively equivalent to SO.

We can obtain a logic "partially alethic" in this sense by classifying vocabulary expressions into *stable* and *unstable* in the sense that the necessity operator would be interpreted as universal quantification over unstable expressions. So considering only relational vocabularies and having $P_1, \dots, P_n$ as all unstable predicates, formulae of the form $\Box\varphi$ will be interpreted as $\forall P_1 \cdots \forall P_n\,\varphi$. In such "partially alethic" modal logic we can easily interpret Σ^1_1 logic by translating formulae of the form

$$\exists P_1 \cdots \exists P_n\,\varphi$$

into $\Diamond\varphi$.[8)] Therefore the Turing degree of the set of such "partially alethic" modal logic is exactly the same as that of second order logic.[9)]

Finally, let us observe that it is not obvious why such a high recursive degree would be an argument against the plausibility of the considered logics. In fact, while Hintikka makes this argument in [5] against "alethic" modal logic, he does no longer make it against IF logic, which has, as we here proved, the same recursive degree as second order logic.

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6) $\mathrm{LOGSPACE}^{\mathrm{NP}}$ is the complexity class of those sets which can be recognized by an oracle Turing machine using $c \log n$ working memory for queries of size n and an oracle belonging to the class NP.

7) This argument was explicitly applied in an unpublished work (master thesis) of the second author [9].

8) We guess that a similar line of reasoning motivated Hintikka in [5].

9) Let us observe that **C** can be considered as a "partially alethic" modal logic having all predicates unstable.

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